Unipotent p-adic differential systems

Bernard Le Stum*
Université de Rennes 1

Version of October 18, 2001

*lestum@univ-rennes1.fr

Contents

1 The 1-dimensional case (Dwork) **3**

2 The motivic π_1 (Deligne) **10**

3 The unipotent rig π_1 (Chiarellotto-LS) **17**

4 Slope filtration (Chiarellotto-LS) **22**

1 The 1-dimensional case (Dwork)

Let p be a prime and $\mathbf{C}_p$ be the smaller complete algebraically closed field for an absolute value with $|p| = \frac{1}{p}$.

We set

$$U := D(0, 1^+) \setminus \cup_{i=1}^r D(x_i, 1^-) \subset \mathbf{C}_p$$

with $|x_i| = 1$ and $|x_i - x_j| = 1$, and call $A^\dagger$ the ring of holomorphic functions on U:

$$A^\dagger = \{f(t) := \sum_{k=0}^{\infty} a_{0,k} t^k + \sum_{i=1}^{r} \sum_{k=1}^{\infty} \frac{a_{i,k}}{(t - x_i)^k},$$

$$\exists \lambda > 1, \forall i : 0, \ldots, r, \lambda^k |a_{i,k}| \to 0\}.$$

If $G = [f_{i,j}] \in M_n(A^\dagger)$, we write $\frac{d}{dt}G := [\frac{d}{dt}f_{i,j}]$.

A *differential system in the neighborhood of* U is a system

$$(\mathcal{S}) : \frac{d}{dt}X + GX = 0, G \in M_n(A^\dagger).$$

If $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are given by G_1 and G_2 and $H \in GL_n(A^\dagger)$, we write

$$(\mathcal{S}_1) \sim_H (\mathcal{S}_2) \text{ iff } \frac{d}{dt}H = HG_1 - G_2H$$

and we say that they are *equivalent*.

If $(\mathcal{S})$ is a differential system given by G, we define by induction $G_0 = I_n$ and

$$G_{i+1} = \frac{d}{dt} G_i - G_i G$$

and call

$$\epsilon(\tau) := \sum_{i=1}^{\infty} \frac{1}{i!} G_i \tau^i \in M_n(A^\dagger[[\tau]])$$

the *Taylor series* of $(\mathcal{S})$.

The differential system is said *overconvergent* if the radius of convergence of the Taylor series is (at least) 1.

Back to $U \subset \mathbf{C}_p$, there exists $q = p^f$ such that for all $i = 1, \ldots, r$, we have $|x_i^q - x_i| < 1$. If $f \in A^\dagger$, then its *Frobenius pull back* $f^{(q)}$ defined by $f^{(q)}(t) := f(t^q)$ is also in $A^\dagger$. If $G = [f_{i,j}] \in M_n(A^\dagger)$, we write $G^{(q)} := [f_{i,j}^{(q)}]$.

The *Frobenius pull back* $(\mathcal{S}^q)$ of a differential system $(\mathcal{S})$ is the differential system given by $qt^{q-1}G^{(q)}$.

We say that $\mathcal{S}$ has a *strong* frobenius if $(\mathcal{S}^{(q)}) \sim (\mathcal{S})$. If this is the case, the system is overconvergent.

A system $(\mathcal{S})$ is called *unipotent* if it is equivalent to a system given by G strictly upper triangular.

Questions : Is a unipotent system always overconvergent ? Does it always have a Frobenius ?

The answer to the first question is yes, the answer to the second is no (Chiarellotto-LS).

However, the answer the second question is almost yes since one can always increase the rank of the system in order to get a Frobenius (independently, Chiarellotto, Crew and Deligne).

Example : Take $r = 2$ with $x_1 = 0, x_2 = 1$. Then, one can show that the differential system given by

$$G := \begin{bmatrix} 0 & \frac{1}{t} & \frac{1}{t-1} \\ 0 & 0 & \frac{1}{t} \\ 0 & 0 & 0 \end{bmatrix}$$

does not have a Frobenius.

We can extend it to

$$G_+ := \begin{bmatrix} 0 & 0 & 0 & \frac{1}{t-1} \\ 0 & 0 & \frac{1}{t} & \frac{1}{t-1} \\ 0 & 0 & 0 & \frac{1}{t} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, we have

$$qt^{q-1}G_{+}^{(q)} = \begin{bmatrix} 0 & 0 & 0 & \frac{qt^{q-1}}{t^q-1} \\ 0 & 0 & \frac{q}{t} & \frac{qt^{q-1}}{t^q-1} \\ 0 & 0 & 0 & \frac{q}{t} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the Frobenius is given by

$$H := \begin{bmatrix} 0 & 0 & 0 & u \\ q-1 & 1 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{bmatrix}$$

with $u(t) = \log(1-t^q) - \log(1-t)^q \in A^\dagger$.

2 The motivic π_1 (Deligne)

If G be an affine group scheme over a field K, we call

$$\mathrm{Rep}(G) = \{\text{finite dim. } K\text{-linear rep. of } G\}.$$

We will also write

$$G^{un} = \varprojlim H$$

where H runs through the unipotent quotients of G (subgroups of $U_{n,K}$).

Recall also that an object of a “Tannakian” category is *unipotent* if it is an iterated extension of the unit object. Then,

$$\mathrm{Rep}(G^{un}) \sim \{\text{fin. dim. unipotent } K\text{-linear rep. of } G\}.$$

Now, if π is any group, then

$$\pi^{alg} = \varprojlim G$$

where G runs trough all Zariski dense morphism $\pi \to G(K)$ with G algebraic. Then,

$$\mathrm{Rep}(G) \sim \{\text{finite dim. } K\text{-linear rep. of } \pi\}.$$

As an example, take $K = \mathbf{C}$ and $\pi = \mathbf{Z}$. Then,

$$\mathbf{Z}^{alg} = \mathbf{G}_{a,\mathbf{C}} \times \mathcal{H}om(\underline{\mathbf{C}}^{\times}, \mathbf{G}_{m,\mathbf{C}}),$$

but $(\mathbf{Z}^{alg})^{un} = \mathbf{G}_{a,\mathbf{C}}$.

Now, let V be a connected smooth algebraic variety over $\mathbf{C}$ and $x \in V(\mathbf{C})$. Then,

$$Rep(\pi_1(V(\mathbf{C}), x)^{alg})$$

$$\sim \{\text{fin. dim. } \mathbf{C}\text{-linear rep. of } \pi_1\}.$$

$$\sim \{\text{Integrable differential systems on } V(\mathbf{C})\}.$$

$$\sim \{\text{Int. diff. systems on } V \text{ regular at } \infty\}.$$

Since a unipotent integrable differential system is always regular at infinity, we also get

$$Rep((\pi_1(V(\mathbf{C}), x))^{alg})^{un})$$

$$\sim \{\text{Unipotent int. differential systems on } V\}.$$

In general, if V/K is a connected smooth algebraic variety over a field of characteristic zero and $x \in V(K)$, there exists an affine group scheme $\pi_1^{dR}(V, x)$ such that

$$Rep(\pi_1^{dR}(V, x)) \sim \{\text{Int. diff. systems on } V \text{ reg. at } \infty\}.$$

Of course, when $K = \mathbf{C}$, we get

$$\pi_1^{dR}(V, x) = \pi_1(V(\mathbf{C}), x)^{alg}.$$

Unfortunaltely, when $K \subset K'$, we have

$$K' \otimes_K \pi_1^{dR}(V, x) \neq \pi_1^{dR}(V_{K'}, x)$$

in general as the following example shows

$$\pi_1^{dR}(\mathbf{G}_{mK}, 1) = \mathbf{G}_{aK} \times \mathcal{H}om(\underline{K/\mathbf{Z}}, \mathbf{G}_{mK}),$$

If we consider the unipotent part, we know that

$$Rep(\pi_1^{dR}(V,x)^{un}) \sim \{\text{Unip. int. diff. sys. on } V\}.$$

and it is true that

$$K' \otimes_K \pi_1^{dR}(V,x)^{un} = \pi_1^{dR}(V_{K'},x)^{un}$$

In particular, if

$$[K:\mathbf{Q}] < \infty,$$

we have for each embedding $K \hookrightarrow \mathbf{C}$,

$$\mathbf{C} \otimes_K \pi_1^{dR}(V,x)^{un} = (\pi_1(V(\mathbf{C}),x)^{alg})^{un}.$$

It is therefore natural to study also for each embedding $K \hookrightarrow \mathbf{C}_p$, the group $\pi_1^{dR}(V_{\mathbf{C}_p}, x)^{un}$.

Deligne shows that under suitable geometric hypothesis, $\pi_1^{dR}(V, x)^{un}$ can be endowed with a Frobenius automorphism F.

Question : Is there a natural construction for this Frobenius automorphism ?

The answer is yes and this construction requires the use of overconvergent crystals (Chiarellotto-LS).

3 The unipotent rig π_1 (Chiarellotto-LS)

Let K be a complete field of char 0 with $|p| = 1/p$,

$$\mathcal{V} := \{x \in K, |x| \leq 1\}, \mathfrak{m} := \{x \in K, |x| < 1\},$$

$k := \mathcal{V}/\mathfrak{m}$ and $x \mapsto \bar{x}, \mathcal{V} \to k$ the reduction map.

Let X/k be a smooth algebraic variety and $X \hookrightarrow P$ an embedding in a formal scheme with proper Zariski closure.

The generic fibre P_K of P is a K-analytic variety, there is a *specialization map* $sp : P_K \to P$ and the *tube* of X in P is $]X[_P := sp^{-1}(X)$. Then $\mathcal{O}_P^\dagger$ is the sheaf of analytic functions in the neighborhood of $]X[_P$.

Example : Assume $K = \mathbf{C}_p$ and with the notations of the first paragraph,

$$X = \mathbf{A}^1 \backslash \{\bar{x}_1, \ldots, \bar{x}_r\},$$

and $P = \mathbf{P}^1$. Then $P_K = \mathbf{P}^{1,an}$, the set of rational points of $]X[_P$ is U and

$$\Gamma(]X[, \mathcal{O}_P^\dagger) = A^\dagger.$$

Note that sp takes a point of homogeneous coordinates (a, b) with $\max(a, b) = 1$ to the point of homogeneous coordinates $(\bar{a}, \bar{b})$.

An *overconvergent isocrystal* on X is a coherent $\mathcal{O}_P^\dagger$-module with an "overconvergent" integrable connection. The overconvergence condition is stable under extensions and it follows that a unipotent coherent module with an integrable connection is automatically overconvergent.

If X is connected and $x \in X(k)$, then there exists an affine group scheme $\pi_1^{rig,un}(X,x)$ on K such that

$$Rep(\pi_1^{rig,un}(X,x)) \sim \{\text{Unip. overc. isoc. on } V\}.$$

Assume k perfect and fix a lifiting $\sigma : K \to K$ of $\bar{x} \mapsto \bar{x}^q$(*Frobenius*). By functoriality, the morphism

$$F_X : X \to X, f \mapsto f^q$$

induces an automorphism

$$F : \pi_1^{rig,un}(X, x) \to \pi_1^{rig,un}(X, x).$$

This will give Deligne's Frobenius structure by transport of structure.

Here is how it works : Let $P/\mathcal{V}$ be proper smooth and Z a normal crossing divisor with smooth components in P. Let X (resp. V) be the special (resp. generic) fibre of $P\backslash Z$. Then,

$$\pi_1^{rig,un}(X,x) \simeq \pi_1^{dR}(V,x)^{un}.$$

This gives a natural description of Deligne's Frobenius structure on $\pi_1^{dR}(V,x)^{un}$.

4 Slope filtration (Chiarellotto-LS)

If X is a variety over k, an overconvergent F-isocrystal on X/K is an overconvergent isocrystal endowed with a "Frobenius" isomorphism

$$\Phi : F_X^* E \simeq E.$$

For example, an F-isocrystal on k/K is a finite dimensional K-vector space H endowed with a σ-linear automorphism $\varphi : H \to H$. A *pro-isocristal* is an inverse limit of isocristals.

Assume k is algebraically closed, the valuation of $\mathcal{V}$ is discrete, π a uniformizer such that $\sigma(\pi) = \pi$ and let $d = [K^\sigma : \mathbf{Q}_p]$.

Then we have Dieudonné-Manin decomposition : Any pro-isocristal H has an increasing $\mathbf{Q}$-filtration $Fil_\lambda H$ with $Gr_\lambda H$ *pure of slope* λ (if $\lambda d = r/s$, then $Gr_\lambda H$ has a basis made of u such that $\varphi^s(u) = \pi^r u$).

Theorem 1 : If E is a unipotent overconvergent isocrystal, then E is a quotient of a unipotent overconvergent F-isocrystal.

The above theorem answers Dwork's question. We will prove it later. Before, we want to state another theorem.

An overconvergent F-isocrystal E is *pure of slope* λ if for all $x \in X(k)$ its fibre E_x is pure of slope λ.

Theorem 2 : If E is a unipotent overconvergent F-isocrystal, then E has a filtration Fil_λ with Gr_λ poure of slope λ.

We now come to the proofs.

The group $G := \pi_1^{rig,un}(X, x)$ is unipotent. Therefore, if $\mathfrak{g} := LieG$ and $\mathcal{U}$ is the complete enveloping algebra of $\mathfrak{g}$, we have

$$Rep(G) \sim \{\text{Finite dim. rep. of } \mathfrak{g}\}.$$

$$\sim \{\mathcal{U}\text{-modules of finite dimension over } K\}.$$

By construction $\mathcal{U}$ comes with a Frobenius automorphism F. A $\mathcal{U}$-module endowed with an F^{-1}-linear automorphism is called an F-$\mathcal{U}$-module. Then, we have

$$\{\text{Unip. overc. F-isoc}\} \sim$$

$$\{F\text{-}\mathcal{U}\text{-mod. of fin. dim. over } K\}.$$

Theorem 1 follows easily : any $\mathcal{U}$-module of finite dimension over K is a quotient of a finite sum of $\mathcal{U}/\mathfrak{a}^n$ where $\mathfrak{a}$ is the augmentation ideal. To prove theorem 2, we will need an intermediate result.

Lemma : $\mathcal{U}$ is a pro-isocrystal with negative slopes.

Assume this is true. Any F-$\mathcal{U}$-module H of finite dimension over K is an F-isocrystal and has a slope filtration. Since the slopes of $\mathcal{U}$ are negative, $Fil_\lambda H$ is stable under the action of U and is also an F-$\mathcal{U}$-module.

We still have to prove the lemma : since

$$\mathcal{U} = \varprojlim \mathcal{U}/\mathfrak{a}^n,$$

one easily reduces to the same assertion for $\mathfrak{a}/\mathfrak{a}^2 = \mathfrak{g}^{ab}$. But $\mathfrak{g}^{ab}$ is dual to

$$\mathrm{Hom}(G, \mathbf{G}_a) = \mathrm{Ext}(\mathcal{O}^\dagger, \mathcal{O}^\dagger) = H^1_{rig}(X)$$

which has positive slopes.