Cohomology and sheaves
(Centre Lebesgue – Rennes – 2014)

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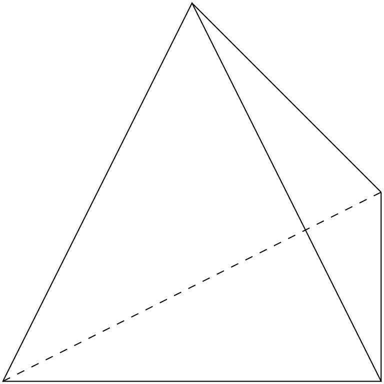
Section 1

Introduction

Euler formula



number of vertices - number of edges + number of faces = 2

More generally, this formula holds for any polyhedron of genus 0 (no hole).

Betti number

Usual (meaning triangulable) topological spaces may be obtained by glueing simplices (such that the above tetrahedra) and one can define their Betti numbers by combinatorial arguments. This may be shown to be independent of the triangulation.

Examples

1. For the sphere, we have $h_0 = h_2 = 1$ and $h_1 = 0$ so that $h_0 - h_1 + h_2 = 2$ (a sphere is homeomorphic to a connected polyhedron of genus 0).
2. For a torus, we have again $h_0 = h_2 = 1$ but now $h_1 = 2$ so that $h_0 - h_1 + h_2 = 0$.
3. More generally, for a compact surface of genus g (g holes) with c components, we have $h_0 = h_2 = c$ and $h_1 = 2g$ so that $h_0 - h_1 + h_2 = 2c - 2g$.

Homology/cohomology

If X is a triangulable topological space, one can define its simplicial homology groups $\mathrm{H}_i(X)$ in such a way that that $h_i = \operatorname{rank} \mathrm{H}_i(X)$.

We will actually use singular homology (considering all continuous maps from simplices to X) instead of simplicial homology: they are isomorphic but there is more flexibility in the later (which is always defined).

In fact, we will consider the cohomology groups $\mathrm{H}^i_{\mathrm{sing}}(X)$ (which are essentially dual to homology groups). More generally, when X is sufficiently connected, one can define $\mathrm{H}^i_{\mathrm{sing}}(X, V)$ where V is any representation of $\pi_1(X, x)$.

A sheaf on a topological space X is a collection of objects defined on open subsets of X that one can glue locally (more precise definition later). Also, one can define the cohomology of a sheaf.

Then, the representations V of $\pi_1(X, x)$ correspond bijectively to locally constant sheaves $\mathcal{F}$ on X and cohomology coincides.

If X is a (real) differential manifold, one can define its de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X)$ as the cohomology of the complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \longrightarrow \cdots$$

of $\mathcal{C}^\infty$ forms on X (with $\mathrm{H}^i := \ker d^i / \mathrm{im}\, d^{i-1}$). Then, de Rham theorem states that $\mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{R}) \simeq \mathrm{H}^i_{\mathrm{dR}}(X)$.

Here is a sheaf-theoretic interpretation of de Rham theorem: one can define a de Rham complex which is a complex of sheaves $\Omega^\bullet_X$ on X and show that $\mathrm{H}^i(X, \Omega^\bullet_X) \simeq \mathrm{H}^i_{\mathrm{dR}}(X)$. On the other hand we know that $\mathrm{H}^i(X, \mathbb{R}_X) \simeq \mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{R})$. de Rham theorem then follows from the fact that $\Omega^\bullet_X$ is a resolution of $\mathbb{R}_X$ (more on this later).

The same results hold for analytic varieties over $\mathbb{C}$. Moreover, there exists a bijective correspondence between locally constant sheaves of finite $\mathbb{C}_X$-vector spaces and coherent $\mathcal{O}_X$-modules with an integrable connection. And de Rham theorem extends to these coefficients.

If X is a smooth algebraic variety over a field k, we can define a de Rham complex $\Omega^\bullet_X$ on X (a complex of sheaves for the Zariski topology) and the de Rham cohomology of X as $\mathrm{H}^i_{\mathrm{dR}}(X) := \mathrm{H}^i(X, \Omega^\bullet_X)$.

When $k = \mathbb{C}$, one has $\mathrm{H}^i_{\mathrm{dR}}(X) \simeq \mathrm{H}^i_{\mathrm{dR}}(X^{\mathrm{an}})$. More generally, there exists an equivalence between coherent modules with a regular integrable connection on both sides and cohomology coincides.

It is also convenient to introduce the sheaf of differential operators $\mathcal{D}_X$ on X. This is a non commutative ring. But a module with an integrable connection is simply a left $\mathcal{D}_X$-module. Moreover, we have

$$\mathrm{H}^i_{\mathrm{dR}}(X, \mathcal{F}) \simeq \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F}).$$

Also, this is beyond the scope of this course, but constructible $\mathbb{C}_X$-vector spaces correspond to regular holonomic $\mathcal{D}_X$-modules once we work in the derived categories. And again, cohomology coincides.

Étale

If X is a topological space, we may consider all local isomorphisms $X' \to X$. This is a site and a sheaf on this site is a collection of objects on these X' that glue locally. Actually, such a sheaf corresponds to a usual sheaf on X and cohomology coincides.

When X is an algebraic variety over a field k, the étale site is made of all étale (meaning flat and unramified) maps $X' \to X$. This is not equivalent to Zariski topology. This is better: when $k = \mathbb{C}$, finite locally constant sheaves on the étale site of X correspond to finite locally constant sheaves on (the site of local isomorphisms of) X^{an} and cohomology coincides.

Moreover, as long as the order of the sheaves are not divisible by the characteristic p of k, the étale cohomology behaves very well. This gives rise to various ℓ-adic cohomologies for $\ell \neq$ ăp. They provide a tool for proving Weil conjectures (Grothendieck school).

Crystalline

If X is an algebraic variety, we may consider the infinitesimal site of all "nilpotent" immersions $U \hookrightarrow T$ where U is an open subset of X. There exists a sheaf of commutative rings $\mathcal{O}_{X,\mathrm{inf}}$ on this site (on $U \hookrightarrow T$, this is the ring of functions defined on T).

In characteristic zero, if X is smooth, then locally finitely presented $\mathcal{O}_{X,\mathrm{inf}}$-modules correspond to coherent $\mathcal{O}_X$-modules with an integrable connection. And again cohomology coincides.

In characteristic $p > 0$, the differential approach makes sense but does not give the correct objects. However, we may define the crystalline site exactly as the infinitesimal site by using divided powers instead of nilpotent immersions. This gives a good theory in the proper smooth case. We may also use Monsky-Washnitzer cohomology in the smooth affine case, or more generally rigid cohomology by doing de Rham cohomology on a suitable lifting. There exists also a theory of arithmetic $\mathcal{D}$-modules.

Section 2

Categories and functors

Warning

We will consider here collections of objects that do not make a set. But we will still use the notations and the vocabulary of set theory.

We need to do that because we want to consider, for example, the collection of all sets. And this is not a set as Russell's paradox shows: let

$$y := \{x, \quad x \notin x\}.$$

Then, if $y \in y$, we have $y \notin y$ and conversely.

In order to get around this problem and stay inside classical set theory, one may also work in a fixed universe. Then a set means a "set that belongs to our universe" and a collection is a "set that might not belong to our universe".

One may also think that set theory is not a good model for category or topos theory and forget about it for a while. . . we'll be careful however.

Quiver

A quiver $\mathcal{C}$ consists in

1. A collection of objets X,
2. For all $X, Y \in \mathcal{C}$, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$,
3. For any object X, an identity morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we say that X is the domain of f, that Y is its codomain and we write $f : X \to Y$.

We will also write $\mathrm{End}_{\mathcal{C}}(X) := \mathrm{Hom}_{\mathcal{C}}(X, X)$ and call it the set of endomorphisms of X. This is a pointed set.

Note that the data of 1) and 2) gives what is called a (directed) graph. Then one usually say vertices and edges instead of objects and morphisms.

Category

A category $\mathcal{C}$ is a quiver endowed with a composition rule

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

for all $X, Y, Z \in \mathcal{C}$, which is associative:

$$\forall f : X \to Y, g : Y \to Z, h : Z \to T, \quad h \circ (g \circ f) = (h \circ g) \circ f.$$

with unit:

$$\forall X \in \mathcal{C}, \quad \begin{cases} \forall f : X \to Y, & f \circ \mathrm{Id}_X = f \\ \forall f : Y \to X, & \mathrm{Id}_X \circ f = f \end{cases}$$

Note that the identity is uniquely determined by this property. Note also that $\mathrm{End}_{\mathcal{C}}(X)$ is now a monoid.

Examples

1. **Set**: sets, maps and usual composition.
2. **FSet**: finite sets and maps between them.
3. G-**Set**: sets endowed with a (left) action of a fixed monoid G (and compatible maps).
4. **Set**-G: sets endowed with a right action of G.

Examples

1. **Mon**: monoids and monoid homomorphisms (preserving the unit).
2. **Gr**: groups and group homomorphisms.
3. **FGr**: finite groups and group homomorphisms between them.
4. **Ab**: abelian groups and group homomorphisms.
5. **Rng**: (associative unitary) rings and (unitary) ring homomorphisms.
6. **CRng**: commutative (associative unitary) rings and (unitary) ring homomorphisms.

Examples

1. G-**Mod**: abelian groups with a linear action of a fixed monoid G.
2. **Mod**-G: abelian groups with a linear action of G on the right.
3. A-**Mod**: (left) A-modules (where A is a fixed ring).
4. k-**Vec**: k-vector spaces, same as A-**Mod** when $A = k$ is a field.
5. **Mod**-A: right A-modules.
6. k-**Alg**: k-algebras (where k is a fixed commutative ring).
7. k-**CAlg**: commutative k-algebras.
8. $\mathbf{Rep}_k(G)$: k-linear representations of G.

Example

A-**Op**: an object is a pair (M, u) with $M \in A$-**Mod** (where A is a fixed ring) and $u \in \mathrm{End}_A(M)$; a morphism is an A-linear map $f : M \to M'$ such that $u' \circ f = f \circ u$.

Examples

1. **Act**: an object is a pair (G, E) made of a monoid G and a G-set E. A morphism is a couple made of a homomorphism $\varphi : G \to G'$ and a map $f : E \to E'$ satisfying

$$\forall g \in G, x \in E, f(gx) = \varphi(g)f(x).$$

2. **Mod**: an object is a pair (A, M) made of a ring A and an A-module M. A morphism is a couple made of a ring homomorphism $\varphi : A \to A'$ and an additive homomorphism $f : M \to M'$ satisfying

$$\forall a \in A, s \in M, f(as) = \varphi(a)f(s).$$

3. **Alg**: an object is a pair (k, A) made of a commutative ring k and a k-algebra A. A morphism is a couple made of a ring homomorphism $\varphi : k \to k'$ and another ring homomorphism $f : A \to A'$ satisfying

$$\forall \alpha \in k, a \in A, f(\alpha a) = \varphi(\alpha)f(a).$$

Examples

1. **Top**: topological spaces and continuous maps.
2. **TGr**: topological groups and continuous group homomorphisms.
3. **Sch**: schemes.
4. $\mathbf{Var}_{/k}$: algebraic varieties over a field k.
5. **An**: analytic varieties over $\mathbb{C}$.

Example

Δ: this is the simplex category whose objects are the positive finite ordinals $[n]$ for $n \in \mathbb{N}$ and morphisms are order preserving maps. Recall that $[n]$ is the set $\{0, \ldots, n\}$ totally ordered by $0 < 1 < 2 < \cdots < n$. The injective (resp. surjective) maps

$$\delta_i^n : [n-1] \to [n] \quad (\text{resp. } \sigma_i^n : [n+1] \to [n])$$

are called the face (resp. degeneracy) maps. Any morphism in Δ is a composition of some face and degeneracy maps.

Examples

1. Any ordered set $(I, \leq)$ may (and will) be seen as a category: the objects are the elements of I and for all $i, j \in I$, there exists a unique morphism $i \to j$ if $i \leq j$ and none otherwise.
2. As a particular case, any finite ordinal will be seen as a category. For example:
   1. $\mathbf{0} = \emptyset = \{\}$: the category with no object (and no morphisms).
   2. $\mathbf{1} = [0] := \{0\}$: the category with exactly one morphism (and one object).
   3. $\mathbf{2} = [1] = \{0 \to 1\}$: the category with one non trivial morphism (and two objects).
3. $\mathbf{Open}(X)$ where X is a topological space: this is the set of open subsets of X ordered by inclusion.
4. Any (unordered) set may be ordered by equality and becomes a category (with only trivial morphisms).

Metacategory

A metacategory is a collection of morphisms endowed with a partial associative composition: the composite $g \circ f$ is not always defined but $h \circ (g \circ f)$ is defined if and only if $(h \circ g) \circ f$ is defined, and then, they are equal.

One defines an object of the metacategory as a morphism X that satisfies $f \circ X = f$ and $X \circ f = f$ whenever this is defined.

One requires that for any morphism f, there exists a (unique) object $X(f)$ (its domain) such that $f \circ X(f)$ is defined and a (unique) object $Y(f)$ (its codomain) such that $Y(f) \circ f$ too is defined. Finally one requires also that $g \circ f$ is defined if and only if $Y(f) = X(g)$.

Then, if the collections

$$\mathrm{Hom}(X, Y) := \{f, X(f) = X \text{ and } Y(f) = Y\}$$

are actually sets, we obtain a category. And conversely, in any category, the morphisms form a metacategory.

Examples

1. $\mathbf{Mat}_A$ where A is a fixed ring: this is the metacategory of matrices of any size and multiplication of matrices (whenever this is defined). This is actually a category.
2. A monoid G is a metacategory. This is actually a category with exactly one object.
3. There exists metacategories that are not categories (the metacategory of all functors for example - see below).

Many authors give a broader meaning to the notion of category (with no set-theoretic condition) which makes it equivalent to what we call a metacategory. And then, they call "locally small category" what we did define as a category. Actually, we will make a tremendous use of the fact that the morphisms between two objects do make a set. This is why we believe our definition is the best for our purpose. However, most definitions below apply also to metacategories.

Small and finite categories

A category is said to be small (resp. finite) if the corresponding metacategory is a set (resp. a finite set).

Examples

1. The category **Set** is not small.
2. The simplex category Δ is small but not finite.
3. An ordered set $(I, \leq)$ is a small category. It is finite if I is finite.
4. If X is a topological space, then $\mathbf{Open}(X)$ is a small category.
5. A monoid G is a small category. It is finite if G is finite.
6. If A is a ring, then $\mathbf{Mat}_A$ is a small category.
7. If n is an integer, then $\mathbf{n}$ is a finite category. This applies in particular to the categories $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.

Some authors use a restricted notion of category and assume from the beginning that they are small. We will see later why we cannot do that.

Product and dual

1. If $\mathcal{C}$ and $\mathcal{C}'$ are two categories, then the product category $\mathcal{C} \times \mathcal{C}'$ is the category whose objects are couples (X, Y) with $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, a morphism is a couple of morphisms and composition is defined component-wise.
2. The opposite category or dual category of a category $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$, for all objects X, Y,

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X),$$

and composition is defined by reversing the order.

Examples

1. The dual category to $(I, \leq)$ is $(I, \geq)$.
2. The dual category to $\mathbf{0}$ or $\mathbf{1}$ is itself.
3. If G is a monoid, then the dual category is the opposite monoid.
4. We always have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$

Subcategory

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a collection of objects X of $\mathcal{C}$, and for all $X, Y \in \mathcal{C}'$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ satisfying the following properties: If $X \in \mathcal{C}'$, then Id_X is a morphism in $\mathcal{C}'$ and if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathcal{C}'$, then $g \circ f$ is a morphisms in $\mathcal{C}'$. It is called a full subcategory if we actually have

$$\forall X, Y \in \mathcal{C}', \mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Examples

1. The category **Gr** is not a subcategory of **Set**.
2. The category **Ab** (resp. **CRng**) is a full subcategory of **Gr** (resp. **Rng**).
3. A nonempty subcategory of a monoid G is a submonoid.
4. A subcategory of an ordered set $(I, \leq)$ is an ordered subset.

Given a collection of objects of $\mathcal{C}$, there exists a unique full subcategory $\mathcal{C}'$ of $\mathcal{C}$ with exactly these objects. This is the full subcategory generated by the given collection of objects. If this collection is defined by a property P, we will often write $\mathcal{C}' = \mathcal{C}^P$ (decoration).

Recall that a category $\mathcal{C}$ is a directed graph with identities and a rule for composition. And the notion of subcategory is stable under intersection. Therefore, if S is a subgraph of $\mathcal{C}$, there exists a smaller subcategory $\mathcal{C}'$ containing this subgraph. This is the category generated by the subgraph.

Examples

1. The full subcategory of **Set** generated by finite sets is **FSet**.
2. If A is a ring, we may consider the full subcategory A-$\mathbf{Mod}^{\mathrm{ffr}}$ or free A-modules of finite rank.
3. The category generated by a subgraph has the vertices of the graph as objects and finite compositions of edges as morphisms.
4. The ordered set $\mathbb{Z}$ is generated by the $d^n : n \to n+1$.
5. The simplex category is generated by the face and degeneracy maps.

Retraction/section

Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. A retraction or left inverse for f is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$.

A section or right inverse for f is a retraction for f in $\mathcal{C}^{\mathrm{op}}$.

Theorem

If f has both a retraction g and a section h, then they are unique and we have $h = g$.

When this is the case, f is called an isomorphism. We will then write $f : X \simeq Y$ and set $f^{-1} := g = h$. This is the inverse of f and we might as well say that f is invertible.

Examples

1. **Set**: A map has a section if and only if it is surjective. And a map with non empty domain has a retraction if and only if it is injective. Finally, a map is an isomorphism if and only if it is a bijection.
2. **Ab**: An injective (resp. surjective) morphism does not always have a retraction (resp. section). But a morphism is an isomorphism if and only if it is bijective.
3. **Top**: Although the converse is true, a continuous bijective map is not an isomorphism (called homeomorphism here) in general.
4. k-**Vec**: A k-linear map has a retraction (resp. section) if and only if it is injective (resp. surjective).
5. $\mathbf{Mat}_k$: A matrix has a retraction or a section if and only if its rank is maximal.
6. If G is a monoid, then the terminology of left or right inverse as well as invertible element are the same as the usual ones.
7. In an ordered set, only the identities have a retraction or a section.

Functor

A (covariant) functor $F : \mathcal{C} \to \mathcal{C}'$ is a map that associates to any $X \in \mathcal{C}$ an object

$$F(X) \in \mathcal{C}'$$

and to any morphism $f : X \to Y$ a morphism

$$F(f) : F(X) \to F(Y).$$

One requires the following properties to be satisfied:

1. $\forall X \in \mathcal{C}, \quad F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$.
2. $\forall f : X \to Y, g : Y \to Z, \quad F(g \circ f) = F(g) \circ F(f)$.

Note that a functor is uniquely determined by its values on a generating graph. Also, any functor preserves sections, retractions and isomorphisms.

A contravariant functor $F : \mathcal{C} \to \mathcal{C}'$ is a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$.

Examples

1. The forgetful functors
$$\mathbf{Rng} \to \mathbf{Ab}, \quad \mathbf{Rng} \to \mathbf{Mon}, \quad \mathbf{Mon} \to \mathbf{Set}, \quad \mathbf{Top} \to \mathbf{Set} \ldots$$
2. The forgetful functors
$$G\text{-}\mathbf{Set} \to \mathbf{Set}, \quad A\text{-}\mathbf{Mod} \to \mathbf{Ab}, \quad k\text{-}\mathbf{Alg} \to \mathbf{Rng} \ldots$$
3. The forgetful functors
$$\mathbf{Act} \to \mathbf{Mon} \times \mathbf{Set}, \quad \mathbf{Mod} \to \mathbf{Rng} \times \mathbf{Ab}, \quad \mathbf{Alg} \to \mathbf{CRng} \times \mathbf{Rng}.$$
4. The forgetful functors
$$\mathbf{Sch} \to \mathbf{Top}, \quad \mathbf{Var}_{/k} \to \mathbf{Top}, \quad \mathbf{An} \to \mathbf{Top}.$$

Examples

1. The free module functor (when A is a ring)

$$E \mapsto AE, \quad \mathbf{Set} \to A\text{-}\mathbf{Mod}$$

2. The free abelian monoid functor

$$E \mapsto \mathbb{N}E, \quad \mathbf{Set} \to \mathbf{Mon}.$$

3. The monoid algebra functor (when k is a commutative ring)

$$G \mapsto k[G], \quad \mathbf{Mon} \to k\text{-}\mathbf{Alg}$$

4. The polynomial ring functor (when k is a commutative ring)

$$E \mapsto k[\mathbb{N}E], \quad \mathbf{Set} \to k\text{-}\mathbf{CAlg}.$$

Examples

1. The abelianization functors
$$\mathbf{Gr} \to \mathbf{Ab}, \quad G \mapsto G^{\mathrm{ab}} := G/[G,G] \quad \text{and}$$
$$\mathbf{Rng} \to \mathbf{Com}, \quad A \mapsto A^{\mathrm{com}} := A/(\{ab - ba\}_{a,b \in A}).$$
2. The functors
$$G \mapsto G[G^{-1}], G \mapsto G^{\times} \quad : \mathbf{Mon} \to \mathbf{Gr}$$
that send a monoid to the group generated by G or to the invertible elements of G.
3. The functors
$$E \mapsto E^{\mathrm{disc}}, E \mapsto E^{\mathrm{coarse}}, \quad \mathbf{Set} \to \mathbf{Top}$$
obtained by endowing a set with its discrete or coarse topology.

Examples

1. The contravariant functor
$$\mathrm{Spec} : \mathbf{CRng} \to \mathbf{Sch} \quad (\text{or } \mathrm{Spm} : k\text{-}\mathbf{CAlg}^{\mathrm{fp}} \to \mathbf{Var}_{/k})$$
that send a ring (or a finitely presented commutative algebra) to the set of its prime (resp. maximal ideals) with additional structure.
2. The contravariant functor
$$X \mapsto \Gamma(X, \mathcal{O}_X), \quad \mathbf{Sch} \to \mathbf{CRng}$$
(or $\mathbf{Var}_{/k} \to k\text{-}\mathbf{CAlg}^{\mathrm{fp}}$, or $\mathbf{An} \to \mathbb{C}\text{-}\mathbf{CAlg}$)
3. The contravariant functor $E \mapsto M^E, \mathbf{Sets} \to \mathbf{Ab}$ if M is an abelian group.

Example

The analytification functor $X \mapsto X^{\mathrm{an}}, \mathbf{Var}_{/\mathbb{C}} \to \mathbf{An}$.

Example

There exists a functor $\Delta^\bullet : \Delta \to \mathbf{Top}$ sending $[n]$ to the standard topological simplex

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0}, \sum_{i=0}^{n} x_i = 1 \right\}$$

and $u : [n] \to [m]$ to the unique linear map sending e_i to $e_{u(i)}$ if $(e_0, \ldots, e_n)$ denotes the usual basis.
In particular, we may consider the face maps given by

$$\delta_k(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_{n-1})$$

and the degeneracy maps given by

$$\sigma_k(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_{k-1}, x_k + x_{k+1}, \ldots, x_{n+1}).$$

Examples

1. If A is a ring, we have

$$\mathrm{Hom} : A\text{-}\mathbf{Mod}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Mod}\text{-}A \quad \text{and}$$

$$\otimes_A : \mathbf{Mod}\text{-}A \times A\text{-}\mathbf{Mod} \to \mathbf{Ab}.$$

   When A is commutative, we can do a little better with

$$\mathrm{Hom}_A : A\text{-}\mathbf{Mod}^{\mathrm{op}} \times A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \quad \text{and}$$

$$\otimes_A : A\text{-}\mathbf{Mod} \times A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}.$$

2. If $A \to B$ is a homomorphism of rings, we have obvious scalar restriction $B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ but also scalar extension and scalar evaluation

$$A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}, \quad M \mapsto B \otimes_A M \text{ and } M \mapsto \mathrm{Hom}_A(B, M).$$

Examples

1. $I_n \mapsto A^n, \quad \mathbf{Mat}_A \to A\text{-}\mathbf{Mod}$.
2. The linear group functor $\mathrm{GL}_n : \mathbf{Rng} \to \mathbf{Gr}$. Note that in the case $n = 1$, this is simply the functor $A \mapsto A^\times$.
3. The duality contravariant functor
$$M \mapsto \check{M} := \mathrm{Hom}_A(M, A)$$
inside A-**Mod** (where A is a commutative ring).
4. A functor $G \to H$ between two monoids is a homomorphism.
5. A functor $(I, \leq) \to (J, \leq)$ is an order preserving map.
6. Any continuous map $f : X \to Y$ induces a functor
$$f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X).$$

Example

Sending a group G to its center $Z(G)$ is not functorial.

Examples

1. The inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ when $\mathcal{C}' \subset \mathcal{C}$.
2. The projections
$$\mathcal{C} \times \mathcal{C}' \to \mathcal{C} \quad \text{and} \quad \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$$
when $\mathcal{C}$ and $\mathcal{C}'$ are two categories.
3. The flip functor
$$\mathcal{C} \times \mathcal{C}' \to \mathcal{C}' \times \mathcal{C}.$$
4. The functors
$$\mathcal{C}' \to \mathcal{C} \times \mathcal{C}', Y \mapsto (X, Y), f \mapsto (\mathrm{Id}_X, f) \quad \text{and}$$
$$\mathcal{C} \to \mathcal{C} \times \mathcal{C}', X \mapsto (X, Y), f \mapsto (f, \mathrm{Id}_Y)$$

Composition

The identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is given by

$$\mathrm{Id}_{\mathcal{C}}(X) = X \quad \text{and} \quad \mathrm{Id}_{\mathcal{C}}(f) = f.$$

The composition of two (covariant) functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ is given by

$$(G \circ F)(X) = G(F(X)) \quad \text{and} \quad (G \circ F)(f) = G(F(f)).$$

Any (covariant) functor $F : \mathcal{C} \to \mathcal{C}'$ defines a (covariant) functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$. We may still write F instead of F^{op}. In particular, if G is a contravariant functor, one usually writes $G \circ F := G \circ F^{\mathrm{op}}$.

Note also that whenever it has a meaning, we have:

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

In other words, all functors together form a metacategory. Actually, small categories and functors between them form a category **Cat**.

Examples

1. The functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ sends (X, Y) to $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $(f : X' \to X, g : Y \to Y')$ to

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X', Y'), h \mapsto g \circ h \circ f.$$

2. By composition, we obtain for all $X \in \mathcal{C}$ the very important functor

$$h_{\mathcal{C}}^{X} : \mathcal{C} \to \mathbf{Set}, Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

We will write $h_X^{\mathcal{C}} := h_{\mathcal{C}^{\mathrm{op}}}^{X}$ so that

$$h_X^{\mathcal{C}}(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

Example

There exists a contravariant functor

$$\mathbf{Top} \to \mathbf{Cat}, \quad X \mapsto \mathrm{Open}(X), f \mapsto f^{-1}.$$

Fiber category

If $F : \mathcal{C} \to \mathcal{C}'$ is any functor, the fiber category $\mathcal{C}_Y$ of F at $Y \subset \mathcal{C}'$ is the subcategory of $\mathcal{C}$ whose objects satisfy $F(X) = Y$ and morphisms satisfy $F(f) = \mathrm{Id}_Y$.

Examples

1. The fiber at a set E of a forgetful functor $\mathcal{C} \to \mathbf{Set}$ classifies the different ways of promoting E to an object of $\mathcal{C}$.
2. The fibers of the forgetful functor $\mathbf{Act} \to \mathbf{Mon}$, $(G, E) \mapsto G$ are the categories G-$\mathbf{Set}$.
3. The fibers of the forgetful functor $\mathbf{Mod} \to \mathbf{Rng}$, $(A, M) \mapsto A$ are the categories A-$\mathbf{Mod}$.
4. The fibers of the forgetful functor $\mathbf{Alg} \to \mathbf{CRng}$, $(k, A) \mapsto k$ are the categories k-$\mathbf{Alg}$.

Slice category

The category of morphisms of a category $\mathcal{C}$ is the category $\mathbf{Mor}(\mathcal{C})$ whose objects are morphisms $X \to Y$ in $\mathcal{C}$ and morphisms are pairs of compatible morphisms. Then, there exists two obvious domain and codomain functors

$$\mathrm{Dom}, \mathrm{Cod} : \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}.$$

If $X \in \mathcal{C}$, the slice (or comma) category $\mathcal{C}_{/X}$ of $\mathcal{C}$ at X is the fiber at X of Cod (the category of objects above X). An object is a pair (Y, g) made of an object $Y \in \mathcal{C}$ and a structural morphism $g : Y \to X$. If Z has structural morphism $h : Z \to X$, then a morphism $Z \to Y$ over X is simply a morphism $f : Z \to Y$ such that $h = g \circ f$.

Note that any morphism $f : Y \to X$ in $\mathcal{C}$ will induce a functor $\mathcal{C}_{/Y} \to \mathcal{C}_{/X}$.

Examples

1. If S is a scheme, then $\mathbf{Sch}_{/S}$ is the usual category of S-schemes.
2. When k is a commutative ring, we have $(\mathbf{CRng}^{\mathrm{op}}_{/k})^{\mathrm{op}} = k\text{-}\mathbf{CAlg}$.

Faithfulness

A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be faithful (resp. fully faithful) or called a forgetful functor (resp. an embedding) if for all $X, Y \in \mathcal{C}$, the map

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

is injective (resp. bijective). It is said to be essentially surjective if any object of $\mathcal{C}'$ is isomorphic to some $F(X)$.

A concrete category is a category $\mathcal{C}$ endowed with a faithful functor $\mathcal{C} \to \mathbf{Set}$ called the forgetful functor.

Examples

1. The categories **Mon**, A-**Mod**, **Rng**, **Top**. . . are concrete.
2. The functors $E \mapsto E^{\mathrm{disc}}$ and $E \mapsto E^{\mathrm{coarse}}$ are fully faithful.
3. The functor $\mathbf{Mat}_A \to A\text{-}\mathbf{Mod}$, $I_n \mapsto A^n$ is fully faithful.
4. The functor $\mathbf{Gr} \to \mathbf{Ab}$, $G \mapsto G^{\mathrm{ab}}$ is essentially surjective.

Natural transformation

A natural transformation $\alpha : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{C}'$ is a collection of morphisms

$$\alpha_X : F(X) \to G(X)$$

for $X \in \mathcal{C}$ such that

$$\forall f : X \to Y, G(f) \circ \alpha_X = \alpha_Y \circ F(f).$$

If $F : \mathcal{C} \to \mathcal{C}'$ is a functor, the natural identity $\mathrm{Id}_F : F \to F$ is given by

$$\forall X \in \mathcal{C}, \mathrm{Id}_{FX} = \mathrm{Id}_{F(X)}.$$

The composite of two natural transformations

$$\alpha : F \to G \quad \text{and} \quad \beta : G \to H$$

is the natural transformation $\beta \circ \alpha$ given by

$$\forall X \in \mathcal{C}, (\beta \circ \alpha)_X = \beta_X \circ \alpha_X.$$

Examples

1. The determinant

$$\det_A : GL_n(A) \to A^\times$$

defines a natural transformation between GL_n and the functor $A \mapsto A^\times$ from **Rng** to **Gr**.

2. The projection $G \twoheadrightarrow G^{ab}$ is natural (between the identity functor $\mathrm{Id}_{\mathbf{Gr}}$ and the functor composed of abelianization $\mathbf{Gr} \to \mathbf{Ab}$ and inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Gr}$).

3. If A denotes a commutative ring, the morphism

$$M \to \check{\check{M}}, m \mapsto (u \mapsto u(m))$$

is natural (between the identity functor on A-**Mod** and the bidual functor obtained by composing the dual with itself).

Example

If $f : X \to Y$ is any morphism in a category, there exists a natural transformation $h^f : h^Y \to h^X$ given by

$$\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z), \quad g \to g \circ f,$$

and a natural transformation $h_f : h_X \to h_Y$ given by

$$\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y), \quad g \to f \circ g.$$

Note that the composition of natural transformations is associative in the sense that, whenever this has a meaning, we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

Actually, the natural transformations between all functors from $\mathcal{C}$ to $\mathcal{C}'$ form a metacategory $\mathbf{Hom}(\mathcal{C}, \mathcal{C}')$. We obtain a category when $\mathcal{C}$ is small.

Finally, giving $\alpha : F \to G$ is equivalent to giving $\alpha^{\mathrm{op}} : G^{\mathrm{op}} \to F^{\mathrm{op}}$.

Natural isomorphism

A natural isomorphism $\alpha : F \xrightarrow{\simeq} G$ is a natural transformation such that there exists a natural transformation $\beta : G \to F$ (necessarily unique) with

$$\beta \circ \alpha = \mathrm{Id}_F \quad \text{et} \quad \alpha \circ \beta = \mathrm{Id}_G.$$

One can check that α is a natural isomorphism if and only if for all $X \in \mathcal{C}$, α_X is an isomorphism.

Examples

1. If A is a commutative ring and M a projective A-module of finite type, then the isomorphism $M \simeq \check{\check{M}}$ is natural.
2. Let $\mathcal{C}$ be a category with one object X and one non trivial morphism σ. Let F be the identity of $\mathcal{C}$ and G the only non trivial endofunctor of $\mathcal{C}$. Then, the isomorphism $F(X) = X = G(X)$ is not natural.

An equivalence of categories $F : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}'$ is a functor such that there exists a functor $G : \mathcal{C}' \to \mathcal{C}$, called a quasi-inverse, with

$$G \circ F \simeq \mathrm{Id}_{\mathcal{C}} \quad \text{et} \quad F \circ G \simeq \mathrm{Id}_{\mathcal{C}'}.$$

Theorem

A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.

In particular, if a functor $\mathcal{C} \to \mathcal{C}'$ is fully faithful, then $\mathcal{C}$ is equivalent to a full subcategory of $\mathcal{C}'$.

A category is said to be essentially small (resp. essentially finite) if it is equivalent to a small (resp. finite) category.

Example

If G is a monoid, there exists two essentially surjective functors $G \to \mathbf{1}$ and $\mathbf{1} \to G$ but the categories are not equivalent unless G is trivial.

Examples

1. Two monoids are equivalent if and only if they are isomorphic.
2. Two ordered sets I and J are equivalent if and only if they are isomorphic as ordered sets.

Examples

1. There exists an equivalence of categories $\mathbf{Mat}_A \simeq A\text{-}\mathbf{Mod}^{\mathrm{ffr}}$. In particular, the later is essentially small.
2. There exists an equivalence of categories $\mathbf{2} \simeq \mathbf{Set}^{\leq 1}$ where the decoration means sets with at most one element. In particular, the later is essentially finite (but not small).

Examples

1. We have $\mathbf{Ab} \simeq \mathbb{Z}$-$\mathbf{Mod}$ (this is even an equality).
2. We have G-$\mathbf{Mod} \simeq \mathbf{Rep}_{\mathbb{Z}}(G) \simeq \mathbb{Z}[G]$-$\mathbf{Mod}$ and more generally $\mathbf{Rep}_k(G) \simeq k[G]$-$\mathbf{Mod}$.
3. We have A-$\mathbf{Op} \simeq A[X]$-$\mathbf{Mod}$.

Examples

1. We always have Set-$G \simeq G^{\mathrm{op}}$-Set and Mod-$A \simeq A^{\mathrm{op}}$-Mod.
2. The map $n \mapsto -n$ is an equivalence $(\mathbb{Z}, \leq) \simeq (\mathbb{Z}, \geq)$ (sending d^n to $d_n := d^{-n-1}$).
3. The category $\mathbf{2}^{\mathrm{op}}$ is equivalent to $\mathbf{2}$.
4. The functor $X \mapsto X^{\mathrm{op}}$ is an autoequivalence of $\mathbf{Mon}$, $\mathbf{Rng}$ or $\mathbf{Cat}$ which is its own (quasi-) inverse. This is a symmetry.

Example

The flip functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{C}' \times \mathcal{C}$ is an equivalence (a symmetry again).

Let $\mathcal{C}$ be a metacategory and S a collection of morphisms of $\mathcal{C}$. Then, there exists, up to equivalence, a unique metacategory $\mathcal{C}[S^{-1}]$, together with a metafunctor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$, such that

1. The metafunctor Q sends morphisms of S to isomorphisms in $\mathcal{C}[S^{-1}]$.
2. If a metafunctor $F : \mathcal{C} \to \mathcal{D}$ sends morphisms of S to isomorphisms in $\mathcal{D}$, it will factor through Q.
3. The metafunctor $Q^{-1} : \mathbf{Hom}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D})$ is fully faithful for any metacategory $\mathcal{D}$.

The metacategory $\mathcal{C}[S^{-1}]$ is then called the localization of $\mathcal{C}$ with respect to S. When $\mathcal{C}$ is a small category, so is $\mathcal{C}[S^{-1}]$. However, it is not always true that $\mathcal{C}[S^{-1}]$ is a category when $\mathcal{C}$ is a category: the collection of all morphisms between two objects of $\mathcal{C}[S^{-1}]$ do not necessarily make a set.

Example

A category $\mathcal{C}$ admits right calculus of fractions with respect to a subcategory S if

1. Given any $f : X \to Y$ in $\mathcal{C}$ and $\varphi : Y' \to Y$ in S, there always exists $f' : X' \to Y'$ and $\varphi' : X' \to X$ in S with $\varphi \circ f' = f \circ \varphi'$.
2. Given any $f, g : X \to Y'$ in $\mathcal{C}$ and $\varphi : Y' \to Y$ in S such that $\varphi \circ f = \varphi \circ g$, there exists $\varphi' : X' \to X$ in S such that $f \circ \varphi' = g \circ \varphi'$.

Then, we may see $\mathcal{C}[S^{-1}]$ as the (*a priori* only meta-) category having the same objects as $\mathcal{C}$ with morphisms and composition described, up to equivalence, by the following diagram

$$\begin{tikzcd}
X'' \arrow[d, "s"'] \arrow[dr] & & \\
X' \arrow[d, "s"'] \arrow[dr] & Y' \arrow[d, "s"'] \arrow[dr] & \\
X \arrow[r, dashed] & Y \arrow[r, dashed] & Z.
\end{tikzcd}$$

Section 3

Universal constructions

Diagram

Let $\mathcal{C}$ be category. Recall that when I is a small category, the functors from I to $\mathcal{C}$ form a category $\mathbf{Hom}(I, \mathcal{C})$ also denoted by $\mathcal{C}^I$.

Examples

1. $\mathcal{C}^{\mathbf{0}} \simeq \mathbf{1}$, $\mathcal{C}^{\mathbf{1}} \simeq \mathcal{C}$ and $\mathcal{C}^{\mathbf{2}} \simeq \mathbf{Mor}(\mathcal{C})$.
2. If G is a monoid, then $\mathbf{Set}^G \simeq G\text{-}\mathbf{Set}$.

An object D of $\mathcal{C}^I$ is also called a (commutative) diagram on I in $\mathcal{C}$. A codiagram on I is a diagram on I^{op}.

Examples

1. A diagram on the ordered set $\mathbb{Z}$ is given by a family of objects X^n and morphisms $d^n : X^n \to X^{n+1}$.
2. A diagram (resp. codiagram) on the simplex category Δ is called a cosimplicial (resp. simplicial) object of $\mathcal{C}$.
3. $\Delta^\bullet$ is a cosimplicial topological space.

Giving a diagram on I in $\mathcal{C}$ amounts to giving a family $(X_i)_{i\in I}$ in $\mathcal{C}$ and for each non trivial $u : i \to j$, a morphism $f_u : X_i \to X_j$ such that

$$\forall v : j \to k, f_{v\circ u} = f_v \circ f_u.$$

And a morphism of diagrams $(X_i, f_u) \to (Y_i, g_u)$ is given by morphisms $h_i : X_i \to Y_i$ satisfying for all $u : i \to j$, $g_u \circ h_i = h_j \circ f_u$.

Examples

1. A diagram on a directed set $(I, \leq)$, is defined by a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$ and whenever $i < j$, a morphism $X_i \to X_j$ such that whenever $i < j < k$, the map $X_i \to X_k$ is exactly the composite of $X_i \to X_j$ and $X_j \to X_k$.
2. A diagram on a set I is given by a family of elements of $\mathcal{C}$ indexed by I.
3. A simplicial object is given by a family of objects X_n, face morphisms $d_{n,i} : X_n \to X_{n-1}$ and degeneracy morphisms $s_{n,i} : X_n \to X_{n+1}$ satisfying explicit commutation rules.

If I is a small category, any functor $F : \mathcal{C} \to \mathcal{C}'$ extends by composition to a functor $F^I : \mathcal{C}^I \to \mathcal{C}'^I$.

If $\mathcal{C}$ is a category, any functor $\lambda : I \to J$ between small categories will induce by composition a functor $\lambda^{-1} : \mathcal{C}^J \to \mathcal{C}^I$ on diagrams.

Examples

1. The unique functor $I \to \mathbf{1}$ will induce the constant diagram functor $\mathcal{C} \simeq \mathcal{C}^{\mathbf{1}} \to \mathcal{C}^I, X \mapsto \underline{X}$.
2. There exists two functors $\mathbf{1} \to \mathbf{2}$ and they will induce the domain and codomain functors $\mathrm{Dom}, \mathrm{Cod} : \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$.
3. By composition, if X is any topological space, we may consider the simplicial set $S_\bullet(X) = h_X \circ \Delta^\bullet$ so that $S_n(X) = \mathrm{Hom}_{\mathrm{cont}}(\Delta^n, X)$.
4. By composition again, if X is any topological space, we may consider the simplicial and cosimplicial groups

$$C_\bullet(X) := \mathbb{Z}S_\bullet(X) \quad \text{and} \quad C^\bullet(X) := \mathbb{Z}^{S_\bullet(X)}.$$

If I is a small category, we saw that there exists a functor

$$\mathcal{C} \to \mathcal{C}^I, X \mapsto \underline{X}$$

sending $X \in \mathcal{C}$ to the constant diagram $\underline{X}$ given by $X_i = X$ and $f_u = \mathrm{Id}_X$.

A commutative diagram D on I in $\mathcal{C}$ has an (inverse) limit $\varprojlim D$ if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}^I}(\underline{X}, D) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \varprojlim D).$$

The limit of a diagram D in $\mathcal{C}^{\mathrm{op}}$ is called a colimit (or direct limit) and written $\varinjlim D$. It is therefore characterized by a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim D, Y) \simeq \mathrm{Hom}_{\mathcal{C}^I}(D, \underline{Y}).$$

A limit or a colimit is said to be finite if I is a finite category.

Description

A diagram (X_i, f_u) has a limit X if and only if there exists morphisms $p_i : X \to X_i$ in $\mathcal{C}$ for all $i \in I$ such that

$$\forall u : i \to j, f_u \circ p_i = p_j,$$

and such that whenever we are given morphisms $g_i : Y \to X_i$ in $\mathcal{C}$ for all $i \in I$ such that

$$\forall u : i \to j, f_u \circ g_i = g_j,$$

then there exists a unique morphism $g : Y \to X$ such that for all $i \in I$, we have $g_i = p_i \circ g$ (and dual):

$$\begin{tikzcd}
 & & X_i \arrow[d, "f_u"] \\
Y \arrow[r, dashed, "g"] \arrow[urr, bend left, "g_i"] \arrow[drr, bend right, "g_j"'] & X \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & X_j
\end{tikzcd}$$

FINAL/INITIAL OBJECT

A limit on the empty category in $\mathcal{C}$ is a final object 1 of $\mathcal{C}$:

Given $X \in \mathcal{C}$, there exists a unique morphism $X \to 1$.

The dual notion is that of initial object 0.

Examples

1. **Set** and **Top**: a final object is a one element set, such as $1 := \{0\}$, and $\emptyset$ is the only initial object.
2. **Mon**, **Gr**, **Ab** or A-**Mod**: final object $=$ initial object $= 1$.
3. k-**Alg** or k-**CAlg**: final object $= 1$ and initial object $= k$.
4. **Sch**: initial object $= \emptyset$ and final object $= \operatorname{Spec}\mathbb{Z}$.
5. $(I, \leq)$: the final (resp. initial) object is the biggest (resp. smallest) element if it exists.
6. X is the final object of a slice category $\mathcal{C}_{/X}$.
7. If $\mathcal{C}$ has a final object 1, then $\mathcal{C}_{/1} \simeq \mathcal{C}$.

Product/coproduct

A limit of a family $\{X_i\}_{i\in I}$ is a product $X := \prod_I X_i$ of the X_i's:

there exists projections $p_i : X \to X_i$, such that

$$\forall g_i : Y \to X_i, i \in I, \quad \exists! g : Y \to X, \quad \forall i \in I, p_i \circ g = g_i.$$

The dual notion is that of coproduct (or sum) $\coprod_I X_i$.

Examples

1. **Set**: product si cartesian product and coproduct is disjoint union.
2. **Top**: same with appropriate topology.
3. In **Sch**, the product is not the cartesian product and in $\mathbf{Var}_{/k}$, it is the cartesian product but not with the product topology.
4. A-**Mod**: product is cartesian product and coproduct is direct sum.
5. k-**Alg**: product is cartesian product and coproduct is tensor product.
6. $(I, \leq)$: the product (resp. coproduct) is the greatest lower bound (resp. least upper bound) if it exists.

Fibered product/coproduct

A limit of two morphisms $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ is a fibered product $X_1 \times_Y X_2$: there exists two projections $p_1 : X \to X_1, p_2 : X \to X_2$ such that $f_1 \circ p_1 = f_2 \circ p_2$ and

$$\forall g_1 : Z \to X_1, g_2 : Z \to X_2, f_1 \circ g_1 = f_2 \circ g_2 \Rightarrow$$

$$\exists! g : Z \to X, g_1 = p_1 \circ g \quad \text{et} \quad g_2 = p_2 \circ g.$$

We also say that the square

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "f_2"] & Y \end{tikzcd}$$

is a cartesian diagram or that X is the pull back of X_2 along f_1.

The dual notions are fibered coproduct $X_1 \coprod_Y X_2$, cocartesian diagram and pushout.

Examples

1. The fibered product of

$$f_1 : X_1 \to Y, f_2 : X_2 \to Y$$

in many concrete categories such as **Set**, **Top**, **Mon**, or A-**Mod**, or k-**Alg**, is

$$\{(x_1, x_2) \in X_1 \times X_2, f_1(x_1) = f_2(x_2)\}$$

with the induced structure.

2. Also, if $Z \subset Y$, then there exists a cartesian diagram

$$\begin{tikzcd} f^{-1}(Z) \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, hook] & Y. \end{tikzcd}$$

3. The fibered coproduct of two morphisms of rings $f : A \to B$ and $g : A \to C$ is their tensor product $B \otimes_A C$.

Kernel/cokernel

A limit of a pair of morphisms $f_1, f_2 : X \to Y$ is a kernel (or equalizer) $Z := \ker(f, g)$:

there exists a morphism $i : Z \to X$ such that $f_1 \circ i = f_2 \circ i$ and

$$\forall g : T \to X, f_1 \circ g = f_2 \circ g \Rightarrow \exists! h : T \to Z, g = i \circ h.$$

We also say that the diagram

$$Z \longrightarrow X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$$

is (left) exact.

The dual notions are that of cokernel (or coequalizer) $Z := \operatorname{coker}(f, g)$ and (right) exact diagram

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \longrightarrow Z.$$

Examples

1. **Set**: the kernel of $f, g : E \to F$ is

$$\{x \in E, f(x) = g(x)\}$$

   and the cokernel is the quotient of F by the smallest equivalence relation such that $f(x) \sim g(x)$ whenever $x \in E$.

2. **Top**: same with appropriate topology.

3. A-**Mod**: $\ker(f, g) = \ker(g - f)$ and

$$\mathrm{coker}(f, g) = N/\mathrm{im}(g - f).$$

4. **Gr**: Given $f, g : G \to H$,

$$\ker(f, g) = \{x \in G, f(x)g(x)^{-1} = 1\}$$

   and $\mathrm{coker}(f, g)$ is the quotient of H by the normal subgroup generated by $\{f(x)g(x)^{-1}, x \in G\}$.

Examples

1. An empty product is the same thing as a final object (and dual).
2. If 1 is a final object, then a product $X \times Y$ is the same thing as a fibered product $X \times_1 Y$ (and dual).
3. Assume that we are given $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ and that $X_1 \times X_2$ exists. Then, if the kernel of $f_1 \circ p_1$ and $f_2 \circ p_2$ exists, it is a fibred product of f_1 and f_2 (and dual).
4. Assume that we are given $f_1, f_2 : X \to Y$ and that $Y \times Y$ exists. Then, if the fibered product of

$$\begin{tikzcd} & X \arrow[d, "{(f_1,f_2)}"] \\ Y \arrow[r, hook] & Y \times Y \end{tikzcd}$$

exists, it is a kernel of f_1 and f_2 (and dual).

Example

Assume $X' := \prod_i X_i$ and $X'' := \prod_{u:i\to j} X_j$ exist. Let $p, f : X' \to X''$ be the morphisms induced respectively by the p_j's and the $f_u \circ p_i$'s. Then, if the kernel of p and f exist, this is a limit for (X_i, f_u).

Theorem

1. *If all kernels and (finite) products exist in $\mathcal{C}$, then all (finite) limits exist in $\mathcal{C}$ (and dual).*
2. *If there exists a final object and all fibered products exist in $\mathcal{C}$, then all finite limits exist in $\mathcal{C}$ (and dual).*

Examples

1. All limits and colimits exist in **Set**, **Top**, **Mon**, **Gr**, **Ab**, A-**Mod**, **Rng**, **CRng**, k-**Alg** and k-**CAlg**.
2. Only finite limits and colimits exist in **FSet** or **FGr**.
3. All finite limits and coproducts exist in **Sch**, $\mathbf{Var}_k$ or **An**.
4. Not all limits or colimits exist in **Sch**, $\mathbf{Var}_k$ or **An**.

A monomorphism $\iota : X \hookrightarrow Y$ is a morphism such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X"] \arrow[d, "\mathrm{Id}_X"] & X \arrow[d, "\iota"] \\ X \arrow[r, "\iota"] & Y \end{tikzcd}$$

is cartesian. We might also say that X (endowed with ι) is a subobject of Y. It means that

$$\forall f, g : Z \to X, \quad \iota \circ f = \iota \circ g \Rightarrow f = g.$$

An epimorphism $\pi : X \twoheadrightarrow Y$ in $\mathcal{C}$ is a monomorphism of $\mathcal{C}^{\mathrm{op}}$ and we might also say that Y (endowed with π) is a quotient of X.

Examples

1. **Set**, **Top**, **Gr**, **Ab**, A-**Mod**: A morphism is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
2. **Mon**, **Rng**, **CRng**, k-**Alg**, k-**CAlg** : A monomorphism is an injective map. However, the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a bimorphism (both a monomorphism and an epimorphism) but it is not surjective.
3. In **Sch**, the inclusion of the disjoint union of the closed points into the affine line $\operatorname{Spec} k[T]$ is an bimorphism which is not surjective. And $\operatorname{Spec} k \to \operatorname{Spec} k[T]/T^2$ is bijective but this is not an epimorphism.

If a morphism has a retraction, then it is necessarily a monomorphism (and dual) - but the converse is not true.

If F is faithful and $F(f)$ is a monomorphism, then f also (and dual).

As a consequence, in a concrete category, any injective map is a monomorphism (and dual).

Assume that $\mathcal{C}$ has finite products.

A monoid of $\mathcal{C}$ is a triple made of an object G, a (multiplication) morphism $\mu : G \times G \to G$ and a (unit) morphism $\epsilon : 1 \to G$, subject to the conditions

$$(\mu \times \mathrm{Id}) \circ \mu = (\mathrm{Id} \times \mu) \circ \mu \quad \text{and} \quad \mu \circ (\mathrm{Id} \times \epsilon) = \mu \circ (\epsilon \times \mathrm{Id}) = \mathrm{Id}.$$

A morphism of monoids of $\mathcal{C}$ is a morphism in $\mathcal{C}$ that commutes both with multiplications and units. This defines a new category $\mathbf{Mon}(\mathcal{C})$ of monoids of $\mathcal{C}$.

One can define the categories of groups, abelian groups, rings or commutative rings of $\mathcal{C}$ along the same lines (but not the categories of fields).

Of course, when $\mathcal{C} = \mathbf{Set}$, we recover the usual categories of monoids, groups, etc.

A (left) action of a monoid G of $\mathcal{C}$ on an object X of $\mathcal{C}$ is a morphism $\alpha : G \times X \to X$ satisfying

$$\alpha \circ (\mu \times \mathrm{Id}_X) = \alpha \circ (\mathrm{Id}_X \times \mu) \quad \text{and} \quad \alpha \circ (\epsilon \times \mathrm{Id}_X) = \mathrm{Id}_X.$$

We obtain the category of G-$\mathcal{C}$ of objects of $\mathcal{C}$ endowed with an action of G as well as the category $\mathbf{Act}(\mathcal{C})$ of objects with an action of an unspecified monoid.

One can define along the same lines the category of A-modules (resp. k-algebras) of $\mathcal{C}$ as well as the category of modules (resp. algebras) of $\mathcal{C}$.

An algebraic theory P is a small category with products and the algebraic category of type P is the category of all diagrams on P that preserve products.

Then the categories $\mathbf{Mon}(\mathcal{C})$, $\mathbf{Gr}(\mathcal{C})$, $\mathbf{Ab}(\mathcal{C})$, $\mathbf{Rng}(\mathcal{C})$, $\mathbf{CRng}(\mathcal{C})$, $\mathbf{Act}(\mathcal{C})$, $\mathbf{Mod}(\mathcal{C})$ and $\mathbf{Alg}(\mathcal{C})$ will be algebraic.

Moreover, the category G-$\mathcal{C}$ (resp. A-$\mathbf{Mod}$, k-$\mathbf{Alg}$), will be a fiber of the forgetful functor $\mathbf{Act}(\mathcal{C}) \to \mathbf{Mon}(\mathcal{C})$ (resp. $\mathbf{Mod}(\mathcal{C}) \to \mathbf{Rng}(\mathcal{C})$, $\mathbf{Alg}(\mathcal{C}) \to \mathbf{CRng}(\mathcal{C})$).

Examples

1. We have $\mathbf{TGr} = \mathbf{Gr}(\mathbf{Top})$ and G-$\mathbf{Mod} = \mathbf{Ab}(G\text{-}\mathbf{Set})$.
2. If G is a topological group, then G-$\mathbf{Top}$ is the category of topological spaces endowed with a continuous action of G.
3. The category of algebraic (resp. analytic) groups over k is exactly $\mathbf{Gr}(\mathbf{Var}_k)$ (resp. $\mathbf{Gr}(\mathbf{An})$).
4. The category of group schemes is exactly $\mathbf{Gr}(\mathbf{Sch})$.
5. In k-$\mathbf{Alg}$ we may define a coring structure on $k[T]$ with addition given by
$$T \mapsto 1 \otimes T + T \otimes 1 \quad \text{and} \quad T \mapsto 0,$$
and multiplication given by
$$T \mapsto T \otimes T \quad \text{and} \quad T \mapsto 1.$$
6. If X is a scheme, an algebraic variety or an analytic variety, we can consider the category of sheaves of $\mathcal{O}_X$-modules on X (more on this later).

Preservation of limit/colimit

A functor $F : \mathcal{C} \to \mathcal{C}'$ preserves limits on I if whenever a commutative diagram D on I has a limit in $\mathcal{C}$, then $F^I(D)$ has a limit in $\mathcal{C}'$ and

$$\varprojlim F^I(D) = F(\varprojlim D).$$

It is said to preserve colimits if F^{op} preserves limits. It is called a left exact functor (resp. right exact functor, exact functor) if it preserves all finite limits (resp. finite colimits, finite limits and colimits).

Theorem

1. *If all kernels and products (resp. finite products) exist in $\mathcal{C}$ and are preserved by $F : \mathcal{C} \to \mathcal{C}'$, then F preserves all limits (resp. F is left exact) (and dual).*
2. *If $\mathcal{C}$ has a final object which is preserved by F and all finite fibered products exist in $\mathcal{C}$ and are preserved by F, then F is left exact (and dual).*

Examples

1. The forgetful functors $\mathbf{Top} \to \mathbf{Set}$ or $A\text{-}\mathbf{Mod} \to \mathbf{Ab}$, the embedding $\mathbf{Mon} \hookrightarrow \mathbf{Gr}$ as well as scalar restriction $B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$, preserve limits and colimits.
2. The forgetful functors $\mathbf{Gr} \to \mathbf{Set}$, $\mathbf{Rng} \to \mathbf{Ab}\ldots$, the embeddings $\mathbf{Ab} \hookrightarrow \mathbf{Gr}$ or $\mathbf{CRng} \hookrightarrow \mathbf{Rng}$, the functor $X \mapsto X^{\mathrm{coarse}}$, the functor $G \mapsto G^{\times}$, scalar evaluation and the functors $N \mapsto \mathrm{Hom}_A(M, N)$ preserve limits.
3. The functor $E \mapsto E^{\mathrm{disc}}$, scalar extension, the functor $E \mapsto AE$, $E \mapsto \mathbb{N}E$, $G \to A[G]$ or $E \mapsto A[\mathbb{N}E]$, the functors $G \mapsto G^{\mathrm{ab}}$ and $G \mapsto G[G^{-1}]$, and the functors $N \to M \otimes_A N$, preserve colimits.
4. If a category $\mathcal{C}$ admits right calculus of fractions with respect to S, then the localization functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is left exact.
5. The contravariant functors $M \mapsto \mathrm{Hom}_A(M, N)$ turn colimits into limits.
6. The contravariant functors $\mathrm{Spec} : \mathbf{CRng} \to \mathbf{Sch}$ and $X \mapsto \Gamma(X, \mathcal{O}_X)$ turn colimits into limits.

Examples

1. The functor
$$h^X : \mathcal{C} \to \mathbf{Set},\, Y \mapsto \mathrm{Hom}(X, Y)$$
preserves all limits (and h_X turns colimits into limits).
2. We have equivalences $(\mathcal{C}^I)^J \simeq \mathcal{C}^{I\times J} \simeq (\mathcal{C}^J)^I$ and it follows that limits preserve limits (and dual): whenever this has a meaning, we have
$$\varprojlim_J \varprojlim_I X_{ij} = \varprojlim_I \varprojlim_J X_{ij} \quad \text{and} \quad \varprojlim_I X_{ij} = \left(\varprojlim_I \underline{X}_{iJ}\right)_j \quad \text{(and dual).}$$
3. If $F : \mathcal{C} \to \mathcal{C}'$ is left exact, it preserves monomorphisms (and dual).
4. Filtering colimits are exact on **Set**, **Top**, **Mon**, etc. and preserved by the forgetful functors **Top** $\to$ **Set**, **Mon** $\to$ **Set**, etc.
5. The functor $\mathbf{Set}_{/Y} \to \mathbf{Set}_{/X}, Z \mapsto Z \times_Y X$ preserves colimits.

We call a category I filtering if given any $i, j \in I$ there exists $i \to k$ and $j \to k$, and given and any $u, v : i \to j$, there exists $w : j \to k$.

A left exact functor preserves any kind of algebraic structure and in particular, monoid, group, abelian group, ring, commutative ring, object with action, module or algebra.

Example

1. The underlying set of a topological, algebraic or analytic group has a group structure but this is not true for a group scheme.
2. If G is a usual monoid, group..., then both G^{disc} and G^{coarse} are topological monoids, groups...
3. If G is an algebraic monoid, group... over $\mathbb{C}$, then G^{an} is an analytic monoid, group...
4. The coring structure on the ring $\mathbb{Z}[T]$ (resp. the k-algebra $k[t]$) defines a ring structure on the scheme $\mathbb{A}^1 := \operatorname{Spec} \mathbb{Z}[T]$ (resp. the algebraic variety $\mathbb{A}^1_k$).

Note that $E \mapsto E^{\mathrm{disc}}$ is exact but does not preserve infinite limits in general (since the profinite topology is not the discrete topology). Note also that the functor $X \mapsto X^{\mathrm{an}}$ is left exact.

A functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be representable by $X \in \mathcal{C}$ if it is naturally isomorphic to h^X. It means that there exists a natural isomorphism

$$F(Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

And a contravariant functor F is representable by X if and only if is naturally isomorphic to h_X.

Example

A diagram D on I has a limit $\varprojlim D$ if and only if the functor

$$X \mapsto \mathrm{Hom}_{\mathcal{C}^I}(\underline{X}, D)$$

is representable by $\varprojlim D$ (and dual).

Examples

1. The forgetful functor from **Mon**, **Gr**, G-**Set** or A-**Mod** to **Set** is representable by $\mathbb{N}$, $\mathbb{Z}$, G or A respectively.
2. The forgetful functor from **Rng** and k-**Alg** to **Set** are representable by $\mathbb{Z}[T]$ and $k[T]$ respectively.
3. The forgetful functor **Top** $\to$ **Set** is representable by $1 = \{0\}$.
4. The forgetful functor from **FGr** $\to$ **FSet** is not representable.
5. If G is a monoid and $S \subset G$, then $G[S^{-1}]$ represents the functor
$$H \mapsto \{f : G \to H, f(S) \subset H^{\times}\}.$$
6. If M (resp. N) is a right (resp. left) A-module, then $M \otimes_A N$ represents the functor
$$P \mapsto \mathrm{Bil}_A((M, N), P).$$

Yoneda's Lemma

Theorem

If $F : \mathcal{C} \to \textbf{Set}$ is any functor and $X \in \mathcal{C}$, then there exists a bijection

$$\begin{array}{rcl} Hom(h^X, F) & \xrightarrow{\simeq} & F(X) \\ \alpha & \longmapsto & s := \alpha_X(Id_X.) \end{array}$$

The inverse is given as follows: to any $s \in F(X)$, one associates the natural transformation $\alpha : h^X \to F$ defined by

$$\alpha_Y : h^X(Y) \to F(Y), f \mapsto F(f)(s)$$

for $Y \in \mathcal{C}$.

Recall that α is an isomorphism if and only if for all $Y \in \mathcal{C}$, α_Y is bijective. It means that

$$\forall t \in F(Y), \exists! f : X \to Y, F(f)(s) = t.$$

It follows that F is representable by X if and only if

$$\exists s \in F(X), \forall Y \in \mathcal{C}, \forall t \in F(Y), \exists! f : X \to Y, F(f)(s) = t.$$

We will also say that X, endowed with s, is universal for $t \in F(Y)$ in $\mathcal{C}$.

Examples

1. If D is a diagram in $\mathcal{C}$, then $\varprojlim D$ is universal for morphisms $\underline{X} \to D$ (and dual).
2. The group $\mathbb{Z}$, endowed with 1, is universal for elements of G.
3. $G[S^{-1}]$ is universal for morphisms $f : G \to H$ with $f(S) \subset H^\times$.
4. $M \otimes_A N$ is universal for bilinear maps $M \times N \to P$.
5. The algebraic closure of a field is not universal.

Note that if X and X' both represent F through s and s' respectively, then there exists a unique isomorphism $f : X \simeq X'$ such that $F(f)(s) = s'$.

Image/coimage

The image of a morphism $f : X \to Y$ is a subobject $\mathrm{Im} f \hookrightarrow Y$ which is universal among all subobjects Y' of Y such that f factors through Y'. The coimage of f is the image of f in $\mathcal{C}^{\mathrm{op}}$.

If kernels and finite coproducts exist in $\mathcal{C}$, so do images and we have

$$\mathrm{Im}\, f = \ker(Y \rightrightarrows Y \textstyle\coprod_X Y) \quad \text{(and dual).}$$

When both image and coimage exist, then there exists a natural morphism $\mathrm{coim}\, f \to \mathrm{im}\, f$. And f is said to be strict if this is an isomorphism.

Examples

1. Any morphism in **Set**, in **Gr** or in A-**Mod** is strict (In **Gr**, this is the classical isomorphism $G/\ker f \simeq \mathrm{im}\, f$).
2. In **Top**, image and coimage have the same underlying set with quotient and induced topology respectively.

Adjoint functor

A functor $F : \mathcal{C} \to \mathcal{C}'$ is adjoint (on the left) to a functor $G : \mathcal{C}' \to \mathcal{C}$ if there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}'}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

The functor G is then coadjoint (or adjoint on the right) to F: it means that G^{op} is adjoint to F^{op}.

Of course, if F is adjoint to both G and G', then G and G' are necessarily isomorphic (and dual).

If F is adjoint to G, then the image of $\mathrm{Id}_{F(X)}$ under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}'}(F(X), F(X)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(F(X))$$

is called the adjunction morphism $\alpha_X : X \to G(F(X))$. The coadjunction morphism $\beta_Y : F(G(Y)) \to Y$ is (the dual to) the adjunction morphism associated to G^{op}.

Theorem

A functor $F : \mathcal{C} \to \mathcal{C}'$ is adjoint to $G : \mathcal{C}' \to \mathcal{C}$ if and only if there exists

$$\alpha : Id_{\mathcal{C}} \to G \circ F \quad et \quad \beta : F \circ G \to Id_{\mathcal{C}'}$$

such that the composite

$$F \stackrel{F(\alpha)}{\to} F \circ G \circ F \stackrel{\beta_F}{\to} F \quad and$$

$$G \stackrel{\alpha_G}{\to} G \circ F \circ G \stackrel{G(\beta)}{\to} G$$

are the identities of F and G respectively. Then, α and β are the adjunction and coadjunction morphisms.
Moreover, the functor F is faithful (resp. fully faithful) if and only if α is a monomorphism (resp. an isomorphism) (and dual).

An adjoint to a forgetful functor is usually called a free functor.

Examples

Most forgetful functors and embeddings have an adjoint:

$$\mathrm{Hom}_A(AE, M) \simeq \mathrm{Hom}(E, M), \quad \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}E, G) \simeq \mathrm{Hom}(E, G),$$

$$\mathrm{Hom}_{\mathbf{Rng}}(\mathbb{Z}[G], A) \simeq \mathrm{Hom}_{\mathbf{Mon}}(G, A), \quad \mathrm{Hom}_{\mathbf{Ab}}(G^{\mathrm{ab}}, H) \simeq \mathrm{Hom}_{\mathbf{Gr}}(G, H),$$

$$\ldots$$

Examples

1. The tensor algebra functor $M \mapsto T(M) := \bigoplus_{n=0}^{\infty} \bigotimes_k^n M$ is a free functor from k-modules to (graded) k-algebras.
2. The symmetric algebra functor $M \mapsto S(M) := T(M)^{\mathrm{com}}$ is a free functor from k-modules to commutative (graded) k-algebras.
3. The exterior algebra functor $M \mapsto \bigwedge(M) := T(M)/(\{m \otimes m\}_{m \in M})$ is a free functor from k-modules to supercommutative (and also skew-commutative graded) k-algebras.

Examples

Some forgetful functors and embeddings have both an adjoint and a coadjoint:

1. We have
$$\mathrm{Hom}_{\mathbf{Top}}(E^{\mathrm{disc}}, X) \simeq \mathrm{Hom}(E, X)$$
$$\text{and} \quad \mathrm{Hom}(X, E) \simeq \mathrm{Hom}_{\mathbf{Top}}(X, E^{\mathrm{coarse}}).$$
2. We have
$$\mathrm{Hom}_B(B \otimes_A M, N) \simeq \mathrm{Hom}_A(M, N)$$
$$\text{and} \quad \mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)).$$
3. We have
$$\mathrm{Hom}_{\mathbf{Gr}}(G[G^{-1}], H) \simeq \mathrm{Hom}_{\mathbf{Mon}}(G, H)$$
$$\text{and} \quad \mathrm{Hom}_{\mathbf{Mon}}(G, H) \simeq \mathrm{Hom}_{\mathbf{Gr}}(G, H^{\times}).$$

Examples

1. The contravariant functor $M \mapsto \check{M}$ is coadjoint to itself (or more precisely to its categorical dual):

$$\mathrm{Hom}_A(M, \check{N}) \simeq \mathrm{Hom}_A(N, \check{M}).$$

2. We also have the contravariant adjointness:

$$\mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}\, A) \simeq \mathrm{Hom}_{\mathbf{CRng}}(A, \Gamma(X, \mathcal{O}_X)).$$

Examples

1. The adjunction between $\times$ and Hom in **Set**:

$$\mathrm{Hom}(E \times F, G) \simeq \mathrm{Hom}(E, \mathrm{Hom}(F, G)).$$

2. The adjunction between $\otimes$ and Hom in A-**Mod**:

$$\mathrm{Hom}_{\mathbf{Ab}}(M \otimes_A N, P) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

Kan extension theorems

Theorem

Let $\mathcal{C}$ be a category and I a small category. Then the functor $X \mapsto \underline{X}, \mathcal{C} \to \mathcal{C}^I$ has a coadjoint if and only all limits on I exist in $\mathcal{C}$; the coadjoint is then given by $D \mapsto \varprojlim D$ (and dual).

Theorem

A functor $F : \mathcal{C} \to \mathcal{C}'$ has a coadjoint G if and only if, for all $Y \in \mathcal{C}'$, the functor $X \mapsto Hom(F(X), Y)$ is representable; it is then represented by $G(Y)$ (and dual).

Theorem

If a functor $G : \mathcal{C}' \to \mathcal{C}$ has an adjoint, then G preserves all limits; and the converse si true if all limits exist in $\mathcal{C}$ and G satisfies Freyd's condition (and dual).

Section 4

Sites and topos

Warning 2

If $\mathcal{C}$ and $\mathcal{D}$ are two categories, then the functors $F : \mathcal{C} \to \mathcal{D}$ are the objects of the metacategory $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ but they do not make a category in general. This is the case however when $\mathcal{C}$ is small. But then, the new category $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ is not small in general, and the process cannot be iterated.

A solution to this problem is to start working in a given universe and then enlarge it when needed.

Also, for classical sheaf theory, one only needs to consider functors defined on small categories and there is no problem.

But we may again think that set theory is not suitable to do category or topos theory and forget about it... However, we must be more careful than ever.

From now on, we will apply to metacategories most of the vocabulary and notations introduced for categories.

Presheaf

A presheaf on a category $\mathcal{C}$ with values in a category $\mathcal{D}$ is a contravariant functor $T : \mathcal{C} \to \mathcal{D}$. A morphism of presheaves is a natural transformation between them. The metacategory of pre sheaves is thus $\mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

Examples

1. For $E \in \mathcal{D}$, we may consider the constant presheaf still denoted by E that sends any X to E and any f to Id_E.
2. If $X \in \mathcal{C}$, we may consider the representable presheaf of sets on $\mathcal{C}$:
$$\widehat{X} := h_X : Y \mapsto \mathrm{Hom}(Y, X).$$

Any functor $\mathcal{D} \to \mathcal{D}'$ will induce by composition a functor

$$\mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}').$$

When $\mathcal{D}$ is a concrete category and we consider the forgetful functor, we obtain the underlying presheaf of sets.

Examples

1. A presheaf on a small category is the same thing as a codiagram.
2. A presheaf on an ordered set $(I, \leq)$ is given by a family $T(i)$ of elements of I, together with a compatible family of "restriction" morphisms $T(j) \to T(i)$ for $i < j$.
3. If X is a topological space, giving a presheaf T on $\mathbf{Open}(X)$ (also called a presheaf on X) is equivalent to giving
   1. an object $T(U)$ for any open subset U of X, and
   2. compatible restriction morphisms

$$\forall U' \subsetneq U, \quad T(U) \to T(U'), \quad s \mapsto s_{|U'}$$

4. Giving a presheaf on the category $\mathbf{Top}$ is equivalent to giving
   1. a presheaf T_X (its realization) on each topological space X, and
   2. a compatible family of morphisms $T_X \mapsto \widehat{f}_* T_Y$ for all continuous map $f : Y \to X$, where $\widehat{f}_* T_Y$ is the presheaf defined by

$$\widehat{f}_* T_Y(U) = T_Y(f^{-1}(U)).$$

Presheaf of sets

If $\mathcal{C}$ is a category, we denote by

$$\widehat{\mathcal{C}} := \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$$

the metacategory of presheaves of sets on $\mathcal{C}$.

Examples

1. We have $\widehat{\mathbf{0}} = \mathbf{1}$ and $\widehat{\mathbf{1}} = \mathbf{Set}$.
2. If G is a monoid, then $\widehat{G} = \mathbf{Set}\text{-}G$.
3. If $\Delta^{\leq 1}$ denotes the category of simplices of degree at most one, then $\widehat{\Delta^{\leq 1}}$ is the category of small quivers.
4. The category of small graphs is the category of presheaves on the category $\{\, 0 \Lleftarrow 1 \,\}$.
5. If X is a topological space, then $\widehat{\mathbf{Open}(X)}$ is the usual category of presheaves on X.
6. $\widehat{\mathbf{Top}}$ is a metacategory which is not a category.

Yoneda's embedding

In a category $\mathcal{C}$, we can rewrite Yoneda's lemma (contravariant version) as

$$\forall X \in \mathcal{C}, \forall T \in \widehat{\mathcal{C}}, T(X) \simeq \mathrm{Hom}(\widehat{X}, T).$$

In other words, morphisms $p : \widehat{X} \to T$ correspond bijectively to section $s \in T(X)$.

As a corollary, we see that there exists a fully faithful functor called the Yoneda embedding:

$$\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}, \quad X \mapsto \widehat{X} \ (= h_X).$$

Example

If $\mathcal{C} = \mathbf{Open}(X)$, then Yoneda's embedding identifies an open subset U of X with the presheaf (essentially with values in the category $\mathbf{2}$):

$$\widehat{U} : U' \mapsto \begin{cases} 1 & \text{if } U' \subset U \\ 0 & \text{otherwise.} \end{cases}$$

Yoneda's embedding preserves all limits but not colimits in general.

Notation: if $\{X_i\}_{i\in I}$ is a diagram of monomorphisms in a category $\mathcal{C}$ which is indexed by an ordered set, we will write $\cap X_i := \varprojlim X_i$ and $\cup X_i := \varinjlim X_i$, and call them respectively the intersection and the union of the X_i's.

Example

We let $\mathcal{C} = \mathrm{Open}(X)$ where X is a topological space.

1. We have $\widehat{\emptyset} \neq 0$ unless X is empty since $\widehat{\emptyset}(\emptyset) = \{\emptyset\}$ă $\neq$ ă$\emptyset = 0(\emptyset)$.
2. If $X = \cup U_i$ is an open covering, we can also consider the subobject

$$\cup \widehat{U}_i : U' \mapsto \begin{cases} 1 & \text{if } \exists i \in I, U' \subset U_i \\ 0 & \text{otherwise,} \end{cases}$$

   of $\widehat{X}$ which is different from $\widehat{X}$ in general.

It is also important to notice that, since Yoneda's embedding is left exact, it preserves any kind of algebraic structure.

Generalized slice category

If T is a presheaf of sets on a category $\mathcal{C}$, we may consider the slice category $\mathcal{C}_{/T}$ defined as follows: an object of $\mathcal{C}_{/T}$ is a pair made of an object X of $\mathcal{C}$ and a section $s \in T(X)$. A morphism in $\mathcal{C}_{/T}$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $T(f)(s') = s$.

Note that, if we are given a morphism of presheaves $T \to T'$, there exists an obvious restriction functor $\mathcal{C}_{/T} \to \mathcal{C}_{/T'}$.

And Yoneda's lemma states that there exists a fully faithful functor

$$\mathcal{C}_{/T} \hookrightarrow \widehat{\mathcal{C}}_{/T}, \quad (X, s) \mapsto p : \hat{X} \to T.$$

Moreover, if $X \in \mathcal{C}$, we also have an equivalence $\mathcal{C}_{/X} \simeq \mathcal{C}_{/\widehat{X}}$.

Finally, one can check that we also have an equivalence $\widehat{\mathcal{C}_{/T}} \simeq \widehat{\mathcal{C}}_{/T}$ and in particular $\widehat{\mathcal{C}_{/X}} \simeq \widehat{\mathcal{C}}_{/\widehat{X}}$.

Limit/colimit of presheaves

If $\mathcal{C}$ is any category and $X \in \mathcal{C}$, then the global section functor

$$\widehat{\mathcal{C}} \to \textbf{Set}, T \mapsto \Gamma(X, T) := T(X)$$

preserves all limits and colimits. As a consequence, we obtain:

Theorem

All limits and colimits exist in $\widehat{\mathcal{C}}$ and are computed argument by argument.

Examples

1. A morphism $T \to T'$ is a monomorphism (resp. an epimorphism) in $\widehat{\mathcal{C}}$ if and only if all $T(X) \to T'(X)$ are injective (resp. surjective).
2. If $\varphi : T \to T'$ is any morphism, then $\operatorname{im} \varphi$ exists in $\widehat{\mathcal{C}}$ and we have for all $X \in \mathcal{C}$, $(\operatorname{im} \varphi)(X) = \operatorname{Im} \varphi_X$ (and dual).

Since the global section functor is left exact, if G is a monoid of $\widehat{\mathcal{C}}$ and $X \in \mathcal{C}$, then $G(X)$ is a usual monoid and G lifts to a presheaf of monoids. Conversely, if G is a presheaf of monoids, then the family of maps

$$\mu_X : G(X) \times G(X) \to G(X) \quad \text{and} \quad \epsilon_X : 0 \to G(X)$$

define a monoid structure on the underlying presheaf of sets of G. In other words, we have an equivalence $\mathbf{Mon}(\widehat{\mathcal{C}}) \simeq \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Mon})$.

More generally, the category of presheaves of monoids (resp. groups, abelian groups, rings, commutative rings, sets with action, modules, algebras) is equivalent to the category of monoids (resp. groups, abelian groups, rings, commutative rings, objects with action, modules, algebras) of $\widehat{\mathcal{C}}$.

Also, since the Yoneda embedding is fully faithful and left exact, giving an algebraic structure on an object X of $\mathcal{C}$ is equivalent to extending the presheaf of sets $\widehat{X}$ to an algebraic presheaf.

Example

The presheaf of rings $X \mapsto \Gamma(X, \mathcal{O}_X)$ is represented by $\mathbb{A}^1$ in **Sch** (or $\mathbb{A}^1_k$ in $\mathbf{Var}_{/k}$) with the ring structure obtained from the coring structure of $\mathbb{Z}[T]$ (resp. $k[T]$).

Also, let G be a presheaf of monoids on a category $\mathcal{C}$. Then, giving an action of G on a presheaf of sets T is equivalent to giving an action of $G(X)$ on $T(X)$ for all X in $\mathcal{C}$ making commutative the diagram

$$\begin{tikzcd} G(Y) \times T(Y) \arrow[r] \arrow[d] & T(Y) \arrow[d] \\ G(X) \times T(X) \arrow[r] & T(X) \end{tikzcd}$$

whenever $f : X \to Y$ is a morphism in $\mathcal{C}$.

We have analogous assertion for modules or algebras.

Functoriality

Theorem

If $g : \mathcal{C} \to \mathcal{C}'$ is any functor, then the functor

$$\widehat{g}^{-1} : \qquad \widehat{\mathcal{C}'} \longrightarrow \widehat{\mathcal{C}}, \qquad T' \longrightarrow T' \circ g.$$

has both an adjoint $\widehat{g}_!$ and a coadjoint $\widehat{g}_*$.

As a consequence, the functor $\widehat{g}^{-1}$ preserves all limits and colimits, $\widehat{g}_!$ preserves all colimits and and a $\widehat{g}_*$ preserves all limits. In particular, that both $\widehat{g}^{-1}$ and $\widehat{g}_*$ preserve algebraic structures. Note also that we can recover g from the equality $\widehat{g(X)} = g_!\widehat{X}$.

Example

Any $X \in \mathcal{C}$ may be seen as a functor $X : \mathbf{1} \to \mathcal{C}$ giving rise to

$$\widehat{\mathcal{C}} \to \widehat{\mathbf{1}} = \mathbf{Set}, T \mapsto \Gamma(X, T).$$

Example

If $f : Y \to X$ is a continuous map, it induces a functor

$$(g =)\ f^{-1} : \mathbf{Open}(X) \to \mathbf{Open}(Y),$$

and by composition, a functor

$$(\widehat{g}^{-1} =)\ \widehat{f}_* : \widehat{\mathbf{Open}(Y)} \to \widehat{\mathbf{Open}(X)},$$

that has adjoint and coadjoint

$$(\widehat{g}_! =)\ \widehat{f}^{-1} \quad \text{and} \quad (\widehat{g}_* =)\ \widehat{f}^! : \widehat{\mathbf{Open}(X)} \to \widehat{\mathbf{Open}(Y)}.$$

Example (continuing)

Explicitly, the functors $\widehat{f}_*$, its adjoint $\widehat{f}^{-1}$ and its coadjoint $\widehat{f}^!$ are given by

$$\widehat{f}_*(T)(U) = T(f^{-1}(U)),$$

$$\widehat{f}^{-1}(T)(V) = \varinjlim_{V \subset f^{-1}(U)} T(U) \quad \text{and}$$

$$\widehat{f}^!(T)(V) = \varprojlim_{f^{-1}(U) \subset V} T(U).$$

In general, we might also write $f^{-1} := g : \mathcal{C} \to \mathcal{C}'$, and consequently:

$$\widehat{f}^{-1} := \widehat{g}_!, \quad \widehat{f}_* := \widehat{g}^{-1}, \quad \widehat{f}^! := \widehat{g}_*,$$

so that now $\widehat{f^{-1}(X)} = \widehat{f}^{-1}(\widehat{X})$.

Let $\mathcal{C}$ be a category. A sieve of $X \in \mathcal{C}$ is a subobject of $\widehat{X}$. Alternatively, this is a collection of morphisms $f : Y \to X$ stable by composition on the left (the correspondence is given by $R \mapsto \mathcal{C}_{/R}$ and $R := \cup \mathrm{Im}\, \widehat{f}$).

Given $R \subset \widehat{X}$ and $f : Y \to X$, we can define a sieve of Y by

$$f^{-1}(R)(Z) := \{s \in \widehat{Y}(Z), f \circ s \in R(Z)\}.$$

A topology on $\mathcal{C}$ is a collection of sets $\mathrm{Cov}(X)$ of sieves of the objects X of $\mathcal{C}$, called covering sieves, such that we always have:

1. $\widehat{X} \in \mathrm{Cov}(X)$.
2. If $R \in \mathrm{Cov}(X)$ and $f : Y \to X$ is any map, then $f^{-1}(R) \in \mathrm{Cov}(Y)$.
3. Let R be a sieve of X. Assume that there exists $S \in \mathrm{Cov}(X)$ such that whenever $f : Y \to X$ belongs to $S(Y)$, then $f^{-1}(R) \in \mathrm{Cov}(Y)$. Then $R \in \mathrm{Cov}(X)$.

A site is a category $\mathcal{C}$ endowed with a topology. It is said to be small if the category $\mathcal{C}$ is small.

Examples

1. We turn $\mathbf{Open}(X)$ into a small site by calling a sieve $R \subset \widehat{U}$ a covering if
$$\{U' \in \mathbf{Open}(X), \quad R(U') \neq \emptyset\}$$
is a set theoretic covering of U.
2. A presheaf $R \subset \widehat{X}$ will be a covering sieve in the (big) site **Top** if
$$\{U \in \mathbf{Open}(X), \quad R(U) \neq \emptyset\}$$
is a (set theoretic) covering of X.

The various topologies on a category $\mathcal{C}$ are ordered by inclusion from coarse (only $\hat{X}$ covers X) to discrete (any R covers X). Any intersection of topologies is a topology and it follows that any family of sieves generates a topology.

Pretopology

A pretopology on a category $\mathcal{C}$ is a set of families $\{X_i \to X\}$ called covering families such that

1. $\{\mathrm{Id}_X : X \to X\}$ is a covering family,
2. if $\{X_i \to X\}$ is a covering family and $f : Y \to X$ is any morphism, then $\{X_i \times_X Y \to Y\}$ (exists and) is a covering family,
3. if $\{X_i \to X\}$ is a covering family and for each i, $\{X_{ij} \to X_i\}$ is also a covering family, then $\{X_{ij} \to X\}$ is a covering family.

A sieve $R \subset X$ is a covering sieve for this pretopology if there exists a covering family $\{f_i : X_i \to X\}_{i \in I}$ such that $\cup \mathrm{Im}\widehat{f_i} \subset R$. They define a topology on $\mathcal{C}$. Conversely, if $\mathcal{C}$ is a site with fibered products, the set of all families $\{f_i : X_i \to X\}_{i \in I}$ such that $\cup \mathrm{Im}\widehat{f_i}$ is a covering sieve of X, is a pretopology that induces the topology of $\mathcal{C}$.

Example

Usual open coverings define the above topology on a topological space.

Examples

1. On the category **Sch**, we can consider the following (pre-) topology:
   1. Zariski topology: A covering family is a family of open immersions $f_i : X_i \hookrightarrow X$ such that $X = \cup f_i(X_i)$.
   2. Étale topology: A covering family is a family of étale maps $f_i : X_i \to X$ such that $X = \cup f_i(X_i)$.
   3. Flat topology: A covering family is a family of flat maps $f_i : X_i \to X$ such that $X = \cup f_i(X_i)$ (with a finiteness condition).

   Of course, the flat topology is finer than the étale topology which is itself finer than the Zariski topology. Actually, the étale (resp. flat) topology is generated by the Zariski topology and the surjective étale (resp. surjective flat) quasi-compact maps.
2. Alternatively, given a scheme X, one may consider the category $\mathbf{Et}(X) := \mathbf{Sch}^{\mathrm{et}}_{/X}$ of all étale maps $Y \to X$ and the étale pretopology. This is a site quite analogous to $\mathbf{Open}(X)$.
3. Of course, we have analogous constructions in $\mathbf{Var}_k$ or $\mathbf{An}$.

Examples

1. We denote by $\mathbf{Inf} := \mathbf{Mor}(\mathbf{Sch})^{\mathrm{nil}}$ the category of nilpotent immersions of schemes $X \hookrightarrow T$. A covering family is made of cartesian squares
$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d] & T_i \arrow[d] \\ X \arrow[r, hook] & T \end{tikzcd}$$
where $\{T_i\}_{i \in I}$ is a Zariski open covering of T. This is the big infinitesimal site.
2. Alternatively, given a scheme X over S, one may consider the small infinitesimal site $\mathbf{Inf}(X/S)$ of all of all nilpotent immersions $U \hookrightarrow T$ over S where U is an open subset of X, with the same pretopology.
3. As usual, we can work as well with $\mathbf{Var}_k$ or $\mathbf{An}$.
4. In order to deal with positive characteristic problem, it might be necessary to consider immersions endowed with a divided power structure. This leads to the big and small crystalline sites.

A sheaf (of sets) on a site $\mathcal{C}$ is a presheaf $\mathcal{F} : \mathcal{C} \to \mathbf{Set}$ such that, for all covering sieve R of X, we have

$$\mathcal{F}(X) = \mathrm{Hom}(\widehat{X}, \mathcal{F}) \simeq \mathrm{Hom}(R, \mathcal{F}).$$

If the topology comes from a pretopology, this is equivalent to

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j)$$

being exact for any covering family.

Example

If E is a set, the presheaf E_X defined on a topological space X by

$$E_X : U \mapsto \mathrm{Hom}_{\mathbf{Top}}(U, E^{\mathrm{disc}})$$

is a sheaf satisfying $E_X(U) = E$ when U is connected.

We denote by $\widetilde{\mathcal{C}} \subset \widehat{\mathcal{C}}$ the full subcategory of sheaves of sets on the site $\mathcal{C}$.

Examples

1. A sheaf on a topological space X is a presheaf $\mathcal{F}$ that satisfies: given an open covering $U = \cup U_i$ of an open subset of X and a family of $s_i \in \mathcal{F}(U_i)$ such that $(s_i)_{|U_j} = (s_j)_{|U_i}$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$.
2. A presheaf $\mathcal{F}$ on **Top** is a sheaf if and only for any topological space X, the realization $\mathcal{F}_X$ of $\mathcal{F}$ is a sheaf on X.
3. For the coarse topology on a category $\mathcal{C}$, any presheaf is a sheaf and consequently $\widetilde{\mathcal{C}} = \widehat{\mathcal{C}}$.
4. The only sheaf for the discrete topology on a category $\mathcal{C}$ is the constant sheaf 0 and consequently $\widetilde{\mathcal{C}} = \mathbf{1}$.
5. The canonical topology on a category $\mathcal{C}$ is the finest topology for which any representable presheaf is a sheaf: in other word, a topology is coarser than the canonical topology if and only if the Yoneda embedding takes values into $\widetilde{C}$ (standard condition).
6. If $\mathcal{C}$ is a site and we endow $\widetilde{\mathcal{C}}$ with its canonical topology, we have $\widetilde{\widetilde{\mathcal{C}}} = \widetilde{\mathcal{C}}$ (note that $\widetilde{\mathcal{C}}$ is only a metacategory in general).

Theorem

If $\mathcal{C}$ is a site, the inclusion functor $\mathcal{H} : \widetilde{C} \hookrightarrow \widehat{C}$ as an adjoint $T \mapsto \widetilde{T}$ which is (also left) exact.

Thus, we have

$$\forall T \in \widehat{\mathcal{C}}, \mathcal{F} \in \widetilde{\mathcal{C}}, \mathrm{Hom}(\widetilde{T}, \mathcal{F}) \simeq \mathrm{Hom}(T, \mathcal{F}).$$

The adjoint is actually obtained by applying twice $\check{\mathcal{H}}$ with

$$\check{\mathcal{H}}(T)(X) = \varinjlim_{R \in \mathrm{Cov}(X)} \mathrm{Hom}(R, T).$$

The sheaf $\widetilde{T}$ is called the sheaf associated to the presheaf T.

As a consequence, all limits and colimits exist in $\widetilde{\mathcal{C}}$ and moreover, limits are computed argument by argument (but not colimits).

Example

The sheaf $\widetilde{E}$ associated to the constant presheaf E on a topological space X is the constant sheaf E_X.

Topos

A (Grothendieck) topos $\mathcal{T}$ is a metacategory equivalent to some $\widetilde{\mathcal{C}}$ where $\mathcal{C}$ is a site. Equivalently, there exists a fully faithful embedding of $\mathcal{T}$ into $\widehat{\mathcal{C}}$ with exact adjoint. Or equivalently again, $\mathcal{T}$ is an exact reflective (to be defined) localization of $\widehat{\mathcal{C}}$ (with respect to local isomorphisms).

A topos is said to be small if there exists such a small $\mathcal{C}$.

Theorem (Giraud)

A category $\mathcal{T}$ is a small topos if and only if

1. *There exists a set of generators,*
2. *All finite limits exist,*
3. *All coproducts exist, are disjoint and are preserved by pull-back,*
4. *All Equivalence relations are effective and quotients are preserved by pull-back.*

Morphism of topos

A morphism of topos $f : \mathcal{T} \longrightarrow \mathcal{T}'$ is a couple of functors

$$\left(f^{-1} : \mathcal{T}' \longrightarrow \mathcal{T}, \quad f_* : \mathcal{T} \longrightarrow \mathcal{T}'\right)$$

with with f^{-1} exact and adjoint to f_*. We say that f is an embedding of topos when f_* is fully faithful.

Examples

1. Any functor $g : \mathcal{C} \to \mathcal{C}'$ induces a morphism of topos $\widehat{g} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$.
2. If $\mathcal{C}$ is a site, there exists an obvious embedding of topos $\widetilde{\mathcal{C}} \hookrightarrow \widehat{\mathcal{C}}$.

Of course, one may compose morphisms of topos. And this is associative with unit as usual. In other words, (morphisms of) topos form a metacategory.

Also note that both f^{-1} and f_* preserve algebraic structures.

If $\mathcal{C}$ and $\mathcal{C}'$ are two sites, a functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$ is said to be continuous if the functor $\widehat{f}_* : T' \mapsto T' \circ f^{-1}$ preserves sheaves. Then, the induced functor $f_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ will have an adjoint

$$f^{-1} : \qquad \begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{C}'} \\ \mathcal{F} & \longrightarrow & \widetilde{\widehat{f}^{-1}(\mathcal{F})} \end{array}$$

If this last functor is exact, then

$$f := (f^{-1}, f_*) : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$$

is a morphism of topos and we will also say that $f : \mathcal{C}' \to \mathcal{C}$ is a morphism of sites. Note that we have $\widetilde{f^{-1}(X)} = f^{-1}(\widetilde{X})$.

A functor $g : \mathcal{C}' \to \mathcal{C}$ between two sites is said to be cocontinuous if $\widehat{g}_*$ preserves sheaves. Then, the induced functor $g_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ extends uniquely to a morphism of toposes $g := (g^{-1}, g_*) : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$.

Examples

1. If we endow two topos $\mathcal{T}$ and $\mathcal{T}'$ with their canonical topology, a morphism of sites $f : \mathcal{T}' \to \mathcal{T}$ is nothing but a morphism of topos.
2. Let $f^{-1} : \mathcal{C} \to \mathcal{C}'$ be a functor between two sites. Assume that $\mathcal{C}$ has fibered products, and that f^{-1} is left exact and preserves covering families. Then f^{-1} induces a morphism of sites $f : \mathcal{C}' \to \mathcal{C}$.
3. If $f : Y \to X$ is a continuous map, the functor f^{-1} defines a morphism of sites. If $\mathcal{G}$ is a sheaf on Y, we have for any open subset U of X,

   $$f_*\mathcal{G}(U) = \mathcal{G}(f^{-1}(U)).$$

   And if $\mathcal{F}$ is a sheaf on X, then $f^{-1}\mathcal{F}$ is the sheaf associated to

   $$V \mapsto \varinjlim_{f(V)\subset U} \mathcal{F}(U).$$

4. Let $g : \mathcal{C}' \to \mathcal{C}$ be a functor between two sites. Assume that, given any $X' \in \mathcal{C}'$, any covering family of $g(X')$ is the image of a covering family of X'. Then, g is cocontinuous.

Induced topology

If $\mathcal{C}'$ is a site, there always exists a finest topology on $\mathcal{C}$ making a functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$ continuous. It is called the induced topology.

For example, if $\mathcal{C}$ is a site and $T \in \widehat{\mathcal{C}}$, we may consider the forgetful functor $j_T : \mathcal{C}_{/T} \to \mathcal{C}$ and endow $\mathcal{C}_{/T}$ with the induced topology. Unfortunately, the adjoint $j_{T!}$ is not left exact in general and we do not get a morphism of sites. However, j_T is also cocontinuous and we do obtain a morphism of toposes

$$j_T : \widetilde{\mathcal{C}_{/T}} \to \widetilde{\mathcal{C}}.$$

Actually, it induces an equivalence $\widetilde{\mathcal{C}_{/T}} \simeq \widetilde{\mathcal{C}}_{/\widetilde{T}}$. Moreover, any morphism $T' \to T$ in $\widehat{\mathcal{C}}$ will induce a morphism of toposes

$$\widetilde{\mathcal{C}_{/T'}} \xrightarrow{j} \widetilde{\mathcal{C}_{/T}}$$

giving rise to a sequence of functors $j_!, j^{-1}, j_*$, each being adjoint to the next. We will usually write $\mathcal{F}_{/T'} := j^{-1}\mathcal{F}$.

Example

If X is a topological space, we may consider the big site $\mathbf{Top}_{/X}$ of all topological spaces over X. The inclusion map

$$\mathbf{Open}(X) \hookrightarrow \mathbf{Top}_{/X}$$

is continuous, cocontinuous and left exact giving rise to two morphisms of topos

$$\widetilde{\mathbf{Top}_{/X}} \underset{\psi_X}{\overset{\varphi_X}{\rightleftarrows}} \widetilde{\mathbf{Open}(X)}$$

with $\varphi_X \circ \psi_X = \mathrm{Id}$ and $\varphi_{X*} = \psi_X^{-1}$. Any continuous map $f : Y \to X$ will provide a commutative diagram of topos

$$\begin{tikzcd}
\widetilde{\mathbf{Top}_{/Y}} \arrow[r, "\varphi_Y"] \arrow[d, "j"] & \widetilde{\mathbf{Open}(Y)} \arrow[d, "f"] \\
\widetilde{\mathbf{Top}_{/X}} \arrow[r, "\varphi_X"] & \widetilde{\mathbf{Open}(X)}.
\end{tikzcd}$$

Example

Let X be any topological space. If $\mathcal{F}$ is a sheaf on $\mathbf{Top}_{/X}$ and Y a topological space over X, the realization of $\mathcal{F}$ on Y is

$$\mathcal{F}_Y := \varphi_{Y*}\mathcal{F}_{/Y}.$$

Any morphism $f : Z \to Y$ over S will induce a morphism

$$\alpha_f : f^{-1}\mathcal{F}_Y \to \mathcal{F}_Z$$

between the realizations.

Giving the sheaf $\mathcal{F}$ on $\mathbf{Top}_{/X}$ is equivalent to giving the collection of all $\mathcal{F}_Y$ and the compatible morphisms α_f. We will call $\mathcal{F}$ crystalline if all the maps α_f are isomorphisms.

One can show that $\mathcal{F}$ is crystalline if and only if it is represented by a local homeomorphism $E \to X$ (an espace étalé).

Moreover, realization $\mathcal{F} \mapsto \mathcal{F}_X$ induces an equivalence between crystalline sheaves on $\mathbf{Top}_{/X}$ and sheaves on X.

Example

If $X \hookrightarrow T$ is a nilpotent immersion of schemes, there exists a diagram

$$\widetilde{\mathbf{Inf}}_{/X\hookrightarrow T} \underset{\psi_{X\hookrightarrow T}}{\overset{\varphi_{X\hookrightarrow T}}{\rightleftarrows}} \widetilde{\mathbf{Open}}(T).$$

Now, assume that X is a scheme over S and $Y \hookrightarrow T$ is a nilpotent immersion over $X \to S$. Then, we can pull back a sheaf $\mathcal{F}$ on $\mathbf{Inf}_{/X\to S}$ (where $X \to S$ is seen as a presheaf on **Inf** by restriction from **Mor**) along

$$j : \widetilde{\mathbf{Inf}}_{/Y\hookrightarrow T} \to \widetilde{\mathbf{Inf}}_{/X\to S}$$

and push along $\varphi_{Y\hookrightarrow T}$ and we obtain the realization $\mathcal{F}_{Y\hookrightarrow T}$ of $\mathcal{F}$ on $Y \hookrightarrow T$. Giving $\mathcal{F}$ is equivalent to giving all $\mathcal{F}_{Y\hookrightarrow T}$ and the morphisms

$$\alpha_{f,u} : u^{-1}\mathcal{F}_{Y\hookrightarrow T} \to \mathcal{F}_{Y'\hookrightarrow T'}$$

for all $(f,u) : (Y' \hookrightarrow T') \to (Y \hookrightarrow T)$. We call $\mathcal{F}$ crystalline if all the maps α_f are isomorphisms.

Local property

Let P be be a property for sheaves. Then a sheaf $\mathcal{F}$ on $\mathcal{C}$ has locally the property P if for any X in $\mathcal{C}$, the set of all $f : Y \to X$ such $\mathcal{F}_{/Y}$ has the property P "is" a covering sieve of X.

In terms of a pretopology, it means that for any $X \in \mathcal{C}$, there exists a covering family $\{X_i \to X\}$ such that $\mathcal{F}_{/X_i}$ has the property P (covering), and that if $\mathcal{F}_{/X}$ has the property P and $Y \to X$ is any morphism, then $\mathcal{F}_{/Y}$ also has the property P (sieve).

A property P is said to be local if satisfying P locally is equivalent to satisfying it globally.

Example

A sheaf $\mathcal{F}$ is (finite) constant if there exists a (finite) set E such that $\mathcal{F} \simeq \widetilde{E}$. Thus, we may consider (finite) locally constant sheaves.

Examples

1. If X is a connected, locally path-connected, semilocally simply-connected topological space and $x \in X$, then there exists an equivalence of categories

$$\widetilde{\mathbf{Open}(X)}^{\mathrm{lc}} \simeq \pi_1(X,x)\text{-}\mathbf{Set}, \quad \mathcal{F} \mapsto \mathcal{F}_x := x^{-1}\mathcal{F}$$

where lc means locally constant.

2. If X is a connected, quasi-compact, quasi-separated scheme and $x \in X$, then there exists an equivalence of categories

$$\widetilde{\mathbf{Et}(X)}^{\mathrm{flc}} \simeq \pi_1^{\mathrm{et}}(X,\bar{x})\text{-}\mathbf{FSet}, \quad \mathcal{F} \mapsto \mathcal{F}_{\bar{x}} := \bar{x}^{-1}\mathcal{F}$$

where flc means finite locally constant and the right hand site denotes the category of finite discrete sets with a continuous action of π_1^{et}.

Examples

1. If X is any topological space, we let $\mathbf{Et}(X)$ be the category of local homeomorphisms $X' \to X$ (*espaces étalés* again). This is a topos. And the obvious morphism of sites $\mathbf{Et}(X) \to \mathbf{Open}(X)$ induces an isomorphism of topos

$$\mathbf{Et}(X) \simeq \widetilde{\mathbf{Open}(X)}.$$

2. Thanks to the implicit function theorem, if X is an algebraic variety over $\mathbb{C}$, there exists a morphism of sites $\mathbf{Et}(X^{\mathrm{an}}) \to \mathbf{Et}(X)$.
3. (Grauert-Remmert) This morphism of sites induces an equivalence between finite topological coverings and and finite étale coverings.
4. As a consequence, there exists an equivalence

$$\widetilde{\mathbf{Et}(X)}^{\mathrm{flc}} \simeq \widetilde{\mathbf{Open}(X^{\mathrm{an}})}^{\mathrm{flc}}$$

and it follows that $\pi_1^{\mathrm{et}}(X, \bar{x})$ is the profinite completion of $\pi_1(X^{\mathrm{an}}, x)$.

Let $\mathcal{C}$ be a site and $\mathcal{D}$ be a concrete category. Then a sheaf on $\mathcal{C}$ with values in $\mathcal{D}$ is a presheaf with values in $\mathcal{D}$ whose underlying presheaf of sets is a sheaf.

Again, we see that he category of monoids (resp. groups, abelian groups, rings, commutative rings, objects with action, modules, algebras) of $\widetilde{\mathcal{C}}$ is equivalent to the category sheaves of monoids (resp. groups, abelian groups, rings, commutative rings, sets with action, modules, algebras).

Let $\mathcal{A}$ be a sheaf of rings on $\mathcal{C}$. An $\mathcal{A}$-module $\mathcal{F}$ is free of finite rank (resp. of finite type, resp. finitely presented) if there exists an isomorphism $\mathcal{A}^n \simeq \mathcal{F}$ (resp. an epimorphism $\pi : \mathcal{A}^n \twoheadrightarrow \mathcal{F}$, resp. such an epimorphism with $\ker \pi$ of finite type). We will be interested in modules that have locally such a property.

An $\mathcal{A}$-module $\mathcal{F}$ is coherent if it is locally of finite type and if for all morphism $f : \mathcal{A}^n_{/X} \to \mathcal{F}_{/X}$, we also have $\ker f$ locally of finite type.

If $\mathcal{A}$ is a sheaf of rings on $\mathcal{C}$ and $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{A}$-modules, their internal hom is defined by

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})(X) := \mathrm{Hom}_{\mathcal{A}_{/X}}(\mathcal{M}_{/X}, \mathcal{N}_{/X}).$$

This is an abelian sheaf, and even an $\mathcal{A}$-module if $\mathcal{A}$ is commutative. Actually,

$$\mathcal{E}nd_{\mathcal{A}}(\mathcal{M}) := \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$$

is naturally a sheaf of rings, and even $\mathcal{A}$-algebras if $\mathcal{A}$ is commutative.

If $\mathcal{M}$ is a left $\mathcal{A}$-modules and $\mathcal{N}$ an abelian sheaf, then $\mathcal{H}om_{\mathbf{Ab}}(\mathcal{M}, \mathcal{N})$ has a natural structure of right $\mathcal{A}$-module. Moreover, this functor has an adjoint $\otimes_{\mathcal{A}}$ called tensor product. More precisely, there exists a natural isomorphism

$$\mathcal{H}om_{\mathbf{Ab}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{P}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{H}om_{\mathbf{Ab}}(\mathcal{N}, \mathcal{P}))$$

when $\mathcal{M}$ is a right $\mathcal{A}$-module, $\mathcal{N}$ is a left $\mathcal{A}$-module and $\mathcal{P}$ is a abelian sheaf. We may replace **Ab** with $\mathcal{A}$-**Mod** when $\mathcal{A}$ is commutative.

Example

If X is a scheme over S (or a variety), then the diagonal embedding $X \hookrightarrow X \times_S X$ is an immersion defined by a sheaf of ideals $\mathcal{I}$ (on some open subset of $X \times_S X$). The n-th infinitesimal neighborhood P_n of X is the subscheme defined by the ideal $\mathcal{I}^{n+1}$ (and the same underlying space as X). We denote by $\mathcal{P}^{(n)}$ the structural sheaf of P_n and by $\mathcal{D}_n$ the $\mathcal{O}_X$-dual of $\mathcal{P}^{(n)}$. The inverse system fo the $\mathcal{P}^{(n)}$ dualizes to a direct system and we call $\mathcal{D} := \cup_{n \in \mathbb{N}} \mathcal{D}_n$ the sheaf of differential operators on X/S.
When X is smooth over S, then $\mathcal{D}$ has a structure of non commutative ring defined as follows: the morphism

$$X \times_S X \times_S X \to X \times_S X$$

that forgets the middle term induces a morphism $P_n \times_X P_m \to P_{n+m}$ that gives $\mathcal{P}^{(n+m)} \to \mathcal{P}^{(n)} \otimes \mathcal{P}^{(m)}$ and finally $\mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_{n+m}$.
We should also mention that $\Omega^1 := \ker\left(\mathcal{P}^{(1)} \to \mathcal{O}_X\right)$ is the sheaf of differential forms on X/S. And its dual $\mathcal{T}$ is the tangent sheaf.

A ringed site is a pair $(\mathcal{C}, \mathcal{O})$ made of a site $\mathcal{C}$ and a ring $\mathcal{O}$ on $\mathcal{C}$.

Example

When X is a topological space and $\mathcal{C} = \mathbf{Open}(X)$, then $(X, \mathcal{O})$ is called a ringed space (and more generally with a Grothendieck topology).

A morphism of ringed sites

$$(\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$$

is a pair made of a morphism of sites $f : \mathcal{C}' \to \mathcal{C}$ and a morphism of rings $f^{-1}\mathcal{O} \to \mathcal{O}'$ (or equivalently, $\mathcal{O} \to f_*\mathcal{O}'$). If $\mathcal{F}'$ is an $\mathcal{O}'$-module, then $f_*\mathcal{F}'$ will have a natural structure of $\mathcal{O}$-module. The induced functor has an adjoint given by

$$\mathcal{F} \mapsto f^*\mathcal{F} := \mathcal{O}' \otimes_{f^{-1}\mathcal{O}} f^{-1}\mathcal{F}.$$

Examples

1. If $\mathcal{C}$ is any site and Λ is a usual ring, we may consider the sheaf of rings $\Lambda_{\mathcal{C}} := \widetilde{\Lambda}$ on $\mathcal{C}$ associated to the constant presheaf of rings Λ and we obtain a ringed site $(\mathcal{C}, \Lambda_{\mathcal{C}})$.
2. If $\mathcal{C}$ is any site, there exists an equivalence $\mathbf{Ab}(\widetilde{\mathcal{C}}) \simeq \mathbb{Z}_{\mathcal{C}}$-$\mathbf{Mod}$.
3. If X is an algebraic variety over $\mathbb{C}$ and Λ is a usual ring, then there exists an equivalence of categories
$$\Lambda_{X^{\mathrm{an}}}\text{-}\mathbf{Mod}^{\mathrm{flc}} \simeq \Lambda_{\mathrm{Et}(X)}\text{-}\mathbf{Mod}^{\mathrm{flc}}.$$

Examples

1. If Λ is a usual ring, the sheaf of rings $\Lambda_{\mathbf{Top}}$ is represented by Λ^{disc}.
2. The presheaf $X \mapsto \Gamma(X, \mathcal{O}_X)$ defines a sheaf of rings $\mathcal{O}$ on $\mathbf{Sch}$ for the Zariski, étale or flat topology (represented by $\mathbb{A}^1$).
3. The presheaf $(X \hookrightarrow T) \mapsto \Gamma(T, \mathcal{O}_T)$ defines a sheaf of rings $\mathcal{O}_{\mathrm{Inf}}$ on the big infinitesimal site $\mathbf{Inf}$.

Examples

1. If X is a scheme (or a variety) and we still denote by X its underlying topological space, then $(X, \mathcal{O}_X)$ is a ringed space.
2. The functor $X \mapsto (X, \mathcal{O}_X)$ is not fully faithful. It becomes so if we replace the target by the subcategory of locally ringed spaces.
3. If X is an algebraic variety over $\mathbb{C}$, there exists a natural morphism of (locally) ringed spaces
$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X).$$
4. (Serre) If X is proper, we have (GAGA)
$$\mathcal{O}_X\text{-}\mathbf{Mod}^{\mathrm{coh}} \simeq \mathcal{O}_{X^{\mathrm{an}}}\text{-}\mathbf{Mod}^{\mathrm{coh}}.$$
5. If X is a smooth analytic variety, there exists a (contravariant) equivalence of categories
$$\mathcal{S}ol := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, -) : \mathcal{D}_X\text{-}\mathbf{Mod}^{\mathcal{O}\text{-}\mathrm{coh}} \simeq \mathbb{C}_X\text{-}\mathbf{Vec}^{\mathrm{fdlc}}.$$

Example

If $X \to S$ is a morphism of schemes and $Y \hookrightarrow T$ in $\mathbf{Inf}_{/X\to S}$, we know that there exists a realization functor

$$\mathcal{O}_{\mathrm{Inf}/X\to S}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_T\text{-}\mathbf{Mod}, \quad \mathcal{F}\text{ă} \longmapsto \mathcal{F}_{Y\hookrightarrow T}.$$

Giving $\mathcal{F}$ is equivalent to giving all $\mathcal{F}_{Y\hookrightarrow T}$'s and the compatible family of

$$\alpha_{f,u} : u^*\mathcal{F}_{Y\hookrightarrow T} \to \mathcal{F}_{Y'\hookrightarrow T'}$$

for all $(f,u) : (Y' \hookrightarrow T') \to (Y \hookrightarrow T)$. We call $\mathcal{F}$ a crystal if all the maps $\alpha_{f,u}$ are isomorphisms (automatic if $\mathcal{F}$ is locally finitely presented).
The same definition works as well on the small infinitesimal site $\mathbf{Inf}(X/S)$ and there exists a pair of morphisms of topos

$$\widetilde{\mathbf{Inf}_{/X\to S}} \underset{r}{\overset{p}{\rightleftarrows}} \widetilde{\mathbf{Inf}(X/S)}$$

inducing an equivalence on crystals.

Examples

1. If X is a scheme over S, we may consider the trivial nilpotent immersion $\mathrm{Id}_X : X \hookrightarrow X$ and the nilpotent immersion $X \hookrightarrow P^{(n)}$ into the n-th infinitesimal neighborhood $P^{(n)} \subset X \times_S X$. There exists two obvious morphisms

$$p_1, p_2 : (X \hookrightarrow P^{(n)}) \to (X \hookrightarrow X).$$

   Therefore, if $\mathcal{F}$ a crystal on X/S, there exists a natural morphism

$$\mathcal{F}_X \hookrightarrow \mathcal{P}^{(n)} \otimes \mathcal{F}_X = p_2^* \mathcal{F}_X \simeq \mathcal{F}_{P^{(n)}} \simeq p_1^* \mathcal{F}_X = \mathcal{F}_X \otimes \mathcal{P}^{(n)}.$$

   If we assume that X is smooth over S, we obtain by duality a morphism $\mathcal{D}_n \times \mathcal{F}_X \to \mathcal{F}_X$. This turns $\mathcal{F}_X$ into a $\mathcal{D}$-module on X/S and this is an equivalence.

2. If X is a smooth scheme over a $\mathbb{Q}$-scheme S, then giving a $\mathcal{D}_{X/S}$-module structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to an integrable connection.

Example

If X is a scheme over S, a differential operator $\mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_S$-linear map that factors through an $\mathcal{O}_X$-linear map $p_2^*\mathcal{F} \to \mathcal{G}$ with $p_2 : P^{(n)} \to X$. One defines in the same way a differential operator between $\mathcal{O}_{\mathrm{Inf}/X\hookrightarrow X}$-modules. Then the morphism

$$\widetilde{\mathbf{Inf}_{/X\hookrightarrow X}} \xrightarrow{\varphi_X} \widetilde{\mathbf{Open}(X)}$$

induces an equivalence between crystals and $\mathcal{O}_X$-modules, even with differential operators as morphisms. Moreover, the functor

$$\widetilde{\mathbf{Inf}_{/X\hookrightarrow X}} \xrightarrow{j} \widetilde{\mathbf{Inf}_{/X\to S}}.$$

sends differential operators to $\mathcal{O}_{\mathrm{Inf}/X\to S}$-linear maps. The linearization of an $\mathcal{O}_X$-module $\mathcal{F}$ is $L(\mathcal{F}) := j_*\varphi_X^*\mathcal{F}$. This defines an equivalence between the category of $\mathcal{O}_X$-modules with differential operators and the category of crystals on the infinitesimal site.

Section 5

Abelian categories

Pre-additive category

A pre-additive category (also called **Ab**-category) is a category $\mathcal{C}$ endowed with a factorization of the Hom functor:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r] \arrow[dr, dashed] & \mathbf{Set} \\ & \mathbf{Ab} \arrow[u] \end{tikzcd}$$

In other words, we require that for all $M, N \in \mathcal{C}$, $\mathrm{Hom}(M, N)$ is endowed with the structure of an abelian group and that for all $M, N, P \in \mathcal{C}$, composition

$$\begin{array}{ccc} \mathrm{Hom}(M,N) \times \mathrm{Hom}(N,P) & \longrightarrow & \mathrm{Hom}(M,P) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

is bilinear. In particular, $\mathrm{End}(M)$ becomes a ring.

Examples

1. The categories **Ab**, A-**Mod**, and more generally $\mathcal{A}$-**Mod** if $\mathcal{A}$ is a sheaf of rings, are pre-additive.
2. The category $\mathbf{Mat}_A$ is pre-additive.
3. The category built on the multiplicative monoid of a ring A, is a pre-additive category. Any pre-additive category with exactly one object has this form.
4. The categories **Set**, **Mon**, **Gr**, **Rng** or **CRng** cannot be endowed with the structure of a pre-additive category.

If $\mathcal{C}$ is a pre-additive category, we will actually consider the functor Hom as a functor with values in **Ab**. Consequently, if $M \in \mathcal{C}$, we will also write

$$h^M : \mathcal{C} \to \mathbf{Ab} \quad \text{and} \quad h_M : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}.$$

If $\mathcal{C}$ is a pre-additive category, then $\mathcal{C}^{\mathrm{op}}$ also, and so does $\mathcal{C}^I$ if I is a small category.

Additive functor

A functor $F : \mathcal{C} \to \mathcal{D}$ between two pre-additive categories is additive if for any $M, N \in \mathcal{C}$, the map

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N))$$

is a group homomorphism.

Examples

1. If $\mathcal{C}$ is any pre-additive category, then the functors h^M and h_M are additive.
2. The composite of two additive functors is additive and F^{op} is additive when F is.
3. If I is a small category and F is additive, then F^I is additive too.
4. If $\mathcal{A}$ is a sheaf of rings, the functors $\mathcal{H}om_{\mathcal{A}}$ and $\otimes_{\mathcal{A}}$ are additive.
5. If $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ is morphism of ringed sites, then both f_* and f^* are additive.

Zero object and direct sum

Let $\mathcal{C}$ be a pre-additive category.

An $M \in \mathcal{C}$ is a zero object if $\mathrm{End}(M) = 1$ (meaning $\mathrm{Id}_M = 0_M$).

An $M \in \mathcal{C}$ is a direct sum of two objects M_1 and M_2 (in which case we write $M = M_1 \oplus M_2$) if there exists

$$p_k : M \to M_k, i_k : M_k \to M, k = 1, 2 \quad \text{such that}$$

$$p_1 \circ i_1 = \mathrm{Id}_{M_1}, \quad p_2 \circ i_2 = \mathrm{Id}_{M_2} \quad \text{and} \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{Id}_M.$$

Both notions are autodual in the sense that the property is satisfied in $\mathcal{C}$ if and only if it is satisfied in $\mathcal{C}^{\mathrm{op}}$.

Theorem

1. *An $M \in \mathcal{C}$ is a zero object if and only if it is a final object (and dual).*
2. *An $M \in \mathcal{C}$ is a direct sum of M_1 and M_2 if and only if it is a product of M_1 and M_2 with projections p_1 and p_2 (and dual).*

Additive category

An additive category is a pre-additive category with a zero object and all direct sums. Equivalently, it means that all finite products or all finite sums exist (and then they both exist and are equal). Moreover, it implies that the factorization of Hom through **Ab** is unique.

Examples

1. The categories **Ab**, A-**Mod** and $\mathbf{Mat}_A$ are additive.
2. If A is a non-zero ring, the category A is not an additive category.

If $\mathcal{C}$ is an additive category then $\mathcal{C}^{\mathrm{op}}$ is also an additive category, and so is $\mathcal{C}^I$ if I is a small category.

Theorem

A functor between two additive categories is additive if and only if it preserves all direct sums and the zero object - or equivalently all finite products (and dual).

Exact sequences

In an additive category $\mathcal{C}$, the kernel of a morphism $f : M \to N$ is $\ker f := \ker(f, 0)$ if it exists. And a sequence

$$0 \to M' \to M \xrightarrow{f} M''$$

is said to be left exact if the sequence

$$\begin{tikzcd} M' \arrow[r] & M \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & M'' \end{tikzcd}$$

is left exact.

The cokernel $\operatorname{coker} f$ of f is the kernel of f in $\mathcal{C}^{\mathrm{op}}$ (if it exists). A sequence is right exact if it is left exact in $\mathcal{C}^{\mathrm{op}}$.

A short exact sequence is a sequence

$$0 \to M' \to M \to M'' \to 0$$

which is exact both on the left and on the right.

Left, right or short exact sequences form an additive category: this is a full subcategory of the category of diagrams on the ordinal category **3** which is stable under products.

A short exact sequence is said to split if it is isomorphic to

$$0\text{ă} \longrightarrow M_1 \xrightarrow{i_1} M_1 \oplus M_2\text{ă} \xrightarrow{p_2} M_2 \longrightarrow 0\ .$$

A short exact sequence

$$0\text{ă} \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

splits if and only if p has a section (and dual).

Finally, M is called an extension of M'' by M' (fiber category) is there exists a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

It is called a trivial extension if the sequence splits.

A pre-abelian category is an additive category $\mathcal{C}$ where any morphism has both a kernel and a cokernel. Equivalently, a pre-additive category $\mathcal{C}$ is pre-abelian if and only if all finite limits and all finite colimits exist in $\mathcal{C}$.

Example

The categories **Ab**, $\mathcal{A}$-**Mod** and $\mathbf{Mat}_A$ are pre-abelian (for the last one, notice for example that $\operatorname{coker}(2) = 0$ in $\mathbf{Mat}_{\mathbb{Z}}$)

If $\mathcal{C}$ is pre-abelian, so are $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}^I$ if I is a small category.

Notation: when we are given a subobject $N \hookrightarrow M$, we will denote by $M \twoheadrightarrow M/N$ the cokernel. Also, if $f : M' \to M$ is any morphism, we will write $f^{-1}(N) := N \times_M M'$.

In a pre-abelian category $\mathcal{C}$, we have for any morphism $f : M \to N$,

$$\operatorname{im} f := \ker \operatorname{coker} f \quad \text{(and dual).}$$

An abelian category is a pre-abelian category satisfying one of the following equivalent properties:

1. Every monomorphism is a kernel, and dual.
2. If $N \hookrightarrow M$ is a monomorphism, then N is the kernel of $M \twoheadrightarrow M/N$, and dual.
3. Any morphism $f : M \to N$ factors uniquely up to isomorphism as an epimorphism followed by a monomorphism.
4. Any morphism $f : M \to N$ is strict (coim $f \simeq$ im ăf).

EXAMPLES

1. The category $\mathcal{A}$-**Mod** is abelian.
2. The category $\mathbf{Mat}_{\mathbb{Z}}$ is not abelian because 2 is a monomorphism which is not a kernel.

If $\mathcal{C}$ is an abelian category, then $\mathcal{C}^{\mathrm{op}}$ is also an abelian category, as well as $\mathcal{C}^I$ if I is a small category.

Theorem

If a functor F between two additive categories is adjoint to a functor G, then both functors are additive and there exists a natural isomorphism of abelian groups

$$Hom(FM, N) \simeq Hom(M, GN)).$$

Theorem

A functor $F : \mathcal{C} \to \mathcal{D}$ between two abelian categories is left exact if and only if it is additive and preserves left exact sequences (and dual).

Theorem

Let $\mathcal{D}$ be an additive (resp. abelian) category. If a fully faithful (resp. and exact) functor $\mathcal{C} \hookrightarrow \mathcal{D}$ has an adjoint or a coadjoint, then $\mathcal{C}$ also is additive (resp. abelian).

Theorem (Quillen)

If $\mathcal{C}$ is an abelian category endowed with its canonical topology, then Yoneda's embedding $\mathcal{C} \hookrightarrow \mathbf{Ab}(\widetilde{\mathcal{C}})$ is exact.

A (long) sequence is a commutative diagram on the ordered set $(\mathbb{Z}, \leq)$:

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \longrightarrow \cdots$$

If $\mathcal{C}$ is additive and $d^n \circ d^{n-1} = 0$ for each $n \in \mathbb{Z}$, we call $K^\bullet$ a (cochain) complex. The complexes of $\mathcal{C}$ form an additive subcategory $\mathbf{C}(\mathcal{C})$. Limits and colimits in $\mathbf{C}(\mathcal{C})$ are computed argument by argument. If $\mathcal{C}$ is abelian, so is $\mathbf{C}(\mathcal{C})$.

The dual notion is that of chain complex. The equivalence $\mathbb{Z} \simeq \mathbb{Z}^{\mathrm{op}}$ induces an equivalence $\mathbf{C}(\mathcal{C}^{\mathrm{op}}) \simeq \mathbf{C}(\mathcal{C})$ between chain complexes and cochain complexes. One usually writes $K_n := K^{-n}$ and $d_n := \pm d^{-n-1}$.

A complex $K^\bullet$ is bounded below if $K^n = 0$ for $n << 0$. They form a full subcategory $\mathbf{C}^+(\mathcal{C})$. A complex is bounded above if it is bounded below in $\mathcal{C}^{\mathrm{op}}$ and we get a category $\mathbf{C}^-(\mathcal{C})$. A complex is bounded if it is bounded both above and below and we get a category $\mathbf{C}^{\mathrm{b}}(\mathcal{C})$.

Examples

1. Any $M \in \mathcal{C}$ gives rise to a unique complex with $K^0 = M$ and $K^n = 0$ otherwise. We obtain a fully faithful functor
$$\mathcal{C} \hookrightarrow \mathbf{C}(\mathcal{C}), M \mapsto [M]$$
that commutes with all limits and colimits.
2. Any morphism $f : M \to N$ gives rise to a unique complex with $d^0 = f$ and $d^n = 0$ otherwise. We obtain
$$\mathbf{Mor}(\mathcal{C}) \hookrightarrow \mathbf{C}(\mathcal{C}), (M \to N) \mapsto [M \to N].$$
3. Same with left, right or short exact sequences.
4. Any simplicial complex $C^\bullet$ in an additive category $\mathcal{C}$ gives rise to a cochain complex with $K^n = C^n$ and $d^n = \sum(-1)^i d_i^n$.
5. If X is a topological space, we may consider the chain and cochain complexes $C_\bullet(X)$ and $C^\bullet(X)$ obtained from the corresponding simplicial and cosimplicial complexes.

Examples

1. A derivation of a graded k-algebra A is a k-linear map $d : A \to A$ of degree 1 satisfying $d \circ d = 0$ and such that
$$\forall a, b \in A, d(ab) = d(a)b + (-1)^{\deg(a)} a d(b).$$
A differential graded algebra is such a pair (A, d). It gives rise to a complex
$$0 \longrightarrow \mathrm{Gr}^0 A \xrightarrow{d} \mathrm{Gr}^1 A \xrightarrow{d} \mathrm{Gr}^2 A \longrightarrow \cdots$$
2. If X is a (smooth) scheme over S, then the derivation $\mathcal{O}_X \to \Omega^1_{X/S}$ extends uniquely to a derivation of $\bigwedge \Omega^1_{X/S} =: \bigoplus \Omega^n_{X/S}$. In particular, we obtain the de Rham complex of X/S:
$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \longrightarrow \cdots$$
3. Same for algebraic, analytic or even differential varieties.

Cohomology

Let $\mathcal{C}$ be an abelian category. Then the n-th cohomology of a cochain complex $K^\bullet$ of $\mathcal{C}$ is

$$\mathrm{H}^n(K^\bullet) = \ker d^n / \mathrm{im}\, d^{n-1}.$$

The n-th homology $\mathrm{H}_n(K_\bullet)$ of a chain complex $K_\bullet$ is its n-th cohomology in $\mathcal{C}^{\mathrm{op}}$.

Examples

1. If $M \in \mathcal{C}$, we have $\mathrm{H}^0([M]) = M$ and $\mathrm{H}^n([M]) = 0$ otherwise.
2. We have $\mathrm{H}^0([M \xrightarrow{f} N]) = \ker f$ and $\mathrm{H}^1([M \xrightarrow{f} N]) = \operatorname{coker} f$.
3. A short exact sequence $0 \to M' \to M \to M'' \to 0$ has 0 cohomology.
4. If X is a topological space, then
$$\mathrm{H}_{n,\mathrm{sing}}(X) := \mathrm{H}_n(C_\bullet(X)) \quad \text{and} \quad \mathrm{H}^n_{\mathrm{sing}}(X) := \mathrm{H}^n(C^\bullet(X)).$$
5. If X is a differential variety, then $\mathrm{H}^n_{\mathrm{dR}}(X) := \mathrm{H}^n(\Omega^\bullet(X))$.

Recall that $\mathcal{C}$ denotes an abelian category. A complex $K^\bullet$ is said to be acyclic (or exact) in degree n if $\mathrm{H}^n(K^\bullet) = 0$. It means that

$$\operatorname{im} d^{n-1} = \ker d^n.$$

An acyclic (or exact) complex $K^\bullet$ is a complex which is acyclic in each degree. We also say that $K^\bullet$ is a long exact sequence.

Note that there exists a family of additive functors

$$\mathrm{H}^n : \mathbf{C}(\mathcal{C}) \to \mathcal{C}$$

which are not exact in general:

Theorem

Any short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ gives rise to a long exact sequence

$$\cdots \to H^n(K^\bullet) \to H^n(L^\bullet) \to H^n(M^\bullet) \to H^{n+1}(K^\bullet) \to \cdots$$

This is a consequence of the snake lemma:

$$\begin{array}{ccccc} & M' & M & M'' & 0 \\ \\ 0 & N' & N & N'' & \end{array}$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K' \arrow[d] & K \arrow[d] & K'' \arrow[d] & \\
 & M' & M & M'' & 0 \\
0 & N' \arrow[d] & N \arrow[d] & N'' \arrow[d] & \\
 & C' \arrow[d] & C \arrow[d] & C'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[d] & \\
 & M' & M & M'' & 0 \\
0 & N' \arrow[d] & N \arrow[d] & N'' \arrow[d] & \\
 & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K' \arrow[r, red] \arrow[d] & K \arrow[r, red] \arrow[d] & K'' \arrow[d] \arrow[dddll, red, out=0, in=180] & \\
 & M' & M & M'' & 0 \\
0 & N' \arrow[d] & N \arrow[d] & N'' \arrow[d] & \\
 & C' \arrow[r, red] \arrow[d] & C \arrow[r, red] \arrow[d] & C'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Injective/projective resolution

Let $\mathcal{C}$ be an abelian category. An $I \in \mathcal{C}$ is called injective if the functor h_I is exact. The category $\mathcal{C}$ has enough injectives if any $M \in \mathcal{C}$ is a subobject of an injective I. The dual notion is that of projective object.

An $I \in \mathcal{C}$ is injective if and only if any monomorphism $I \hookrightarrow M$ has a retraction if and only if any extension by I is trivial (and dual).

Examples

1. The abelian groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective (meaning divisible).
2. A left A-module is projective if and only if it is a direct factor of a free A-module.
3. $\mathbb{Z}/2$ is projective (but not free) over $\mathbb{Z}/6$.
4. If A is a ring, then the category A-**Mod** has enough injectives and projectives.
5. If $\mathcal{A}$ is a sheaf of rings on a small site, then $\mathcal{A}$-**Mod** has enough injectives.

Let $\mathcal{C}$ be an abelian category. A quasi-isomorphism of complexes $f : K^\bullet \to L^\bullet$ is a morphism such that each $\mathrm{H}^n(f)$ is an isomorphism. Note that being quasi-isomorphic is not a symmetric relation. We will also say that $L^\bullet$ is a right resolution of $K^\bullet$. It is called an injective resolution if each L^n is injective.

A left (resp. a projective) resolution is a right (resp. an injective) resolution in $\mathcal{C}^{\mathrm{op}}$.

Examples

1. A complex is acyclic if and only if it is quasi-isomorphic to [0].
2. A sequence $0 \to M \to K^0 \to K^1 \to \cdots$ is exact if and only if $[M] \to K^\bullet$ is a right resolution.
3. Any abelian group M has a projective resolution of length 2: a short exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with L_0 and L_1 free.

If $\mathcal{C}$ is an abelian category with enough injectives, then any complex which is bounded below has an injective resolution (and dual).

Examples

1. We may consider the presheaf of abelian complexes on **Top** defined by
$$C^\bullet : X \mapsto C^\bullet(X).$$
Then there exists a surjective quasi-isomorphism $C^\bullet(X) \to \widetilde{C}^\bullet(X)$ (theorem of small chains below). Moreover, when X is locally contractible, then $\widetilde{C}^\bullet_X$ is a right resolution of $\mathbb{Z}_X$.
2. If X is a smooth analytic variety (resp. a differentiable manifold), then the de Rham complex $\Omega^\bullet_X$ is a right resolution of $\mathbb{C}_X$ (resp. $\mathbb{R}_X$).
3. Let X be a smooth scheme over S. Then the Koszul complex
$$0 \to \bigwedge^d \mathcal{T}_{X/S} \otimes \mathcal{D}_{X/S} \to \cdots \to \mathcal{T}_{X/S} \otimes \mathcal{D}_{X/S} \to \mathcal{D}_{X/S}$$
is a left resolution of $\mathcal{O}_X$.
4. If X is a smooth scheme over a $\mathbb{Q}$-scheme S, then Grothendieck linearized complex $L(\Omega^\bullet_{X/S})$ is a right resolution of $\mathcal{O}_{\mathbf{Inf}_{X\to S}}$.

Homotopy

Let $\mathcal{C}$ be an additive category. A morphism of complexes $f : K^\bullet \to L^\bullet$ is said to be homotopic to 0 if there exists a family of $s^n : K^n \to L^{n-1}$ such that for all n, we have

$$f^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n.$$

Morphisms that are homotopic to 0 form a subgroup of $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(K^\bullet, L^\bullet)$. And composition on both sides with a morphism homotopic to 0 always gives a morphism homotopic to 0.

Two morphisms of complexes $f, g : K^\bullet \to L^\bullet$ are said to be homotopic if $g - f$ is homotopic to 0. This is an equivalence relation $\sim$ compatible with composition on both sides.

A homotopy equivalence is a morphism of complexes f such that there exists g with $f \circ g \sim \mathrm{Id}$ and $g \circ f \sim \mathrm{Id}$.

One defines the category $\mathbf{K}(\mathcal{C})$ of complexes up to homotopy as the category that has the same objects as $\mathbf{C}(\mathcal{C})$ and

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(K^\bullet, L^\bullet)/\{\text{homotopic to } 0\}$$

with the induced composition. This is an additive category and there exists an obvious additive functor $\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C})$.

A morphism of complexes is homotopic to 0 if and only if its image is 0 in $\mathbf{K}(\mathcal{C})$. Two morphisms of complexes are homotopic if and only if they have the same image in $\mathbf{K}(\mathcal{C})$. A morphism of complexes is a homotopy equivalence if and only if its image in $\mathbf{K}(\mathcal{C})$ is an isomorphism.

On defines $\mathbf{K}^+(\mathcal{C})$, $\mathbf{K}^-(\mathcal{C})$ and $\mathbf{K}^{\mathrm{b}}(\mathcal{C})$ in the same way (may also be seen as full subcategories).

Example

If T is a presheaf on **Top**, one may consider the simplicial set $S_\bullet(T) := T \circ \Delta^\bullet$ and define the corresponding abelian complexes $C_\bullet(T)$ and $C^\bullet(T)$. The small chain theorem states that if R is a sieve of X, there exists an injective homotopy equivalence $C_\bullet(R) \hookrightarrow C_\bullet(X)$.

The derived category $\mathbf{D}(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the localization of $\mathbf{C}(\mathcal{C})$ at all quasi-isomorphisms. Note that, by definition, the functors H^n factor through $\mathbf{D}(\mathcal{C})$.

Actually, H^n factors also through $\mathbf{K}(\mathcal{C})$. More precisely, if f is homotopic to 0, then $\mathrm{H}^n(f) = 0$ for all $n \in \mathbb{Z}$. If f is homotopic to g, then $\mathrm{H}^n(f) = \mathrm{H}^n(g)$. And if f is a homotopy equivalence, then it is a quasi-isomorphism.

It follows that the localization functor factors as

$$\mathbf{C}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}).$$

Moreover, we have:

Theorem

The category $\mathbf{K}(\mathcal{C})$ admits both left and right calculus of fractions with respect to quasi-isomorphisms.

If $\mathcal{C}$ is an abelian category, one can also define and describe $\mathbf{D}^+(\mathcal{C})$, $\mathbf{D}^-(\mathcal{C})$ and $\mathbf{D}^b(\mathcal{C})$ along the same lines.

Moreover, there exists a canonical embedding $\mathbf{D}^+(\mathcal{C}) \hookrightarrow \mathbf{D}(\mathcal{C})$ whose essential image is made of complexes such that $\mathrm{H}^n(K^\bullet) = 0$ for $n << 0$ (and analobous statements with $-$ and b).

Note also that the functor $\mathrm{H}^0 : \mathbf{D}(\mathcal{C}) \to \mathcal{C}$ is a retraction of the embedding $\mathcal{C} \hookrightarrow \mathbf{D}(\mathcal{C}), M \mapsto [M]$ (and same with $+$, $-$ and b).

Theorem

Assume that $\mathcal{C}$ is an abelian category with enough injectives and let $\mathcal{I}$ denote the full (additive) subcategory of all injective objects. Then there exists a natural equivalence

$$\mathbf{K}^+(\mathcal{I}) \simeq \mathbf{D}^+(\mathcal{C}) \quad (\textit{and dual}).$$

Of course, this map the composition of inclusion $\mathbf{K}^+(\mathcal{I}) \hookrightarrow \mathbf{K}^+(\mathcal{C})$ and localization $\mathbf{K}^+(\mathcal{C}) \to \mathbf{D}^+(\mathcal{C})$.

If $\mathcal{C}$ is a additive category, then any additive functor $F : \mathcal{C} \to \mathcal{C}'$ extends naturally to an additive functor

$$F : \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C}').$$

And it will automatically preserve homotopies and induce a functor

$$F : \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}').$$

Assume now that $\mathcal{C}$ and $\mathcal{C}'$ are abelian. Then, if F is exact, it will induce a functor

$$F : \mathbf{D}(\mathcal{C}) \to \mathbf{D}(\mathcal{C}'),$$

but this is not the case in general.

Of course, boundedness (left, right or both sides) is preserved under these constructions.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories with $\mathcal{C}$ having enough injectives. Then the right derived functor of F is the unique (up to isomorphism) functor making commutative the following diagram:

$$\begin{tikzcd}
\mathbf{D}^+(\mathcal{C}) \arrow[r, "\mathrm{R}F"] & \mathbf{D}^+(\mathcal{C}') \\
\mathbf{K}^+(\mathcal{I}) \arrow[u, "\simeq"] \arrow[d, hook] & \\
\mathbf{K}^+(\mathcal{C}) \arrow[r, "F"] & \mathbf{K}^+(\mathcal{C}') . \arrow[uu]
\end{tikzcd}$$

And we will set for all $n \in \mathbb{Z}$,

$$\mathrm{R}^n F K^\bullet := \mathrm{H}^n(\mathrm{R}FK^\bullet).$$

If $M \in \mathcal{C}$, we may consider $\mathrm{R}FM := \mathrm{R}F[M] \in \mathbf{D}^+(\mathcal{C}')$, and for all $n \in \mathbb{Z}$, $\mathrm{R}^n FM \in \mathcal{C}'$. Since F is left exact, we have $\mathrm{R}^0 FM = FM$.

One defines dually $\mathrm{L}F$ for F right exact and $\mathcal{C}$ having enough projectives.

Example

Let M be an object of $\mathcal{C}$, an abelian category with enough injectives. Then

$$\mathrm{Ext}^n(M, N) := \mathrm{R}^n\mathrm{Hom}(M, N).$$

We have $\mathrm{Ext}^n(M, N) = 0$ for $n < 0$, $\mathrm{Ext}^0(M, N) = \mathrm{Hom}(M, N)$, and $\mathrm{Ext}^1(M, N)$ classifies all extensions of M by N.

Theorem

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor between abelian categories. Assume that $\mathcal{C}$ has enough injectives. Then, any short exact sequence

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

gives rise to a long exact sequence

$$\cdots \to R^nFK^\bullet \to R^nFL^\bullet \to R^nFM^\bullet \to R^{n+1}FK^\bullet \to \cdots$$

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor between abelian categories with $\mathcal{C}$ having enough injectives. Then, an object $M \in \mathcal{C}$ is said to be (right) acyclic with respect to F if $FM = RFM$ (i.e. $\mathrm{R}^n FM = 0$ for $n \neq$ ă0).

If $K^\bullet$ is a right resolution of an object $M \in \mathcal{C}$ where each K^n is acyclic with respect to F, then we have $RFM = FK^\bullet$. As a consequence

$$\forall n \in \mathbb{N}, \quad R^n FM = H^n(FK^\bullet).$$

Examples

1. An object is injective if and only if it is acyclic with respect to any left exact functor F if and only if it is acyclic with respect to Hom.
2. An abelian sheaf on a topological space X is said to be acyclic if it is acyclic with respect to $\Gamma(X, -)$.
3. The sheaf $\widetilde{C}_X^n$ on a topological space is acyclic (flabby sheaf).
4. The sheaf of $\mathcal{C}^\infty$-functions on a differential manifold X is acyclic (soft sheaf).
5. A scheme X is affine if and only if coherent sheaves are acyclic.

If $\mathcal{C}$ is a site, we already met the global section functor on $X \in \mathcal{C}$:

$$\begin{aligned} \mathbf{Ab}(\widetilde{\mathcal{C}}) &\longrightarrow \mathbf{Ab} \\ \mathcal{F} &\longmapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X). \end{aligned}$$

If $\mathcal{F}^\bullet$ is a complex of abelian groups on $\mathcal{C}$, we write

$$\mathrm{H}^n(X, \mathcal{F}^\bullet) := \mathrm{R}^n\Gamma(X, \mathcal{F}^\bullet).$$

When E is a usual abelian group, we will just write

$$\mathrm{H}^n(X, E) := \mathrm{H}^n(X, E_{\mathcal{C}}).$$

More generally, if $X \in \mathcal{C}_{/T}$ and $p : \hat{X} \to T$ denotes the structural morphism, we set for $\mathcal{F}$ on $\mathcal{C}_{/T}$,

$$\mathrm{H}^n(X/T, \mathcal{F}^\bullet) := \mathrm{R}^n p_* \mathcal{F}^\bullet.$$

(I said "more generally" because the previous case corresponds to T being the final object of $\widehat{\mathcal{C}}$).

Examples

1. If X is a locally contractible topological space, then
$$\mathrm{H}^n(X, \mathbb{Z}) \simeq \mathrm{H}^n(X, \tilde{C}_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X).$$
2. If X is a differential manifold, then
$$\mathrm{H}^n(X, \mathbb{R}) \simeq \mathrm{H}^n(X, \Omega^\bullet_X) \simeq \mathrm{H}^n_{\mathrm{dR}}(X).$$
3. If X is a smooth analytic variety, then, we have
$$\mathrm{H}^n(X, \mathbb{C}) \simeq \mathrm{H}^n(X, \Omega^\bullet_X).$$
4. If X is a smooth algebraic variety over $\mathbb{C}$, we have (GAGA)
$$\mathrm{H}^n(X, \Omega^\bullet_X) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}).$$
5. If X is a smooth scheme over a $\mathbb{Q}$-scheme S, we have
$$\mathrm{H}^n(X, \Omega^\bullet_{X/S}) \simeq \mathrm{Ext}^n_{\mathcal{D}_{X/S}}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{H}^n(X \to S, \mathcal{O}_{\mathrm{inf}}).$$

Examples

1. When **Sch** is endowed with the Zariski, étale or flat topology, one writes $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathcal{F}^\bullet)$, $\mathrm{H}^n_{\mathrm{et}}(X, \mathcal{F}^\bullet)$ or $\mathrm{H}^n_{\mathrm{fl}}(X, \mathcal{F}^\bullet)$ to make the difference.
2. If X is an algebraic variety over k and $\ell \nmid$ ăcar(k), then

$$\mathrm{H}^n_\ell(X) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_m \mathrm{H}^n_{\mathrm{et}}(k^{\mathrm{sep}} \otimes_k X, \mathbb{Z}/\ell^m)$$

3. If X is an algebraic variety over $\mathbb{C}$, we have

$$\mathrm{H}^n_{\mathrm{et}}(X, \mathbb{Z}/\ell^m) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \mathbb{Z}/\ell^m) \quad \text{and} \quad \mathrm{H}^n_\ell(X) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \mathbb{Q}_\ell).$$

More generally, all these isomorphisms in cohomology extend to coefficients. The main tool in order to prove these results is:

Theorem

1. *We have $R(G \circ F) = RG \circ RF$ as long as F-acyclic is sent to G-acyclic.*
2. *If a functor G has an exact adjoint, then it sends injective to injective.*

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