Absolute (prismatic) Calculus

Bernard Le Stum
(joint work with Michel Gros and Adolfo Quirós)

Université de Rennes

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CONTENT

Let us recall the following beautiful theorems:

Theorem (Bhatt, Schloze, 2023)

Let K be a complete discrete valuation field of mixed characteristic with perfect residue field k and $G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$ its absolute Galois group. Then, there exists an equivalence

$$\{\textit{prismatic } F\textit{-crystals on } \mathcal{O}_K\} \simeq \{\textit{lattices in crystalline representations of } G_K\}.$$

Theorem (Berger, Colmez, Fontaine, Wach , 2004)

If K is unramified, then there exists an equivalence

$$\{\textit{Wach modules over } \mathcal{O}_K\} \simeq \{\textit{lattices in crystalline representations of } G_K\}.$$

As a consequence, we also have

$$\{\text{prismatic } F\text{-crystals on } \mathcal{O}_K\} \simeq \{\text{Wach modules over } \mathcal{O}_K\}.$$

Our main theorem

There exists a very recent direct construction of this last equivalence by Abhinandan. One can actually prove a version without Frobenius:

Theorem (Gros, L-S, Quirós)

Let W be the ring of Witt vectors of a perfect field k of (odd) characteristic $p > 0$, ζ a primitive pth root of unity and $\mathcal{O}_K := W[\zeta]$. Then there exists an equivalence

$$\{prismatic\ crystals\ on\ \mathcal{O}_K\} \simeq \{weakly\ nilpotent\ absolute\ \nabla\text{-}modules\ over\ \mathcal{O}_K\}.$$

1. Moving between $W[\zeta]$ and W involves only an action of $\mathbb{F}_p^\times$.
2. There exists an equivalence (with finiteness conditions)

$$\{\text{Wach modules over } \mathcal{O}_K\} \simeq \{\text{absolute } F\text{-}\nabla\text{-modules over } \mathcal{O}_K\}.$$

3. Takeshi Tsuji is able to mix absolute and relative connections and extend our theorem to this setting.

Geometric analog

Our main theorem has the following geometric analog:

$$\{\text{crystals on } X/\mathcal{O}_K\} \simeq \{\text{top. quasi-nilp. int. } \nabla\text{-modules on } \mathcal{X}/\mathcal{O}_K\}$$

for X a smooth variety over k and $\mathcal{X}$ a smooth formal lifting over $\mathcal{O}_K$. More classically, if V is a complex manifold, then we have equivalences (analog to Bhatt-Schloze and Berger theorems)

$$\{\text{infinitesimal crystals on } V\} \simeq \{\text{representation of } \pi_1(V, x))\},$$
$$\{\text{integrable } \nabla\text{-modules on } V\} \simeq \{\text{representation of } \pi_1(V, x))\}$$

and in particular

$$\{\text{infinitesimal crystals on } V\} \simeq \{\text{integrable } \nabla\text{-modules on } V\}.$$

This last equivalence (which is the "classical" analog of our theorem) also holds for smooth algebraic or rigid analytic varieties in characteristic zero. We will briefly review now how this is proved since we shall then follow exactly the same strategy which splits the equivalence in two.

Infinitesimal crystals

We consider (algebraic, complex analytic, rigid analytic...) varieties over a field k.

Definition

A *thickening* $U \hookrightarrow T$ is a nilpotent immersion.

Example

If $\mathcal{I}$ denotes the ideal of the diagonal $V \hookrightarrow V \times V$, then $\mathcal{I}^{n+1}$ defines a subvariety $V^{(n)} \subset V \times V$ and $V \hookrightarrow V^{(n)}$ is a thickening.

Definition

The *infinitesimal site* of a variety V is the set of all thickenings $U \hookrightarrow T$ where U is an open subset of V. An *infinitesimal module* on V is a module on this site. Equivalently, this is a compatible family E of $\mathcal{O}_T$-modules E_T on T (for all thickenings $U \hookrightarrow T$). It is a *crystal* if all transition maps $u^*E_T \to E_{T'}$ are bijective.

Stratifications

The projections induce various maps $p_1, p_2 : V^{(n)} \hookrightarrow V \times V \to V$ as well as $p_{12}, p_{13}, p_{23} : V^{(n)} \times'_V V^{(m)} \to V^{(n+m)}$ (coming from $V \times V \times V \to V \times V$).

Definition

A *stratification* on an $\mathcal{O}_V$-module $\mathcal{E}$ is a compatible family of isomorphisms $\epsilon_n : p_2^*\mathcal{E} \simeq p_1^*\mathcal{E}$ on $V^{(n)}$ satisfying $p_{13}^*(\epsilon_{n+m}) = p_{12}^*(\epsilon_{n+m}) \circ p_{23}^*(\epsilon_{n+m})$.

Proposition

If V is smooth, then there exists an equivalence of categories

$$\{\textit{infinitesimal crystals on } V\} \simeq \{\textit{stratified modules on } V\}$$

Proof.

If E is an infinitesimal crystal, we set $\mathcal{E} = E_V$ and $\epsilon_n : p_2^*E_V \simeq E_{V^{(n)}} \simeq p_1^*E_V$. For the converse, since V is smooth, any thickening $U \hookrightarrow T$ has locally a section $s : T \to U$ and we set $E_T := s^*\mathcal{E}_{|U}$. □

Connections

Definition

A *derivation* $D : \mathcal{O}_V \to \mathcal{E}$ is a linear map satisfying $D(fg) = D(f)g + fD(g)$.

Example

The map $d := p_2^* - p_1^* : \mathcal{O}_V \to \mathcal{I}/\mathcal{I}^2 =: \Omega_V$ is a universal derivation.

Definition

A ∇-*module* on V is an $\mathcal{O}_V$-module $\mathcal{E}$ endowed with a *connection*: a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega_V$ satisfying $\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$. It is said to be *integrable* if $\nabla^2 = 0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Lambda^2\Omega_V$.

Proposition

In characteristic zero, there exists an isomorphism of categories

$$\{\textit{stratified modules on } V\} \simeq \{\textit{integrable } \nabla\textit{-modules on } V\}$$

Proof.

One simply sets $\nabla(s) = \epsilon(1 \otimes s) - s \otimes 1 \in \mathcal{E} \otimes_{\mathcal{O}_V} \Omega_V \subset p_1^*\mathcal{E}$. Conversely, this formula provides ϵ on $V^{(2)}$. It can then be extended to all $V^{(n)}$'s. □

Example

Assume V is one dimensional with coordinate x so that $\Omega_V = \mathcal{O}_V \mathrm{d}x$ and $\mathrm{d}f = \frac{\partial}{\partial x}(f)\mathrm{d}x$. Then, $\nabla(s) = \partial(s) \otimes \mathrm{d}x$ for some $\partial : \mathcal{E} \to \mathcal{E}$ such that

$$\partial(fs) = \frac{\partial}{\partial x}(f)s + f\partial(s).$$

If we set $\xi = p_2^*(x) - p_1^*(x) \in \mathcal{O}_{V \times V}$, then

$$\mathcal{O}_{V^{(n)}} \simeq \mathcal{O}_V[\xi]/\xi^{n+1}$$

and

$$\epsilon(f \otimes s) = \sum_{k=0}^{n} \frac{1}{k!} f\partial^k(s)\xi^k \mod \xi^{n+1}$$

is the *Taylor expansion*.

Recall q-analogs

The *q-analog* of $n \in \mathbb{N}$ is

$$(n)_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}.$$

One may then consider the q-analogs of factorials and binomial coeffcients

$$(n)_q! := (n)_q(n-1)_q(n-2)_q \cdots \quad \text{and} \quad \binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}.$$

Example

Classically, the q-derivation with respect to x is defined as

$$\partial_q(f)(x) := \frac{f(qx) - f(x)}{qx - x} \quad \text{so that } \frac{\partial}{\partial x}(f) = \lim_{q \to 1} \partial_q(f).$$

We have $\partial_q(x^n) = (n)_q x^{n-1}$ and $\partial_q^k(x^n) = (k)_q! \binom{n}{k}_q x^{n-k}$.

Absolute derivations

From now on, we let W be the ring of Witt vectors of a perfect field k of (odd) characteristic $p > 0$ and ζ a primitive pth root of unity. We write $R = W[[q-1]]$ so that $\mathcal{O}_K := W[\zeta] \simeq R/(p)_q R$ ($(R, (p)_q)$ is a "prism" that exhibits $\mathcal{O}_K$). We consider the continuous endomorphism γ of R given by $\gamma(q) = q^{p+1}$.

Definition

If M is an R-module, then a *q^p-derivation* (or a γ-derivation) $D : R \to M$ is a W-linear map satisfying $D(\alpha\beta) = D(\alpha)\beta + \gamma(\alpha)D(\beta)$.

Example

With $M = R$, we may consider 1) $\partial_{q^p}(q^k) = (k)_{q^p} q^{k-1}$ but also 2) $\partial_{\mathbb{\Delta}}(q^k) = (pk)_q q^{k-1}$ (actually $\partial_{\mathbb{\Delta}} = (p)_q \partial_{q^p}$).

Lemma

There exists a universal q^p-derivation $d_{q^p} : R \to \Omega_{q^p} = R\mathrm{d}_{q^p}(q)$ and we have $d_{q^p}(\alpha) = \partial_{q^p}(\alpha) d_{q^p}(q)$. $\square$

Absolute prismatic connections

Definition

An *absolute* $\mathbb{\Delta}$*-connection* on an R-module M is a W-linear map $\nabla : M \to M \otimes_R \Omega_{q^p}$ satisfying

$$\nabla(\alpha s) = (p)_q \, s \otimes \mathrm{d}_{q^p}(\alpha) + \gamma(\alpha)\nabla(s).$$

It is equivalent to give an endomorphism $\partial_{\mathbb{\Delta}}$ of M such that

$$\partial_{\mathbb{\Delta}}(qs) = (p)_q s + q^{p+1}\partial_{\mathbb{\Delta}}(s)$$

via $\nabla(s) = \partial_{\mathbb{\Delta}}(s) \otimes \mathrm{d}_{q^p} q$. This endomorphism provides a γ-linear endomorphism γ_M of M given by $\gamma_M(s) = s + (q^2 - q)\partial_{\mathbb{\Delta}}(s)$.

Example

1. (trivial) $M = R$, $\partial_{\mathbb{\Delta}}$ as above and then $\gamma_M = \gamma$.
2. (Hodge-Tate) $M = \mathcal{O}_K s$, $\partial_{\mathbb{\Delta}}(s) = \frac{p}{\zeta^2 - \zeta} s$ and $\gamma_M(s) = (p+1)s$.
3. (Breuil-Kisin) $M = Re$, $\partial_{\mathbb{\Delta}}(e) = \frac{1}{q^2 - q}\left(\frac{p+1}{(p+1)_q} - 1\right) e$ and $\gamma_M(e) = \frac{p+1}{(p+1)_q} e$.

Twisted divided powers

Lemma

The multiplication rule

$$\omega^{\{n\}}\omega^{\{m\}} = \sum_{0 \leq i \leq \min\{n,m\}} q^{\frac{pi(i-1)}{2}} \binom{n+m-i}{n}_{q^p} \binom{n}{i}_{q^p} (q^2-q)^i \omega^{\{n+m-i\}}$$

defines a commutative ring structure on $R\langle\omega\rangle_{\mathbb{\Delta}} := \bigoplus_{n\in\mathbb{N}} R\omega^{\{n\}}$.

Proof.

Formally follows from the heuristic (twisted divided power)

$$\omega^{\{n\}} := \frac{1}{(n)_{q^p}!} \prod_{k=0}^{n-1} \left(\omega - (k)_{q^p}(q^2-q)\right). \qquad \square$$

We shall denote by $P := \widehat{R\langle\omega\rangle}_{\mathbb{\Delta}}$ the completion for the $(p, q-1)$-adic topology.

Formal groupoid

We shall need the following constant:

$$L(\omega) := 1 + \left(\sum_{k=1}^{p-1} \frac{1}{(k)_{q^p}} \frac{\partial^k}{\partial q^k}((p)_q)(p)_q^{k-1}\omega^{k-1}\right)\omega \in P.$$

Lemma

There exists (each time) a unique continuous map of W-algebras:

1. *(canonical)* $\iota : R \to P, q \mapsto q$,
2. *(Taylor)* $\theta : R \to P, q \mapsto q + (p)_q\omega$,
3. *(augmentation)* $e : P \to R, q \mapsto q, \omega \to 0$,
4. *(comultiplication)* $\Delta : P \to P \otimes'_R P, q \to q \otimes 1, \omega \mapsto L(\omega) \otimes 1 - 1 \otimes \omega$,
5. *(flip)* $\tau : P \to P, q \mapsto \theta(q), \omega \mapsto -L(\omega)^{-1}\omega$.

Proof.

One can build the first three by hand. However, for the last two, it seems necessary to rely on prismatic theory. Note that $\theta((p)_q) = (p)_q L(\omega)$. □

Parenthesis

Replacing R with V and P with $V^{(n)}$, this is dual-analogous to

1. canonical: $p_1 : V^{(n)} \to V, (x, y) \mapsto x$,
2. Taylor: $p_2 : V^{(n)} \to V, (x, y) \mapsto y$,
3. augmentation: $V \hookrightarrow V^{(n)}, x \mapsto (x, x)$,
4. comultiplication: $p_{13} : V^{(n)} \times' V^{(m)} \to V^{(n+m)}, (x, y, z) \mapsto (x, z)$,
5. flip: $V^{(n)} \to V^{(n)}, (x, y) \mapsto (y, x)$.

This is actually (dual to) a groupoid structure, with V (or R) as objects and $V^{(n)}$ (or P) as arrows (a groupoid is a category where all morphisms are isomorphisms):

1. canonical: target (morphism sent to object),
2. Taylor: domain (morphism sent to object),
3. augmentation: identity (object sent to morphism),
4. comultiplication: composition (pair of morphisms sent to morphism),
5. flip: inverse (morphism sent to morphism).

Hyperstratifications

Definition

An *absolute* $\mathbb{A}$*-hyperstratification* on an R-module M is a P-linear isomorphism

$$\epsilon : P\widehat{\otimes}'_R M \simeq M\widehat{\otimes}_R P$$

satisfying $(\epsilon \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \epsilon) \circ (\Delta \otimes \mathrm{Id}) = (\mathrm{Id} \otimes \Delta) \circ \epsilon$.

Definition

An absolute $\mathbb{A}$-connection is *weakly nilpotent* if $\partial_{\mathbb{A}}$ is nilpotent modulo $(p, q-1)$.

Theorem (Gros, L-S, Quirós)

On finite projective R–modules, there exists an isomorphism of categories

$$\{absolute\ \mathbb{A}\text{-}hyperstratifications\} \simeq \{weakly\ nilpotent\ absolute\ \mathbb{A}\text{-}connections\}$$

Proof.

One essentially sets $\nabla(s) = \epsilon(1 \otimes s) - s \otimes 1$. But some work is needed here. □

Prisms

Definition

A δ-*structure* on a (commutative) ring B is a (ring) section $f \mapsto (f, \delta(f))$ of the projection $W_1(B) \twoheadrightarrow B$ (where W_1 denotes the Witt vectors of length 2). The corresponding *frobenius* is then defined as $\phi(f) = f^p + p\delta(f)$. A *prism* is a couple (B, J) where B is a (p, J)-adically complete *bounded* (B/J has bounded p^∞-torsion) δ-ring and $J \subset B$ is an invertible ideal such that $p \in J + \phi(J)$.

Example

1) $(\mathbb{Z}_p, (p))$, 2) $(W[[q-1]], ((p)_q))$, 3) $(A_{\mathrm{inf}}, \ker\theta)$ with Fontaine's ring $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ (that rules them all) and Fontaine's $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C, [x] \mapsto x^\sharp$.

Definition

The *(absolute) prismatic site* of $\mathcal{O}_K$ is the set of all prisms (B, J) endowed with a morphism $\mathcal{O}_K \to B/J$. A *prismatic module* is a module E on this site (for the flat topology). This provides a compatible family E of B-modules E_B. The module E is a *crystal* if all transition maps $B' \otimes_B E_B \to E_{B'}$ are bijective.

Main theorem

Theorem (Gros, L-S, Quirós)

There exists an equivalence (finite locally free and finite free)

$$\{\textit{prismatic crystals on } \mathcal{O}_K\} \simeq \{\textit{weakly nilpotent } \mathbb{\Delta}\textit{-connections on } R\}.$$

Proof.

If E is a prismatic crystal, we set $M := E_R$ and $\epsilon : P \otimes'_R E_R \simeq E_P \simeq E_R \otimes_R P$.
The converse is based on a theory of linearization of differential operators. □

Example

Recall that Breuil-Kisin absolute connection on Re is given by

$$\partial_{\mathbb{\Delta}}(e) = \frac{1}{q^2 - q}\left(\frac{p+1}{(p+1)_q} - 1\right)e.$$

It has a frobenius given by $\phi(e) = \frac{1}{(p)_q}e$. The corresponding representation is the Tate module $\mathbb{Z}_p(1)$ corresponding to the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$.

Do it yourself (Fontaine)

In general, the correspondence is explicitly given by

$$M \mapsto V := (W(C^\flat) \otimes_R M)^{\phi=1}.$$

Here, C denotes a complete algebraic closure of K, $C^\flat$ its tilt (towers of p-th roots), and $W(C^\flat)$ denotes the Witt vectors of $C^\flat$. The map $R \to W(C^\flat)$ sends q to the Teichmüller lifting of $\zeta^\flat \in C^\flat$ where $\zeta^\flat$ is a tower of p-th roots of ζ. Finally, G_K acts on M as γ_M via the quotient $G_K \twoheadrightarrow \Gamma_K := \mathrm{Gal}(K(\mu_{p^\infty})/K)$ through the progenerator γ of Γ_K corresponding to $1+p$ via the cyclotomic isomorphism $\chi : \Gamma_K \simeq 1 + p\mathbb{Z}_p$.

– Thank you –

Post-scriptum: Needless to say, all the above equivalences are compatible with cohomology.

[Ber04] Laurent Berger. "Limits of crystalline representations". French. In: *Compos. Math.* 140.6 (2004).

[BS23] Bhargav Bhatt and Peter Scholze. "Prismatic F-crystals and crystalline Galois representations". In: *Camb. J. Math.* 11.2 (2023).

[Fon90] Jean-Marc Fontaine. *p-adic representations of local fields. I*. French. The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 249-309 (1990). 1990.

[GSQ23] Michel Gros, Bernard Le Stum, and Adolfo Quirós. *Absolute calculus and prismatic crystals on cyclotomic rings*. 2023. arXiv: 2310.13790.