A quantum Simpson correspondence
(Banff – 2017)

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Contents

Comparison and correspondence

Given two different cohomological theories, an *isomorphism (comparison)* in cohomology usually extends to an *equivalence (correspondence)* between the categories of coefficients. The correspondence is the *non-abelian version* of the comparison.

Here is a basic example: if X is a topological space which is connected and locally simply connected, then the isomorphism

$$H^i(X, \mathbb{Z}) \simeq \mathrm{Hom}(H_i(X), \mathbb{Z})$$

(sheaf cohomology and singular cohomology) extends to a correspondence

$$\mathrm{Sh}_{\mathrm{l.c.}}(X) \simeq \pi_1(X, x) - \mathrm{Sets},$$

and in particular, if k is any commutative ring, we obtain

$$k - \mathrm{Mod}_{\mathrm{l.c.}}(X) \simeq \mathrm{Rep}_k(\pi^1(X, x)).$$

Riemann-Hilbert and Simpson correspondences

If X is a smooth complex algebraic variety (or a complex manifold), we may consider *de Rham theorem*

$$H^i_{\mathrm{dR}}(X) \simeq H^i(X^{\mathrm{an}}, \mathbb{C}) \quad (\simeq \mathrm{Hom}(H_i(X^{\mathrm{an}}), \mathbb{C})).$$

The non-abelian version is (srict) *Riemann-Hilbert* correspondence

$$\mathrm{MIC}_{\mathrm{reg}}(X) \simeq \mathbb{C} - \mathrm{Vect}_{\mathrm{l.c.}}(X^{\mathrm{an}}) \quad (\simeq \mathrm{Rep}_{\mathbb{C}}(\pi^1(X^{\mathrm{an}}, x)).$$

If X is projective (or compact Kähler), we may consider *Hodge theorem*

$$H^i_{\mathrm{dR}}(X) \simeq H^i_{\mathrm{Hodge}}(X) \quad (:= \oplus_k H^{i-k}(X, \Omega^k_X)).$$

The non-abelian version is *Simpson correspondence* ([Sim92])

$$\mathrm{MIC}(X) \simeq \mathrm{HIG}_{\mathrm{s.s.},\mathrm{c}_1=0}(X).$$

Local Ogus-Vologodsky correspondence

Assume now that X is a scheme which is smooth over a fixed scheme S of characteristic $p > 0$. We may consider *Cartier isomorphism*

$$C : \mathcal{H}^i(F_*\Omega_X^\bullet) \simeq \Omega_{X'}^i, \quad f^{p-1}\mathrm{d}f \leftrightarrow \mathrm{d}F^*f \text{ for } i = 1,$$

where $F : X \to X'$ denotes the relative Frobenius. When F lifts modulo p^2, Cartier isomorphism extends to an isomorphism in the derived category ([DI87])

$$C : F_*\Omega_X^\bullet \underset{D}{\simeq} \bigoplus_k \Omega_{X'}^k[-k] \quad \left(C^{-1} = \frac{1}{p}\widetilde{F}^* \text{ for } i = 1\right),$$

giving rise to an isomorphism

$$C : H^i_{\mathrm{dR}}(X) \simeq H^i_{\mathrm{Hodge}}(X').$$

It extends to an equivalence ([OV07])

$$C : \mathrm{MIC}(X)_{\mathrm{nilp}} \simeq \mathrm{HIG}(X')_{\mathrm{nilp}}.$$

Comments

- If we only assume that X lifts modulo p^2, one can use local liftings of Frobenius and obtain ([DI87])

$$C : (F_*\Omega_X^\bullet)_{<p} \underset{D}{\simeq} \bigoplus_{k<p} \Omega_{X'}^k[-k],$$

  as well as a correspondence ([OV07])

$$C : \mathrm{MIC}(X)_{\mathrm{nilp}<p} \simeq \mathrm{HIG}(X')_{\mathrm{nilp}<p}.$$

- If X is a proper smooth variety over a p-adic field K, then *Hodge-Tate theorem*

$$H^i_{\mathrm{et}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq H^i_{\mathrm{Hodge}}(X) \otimes_K \mathbb{C}_p$$

  also gives rise to a p-adic Simpson theory ([AGT16]).
- There exists a p-adic Cartier theory but its relation to p-adic Simpson correspondence is not well understood yet ([Shi15]).

q-difference calculus

When A is a "function algebra in one variable x" and $f \in A$, we have

$$\partial(f) = \lim_{h\to 0} \Delta_h(f) = \lim_{q\to 1} \delta_q(f)$$

in which $\partial(f) = \frac{\mathrm{d}}{\mathrm{d}x}(f)$,

$$\Delta_h(f)(x) = \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x}.$$

Calculus with respect to Δ_h (resp. δ_q) instead of ∂ is called *difference* (resp. *q-difference*) *calculus*. Let us formalise this.

We fix from now on a commutative ring R and a *twisted R-algebra*: a commutative R-algebra A endowed with an endomorphism σ (think of $\sigma(x) = x$, $\sigma(x) = x + h$ or $\sigma(x) = qx$). A *twisted derivation* (think of ∂, Δ_h or δ_q) is an R-linear map $A \to M$ that satisfies the *twisted Leibniz rule*

$$\forall f, g \in A, \quad D(fg) = fD(g) + \sigma(g)D(f).$$

Twisted calculus

We set

$$P := A \otimes_R A, \quad I := \ker(P \to A, f \otimes g \mapsto fg), \quad \sigma(f \otimes g) := \sigma(f) \otimes g,$$

$$I^{(n+1)} := I\sigma(I)\cdots\sigma^n(I) \quad \text{and} \quad P_{(n)} := P/I^{(n+1)}.$$

We call $x \in A$ a *σ-coordinate* if $P_{(n)}$ is free on $1, \xi, \ldots, \xi^n$ for all $n \in \mathbb{N}$, with $\xi = 1 \otimes x - x \otimes 1$. We call x a *q-coordinate* if moreover, $\sigma(x) = qx$ with $q \in R$.

Example

If $q^p = 1$, R is a $\mathbb{Z}/p^N\mathbb{Z}$-algebra and x is an étale coordinate on A, then x is also a q-coordinate for a unique σ on A.

There exists a *universal twisted derivation* $d : A \to \Omega_{A,\sigma} := I/I^{(2)}$. When x is a σ-coordinate, there exists a unique twisted derivation ∂_σ (that depends on x) such that $\partial_\sigma(x) = 1$ (think of ∂, Δ_h or δ_q again).

Twisted connections

A *twisted connection* is an R-linear map $\nabla : M \to M \otimes_A \Omega_{A,\sigma}$ such that

$$\forall f \in A, \forall s \in M, \quad \nabla(fs) = s \otimes df + \sigma(f)\nabla(s).$$

We will denote by $\mathrm{MIC}_\sigma(A)$ the category of A-modules endowed with a twisted connection.

A *Higgs field* is an A-linear map $M \to M \otimes_A \Omega_A$. We will denote by $\mathrm{HIG}(A)$ the category of A-modules endowed with a Higgs field.

If x is a σ-coordinate on A, then a twisted connection on M amounts to an R-linear map $\partial_\sigma : M \to M$ such that

$$\forall f \in A, \forall s \in M, \quad \partial_\sigma(fs) = \partial_\sigma(f)s + \sigma(f)\partial_\sigma(s).$$

It is said to be *quasi-nilpotent* if $\forall s \in M, \exists N \in \mathbb{N}, \partial_\sigma^N(s) = 0$.

Similarly, if x is an étale coordinate on A, a Higgs field simply amounts to an A-linear map $M \to M$ (with analogous definition of quasi-nilpotency).

q-Simpson correspondence

We fix from now on an endomorphism F^* of R. A *p-Frobenius* (with respect to a σ-coordinate x) on A is a locally free morphism of rank p $F^* : A' := R_{F^*\nwarrow} \otimes_R A \to A$ such that $F^*(1 \otimes x) = x^p$ and $\partial_\sigma \circ F^* = 0$.

Example

Assume that $q^p = 1$, R is a $\mathbb{Z}/p^N\mathbb{Z}$-algebra with Frobenius lift F^* and x is an étale coordinate on A (and $\sigma(x) = qx$). Then there exists a unique p-Frobenius F^* (with respect to x) on A.

Theorem (Gros, –, Quirós)

Let $q \in R$ be a primitive p-th root of unity with p prime. Let A be a twisted R-algebra with étale q-coordinate x. Then any p-Frobenius on A induces an equivalence

$$C_\sigma : \mathrm{MIC}_\sigma(A)_{\mathrm{nilp}} \simeq \mathrm{HIG}(A')_{\mathrm{nilp}}.$$

Azumaya Splitting

Given a σ-coordinate x on A, the *twisted Weyl algebra* $\mathrm{D}_{A/R,\sigma}$ is the non-commutative polynomial ring in one variable (still written) ∂_σ over A with the commutation rule $\partial_\sigma f = \partial_\sigma(f) + \sigma(f)\partial_\sigma$. We will denote by $\mathrm{Z}_{A/R,\sigma}$ its center and by $\mathrm{Z}A_{A/R,\sigma}$ the centralizer of A.

The twisted *p-curvature theorem* states that if x is actually a q-coordinate such that $q^p = 1$, then $\mathrm{Z}_{A/R,\sigma} = A^{\partial_\sigma=0}[\partial_\sigma^p]$ and $\mathrm{Z}A_{A/R,\sigma} = A[\partial_\sigma^p]$.

Then, q-Simpson correspondence will follow by Morita equivalence from the twisted Azumaya splitting (completion is meant along the augmentation ideal):

Theorem (Gros, –, Quirós)

With the assumptions of the previous theorem, any p-Frobenius on A provides an isomorphism

$$\widehat{\mathrm{D}}_{A/R,\sigma} \simeq \mathrm{End}_{\widehat{\mathrm{Z}}_{A/R,\sigma}}(\widehat{\mathrm{Z}A}_{A/R,\sigma}).$$

Explicit formula

The Azumaya splitting comes from a morphism of rings

$$\mathrm{D}_{A/R,\sigma} \to \mathrm{End}_{\mathrm{Z}_{A/R,\sigma}}(\mathrm{Z}A_{A/R,\sigma}),$$

or, equivalently, a structure of $\mathrm{D}_{A/R,\sigma}$-module on the $\mathrm{Z}_{A/R,\sigma}$-module $\mathrm{Z}A_{A/R,\sigma}$. The action is explicitly given by

$$\forall f \in A, \quad \partial_\sigma \bullet f = \partial_\sigma(f) + \sigma(f)x^{p-1}\partial_\sigma^p.$$

One can show that $\partial_\sigma^k \bullet f = \Phi(\partial_\sigma^k f)$ where Φ is the A-linear map

$$\Phi : \mathrm{D}_{A/R,\sigma} \to \mathrm{Z}A_{A/R,\sigma}, \quad \partial_\sigma^n \mapsto \sum_{k=0}^{n} B_{k,n} x^{pk-n}\partial_\sigma^{pk},$$

in which, using the q-analog notation (explained later), we have

$$B_{k,n} \text{ `` = '' } \frac{(n)_q!}{(k)_{q^p}!(p)_q^k} \sum_{i=0}^{k} (-1)^{k-i} q^{\frac{p(k-i)(k-i-1)}{2}} \binom{k}{i}_{q^p} \binom{pi}{n}_q .$$

Twisted divided powers

We recall that the *q-analog* of an integer n is

$$(n)_q = 1 + q + \cdots + q^{n-1} \quad \left(\text{“} = \text{”} \; \frac{q^n - 1}{q - 1} \right).$$

We define the *twisted divided power polynomial ring* as the free module $A\langle\xi\rangle$ on $\xi^{[n]}, n \in \mathbb{N}$ with multiplication rule

$$\xi^{[m]}\xi^{[n]} = \sum_{i=0}^{\min(m,n)} (q-1)^i q^{\frac{i(i-1)}{2}} \binom{m+n-i}{m}_q \binom{m}{i}_q x^i \xi^{[m+n-i]}.$$

The completion $A\langle\langle\xi\rangle\rangle$ of $A\langle\xi\rangle$ along the $(\xi)^{[n]}$'s has also a (right) structure of A-algebra given by the *twisted Taylor map*

$$f \mapsto \sum_{k=0}^{\infty} \partial_\sigma^k(f)\xi^{[k]}$$

and the comultiplication on $A\langle\langle\xi\rangle\rangle$ is defined by $\xi^{[n]} \mapsto \sum_{i=0}^{n} \xi^{[n-i]} \otimes \xi^{[i]}$.

Duality

As filtered A-modules, $\mathrm{D}_{A/R,\sigma}$ and $A\langle\xi\rangle$ are dual to each other, with dual basis $\{\partial_\sigma^k\}_{k\in\mathbb{N}}$ and $\{\xi^{[n]}\}_{n\in\mathbb{N}}$. Multiplication on $\mathrm{D}_{A/R,\sigma}$ corresponds to comultiplication on $A\langle\langle\xi\rangle\rangle$.

Now, we define $A\langle\omega\rangle$ exactly as $A\langle\xi\rangle$ but with the simpler multiplication rule

$$\omega^{[m]}\omega^{[n]} = \sum_{i=0}^{\min(m,n)} (q-1)^i \binom{m+n-i}{i}\binom{n}{i} x^i \omega^{[m+n-i]}.$$

As filtered A-modules, $\mathrm{Z}A_{A/R,\sigma}$ and $A\langle\omega\rangle$ are dual to each other, with dual basis $\{\partial_\sigma^{pk}\}_{k\in\mathbb{N}}$ and $\{\omega^{[n]}\}_{n\in\mathbb{N}}$. Multiplication on $A[\partial_\sigma^p]$ corresponds to comultiplication on $A\langle\langle\omega\rangle\rangle$ (defined as before). The p-curvature theorem is obtained by duality from the existence of the *twisted divided p-power isomorphism*:

$$A\langle\omega\rangle \simeq A\langle\xi\rangle/\xi, \quad \omega^{[k]} \mapsto \overline{\xi^{[kp]}}.$$

Duality again

The above A-linear map $\Phi : \mathrm{D}_{A/R,\sigma} \to \mathrm{Z}A_{A/R,\sigma}$ comes by duality from the twisted *divided p-Frobenius map*

$$[F]^* : A\langle\omega\rangle \to A\langle\xi\rangle, \quad \omega^{[n]} \mapsto \sum_{i=n}^{pn} B_{n,i}(q) x^{pn-i} \xi^{[i]}.$$

The Azumaya splitting follows from the fact that $[F]^*$ induces an isomorphism

$$A[\xi]/\xi^{(p)} \otimes_A A\langle\omega\rangle \simeq A\langle\xi\rangle,$$

in which $\xi^{(n)} := \xi(\xi + (1-q)x)\cdots(\xi + (1-q^{n-1})x)$. Everything actually follows from the fact that the twisted divided p-Frobenius map is natural in the sense that

$$[F]^*(\omega^{[n]}) \text{ “}=\text{” } \frac{1}{(n)_{q^p}!(p)_q^n} F^*(\omega^{(n)})$$

with $\omega^{(n)} = \omega(\omega + (1-q^p)x)\cdots(\omega + (1-q^{(n-1)p})x)$ and

$$F^*(\omega) = (\xi + x)^p - x^p.$$

Quantum Cartier isomorphism

The q-Simpson correspondence is given by

$$\begin{array}{ccc} \mathrm{MIC}_\sigma(A)_{\mathrm{nilp}} & \xrightarrow{\simeq} & \mathrm{HIG}(A')_{\mathrm{nilp}} \\ (M, \partial_\sigma) & \longmapsto & \left(M^{\Phi=1}, \partial_\sigma^p\right) \\ \left(A \otimes_{A'} H, \partial_\sigma \otimes 1 + x^{p-1}\sigma \otimes \theta\right) & \longleftarrow\!\shortmid & (H, \theta) \end{array}.$$

Morita equivalence also provides an isomorphism (see also [Sch16]):

$$\mathrm{R}\Gamma_{\mathrm{Hodge}}(H) := [H \xrightarrow{\theta} H] \simeq [M \xrightarrow{\partial_\sigma} M] =: \mathrm{R}\Gamma_{\mathrm{dR},\sigma}(M).$$

As a consequence, we recover the q-Cartier inverse isomorphisms:

Corollary

We have

$$A' \overset{F^*}{\simeq} A^{\partial_\sigma = 0} \quad \text{and} \quad A' \overset{F^*}{\simeq} A/\partial_\sigma A.$$

And then ?

Assume that $q^p = 1$ (with p prime), R is a $\mathbb{Z}/p^N\mathbb{Z}$-algebra with Frobenius lift F^*, x is an étale coordinate on A and $\sigma(x) = qx$.

Then, there exists a natural morphism of A-algebras (*confluence*)

$$\begin{array}{ccc} D_{A/R,\sigma} & \longrightarrow & \widehat{D}_{A/R} \\ \partial_\sigma & \longmapsto & \sum \frac{(q-1)^{k-1}}{(k)_q!} x^{k-1} \partial^k \end{array}$$

(formally $\sigma = q^{x\partial}$). It induces a functor

$$\mathrm{MIC}(A)_{\mathrm{nilp}} \to \mathrm{MIC}_\sigma(A)_{\mathrm{nilp}} \simeq \mathrm{HIG}(A')_{\mathrm{nilp}}.$$

Unfortunately (but expectedly), its image falls inside $\mathrm{HIG}(A')_{\mathrm{triv}}$ (twisted p-curvature is zero).

It will actually be necessary to consider the more general case $q^{p^n} = 1$ and lift the quantum Simpson correpondence much as in [Shi15] (work in progress with Gros and Quirós).

What else ?

- The condition p prime may be replaced everywhere by the conditions
  1. R has q-characteristic p which means that p is the smallest positive integer such that $(p)_q = 0$,
  2. R is q-divisible which means that for all $n \in \mathbb{N}$, we have $(n)_q \in R^\times \cup 0$.

  This holds in particular if q is a primitive p-th root of unity in a *field* $K \subset R$ (p not necessary prime !). This holds also in the case $q = 1$ and $\mathrm{Char}(R) = p$ (prime): we recover Ogus-Vologodsky correspondence.
- The theory of twisted connections extends in a staightforward way to the case of complete adic rings and we get an equivalence between
  1. finite A-modules with a topologically quasi-nilpotent σ-connection and
  2. finite A'-modules with a topologically quasi-nilpotent Higgs field.
- The theory of twisted differential operators also extends to several variables and our results should also extend to this

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