$\mathbb{M}$-modules

Bernard Le Stum

Université de Rennes

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Introduction

Recall the Weyl algebra ($k, n \in \mathbb{N}$)

$$\mathcal{D} = \left\{ \sum_{\text{finite}} a_{kn} x^k \partial^n, a_{kn} \in \mathbb{C} \right\} \text{ with } \partial x = x\partial + 1.$$

(1) There exists a sequence of inclusions

$$\mathcal{O} = \left\{ \sum_{\text{finite}} c_n x^n, c_n \in \mathbb{C} \right\} \subset \mathcal{D} = \left\{ \sum_{\text{finite}} f_n \partial^n, f_n \in \mathcal{O} \right\}$$

$$\subset \widehat{\mathcal{D}} = \left\{ \sum_{\text{infinite}} f_n \partial^n, f_n \in \mathcal{O} \right\} \simeq \mathrm{End}_{\mathbb{C}}(\mathcal{O}).$$

(2)There exists a presentation $0 \to \mathcal{D} \xrightarrow{\cdot\partial} \mathcal{D} \to \mathcal{O} \to 0$.

(3) Left $\mathcal{D}$-modules form a (Grothendieck) closed symmetric monoidal abelian category with unit $\mathcal{O}$.

(4) The category of fundamental representations of an open subset embeds into that of $\mathcal{D}$-modules.

One can introduce in the same way the ring of matrices ($k, n \in \mathbb{N}$)

$$\mathbb{M} = \left\{ \sum_{\text{infinite in } t} c_{kn} e_k t^n, c_{kn} \in \mathbb{Z} \right\}, \text{ with } te_{n+1} = e_n \text{ et } e_k e_n = \begin{cases} e_k \text{ if } n = 0 \\ 0 \text{ otherwise.} \end{cases}$$

(1) One can show that

$$\mathbb{Z}_\bullet := \left\{ \sum_{\text{finite}} a_n e_n, a_n \in \mathbb{Z} \right\} \subset \mathbb{M}^\vee := \left\{ \sum_{\text{finite}} a_n t^n, a_n \in \mathbb{Z}_\bullet \right\}$$

$$\subset \mathbb{M} = \left\{ \sum_{\text{infinite}} a_n t^n, a_n \in \mathbb{Z}_\bullet \right\} \simeq \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_\bullet).$$

(2) There exists a presentation $0 \to \mathbb{M} \xrightarrow{\cdot t} \mathbb{M} \to \mathbb{Z}_\bullet \to 0$.

(3) $\mathbb{M}$-modules form a (Grothendieck) closed symmetric monoidal abelian category with unit $\mathbb{Z}_\bullet$.

(4) The category of complete metrizable linearly topologized abelian groups embeds into that of $\mathbb{M}$-modules.

The ring $\mathbb{M}$ of Matrices

Definition

The ring $\mathbb{M}$ of matrices and the module $\mathbb{Z}^{\bullet}$ of vectors are repectively the set of

1. $\mathbb{N} \times \mathbb{N}$-matrices with finite columns and coefficients in $\mathbb{Z}$ (endowed with usual multiplication),
2. finite columns of integers (endowed with usual action):

$$\mathbb{M} = \left\{ \begin{bmatrix} * & * & \cdots \\ \vdots & \vdots & \\ * & \vdots & \\ 0 & * & \\ \vdots & 0 & \\ \vdots & \vdots & \end{bmatrix} \right\} \quad \text{and} \quad \mathbb{Z}^{\bullet} := \left\{ \begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{bmatrix} \right\}.$$

If we identify $\mathbb{Z}^{\bullet}$ with the first column of $\mathbb{M}$, it becomes a projective principal (left) ideal and $\mathbb{M} \simeq \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{\bullet}) \simeq \mathbb{Z}^{\mathbb{N}}_{\bullet}$ (as $\mathbb{M}$-modules).

If we set

$$e_n := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad t := \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & & \ddots & \ddots \end{bmatrix},$$

we then recover the description of $\mathbb{M}$ from the introduction. Actually, there exists a split exact sequence

$$0 \longrightarrow \mathbb{M} \xrightarrow{\cdot t^n} \mathbb{M} \longrightarrow \mathbb{Z}_\bullet^n \longrightarrow 0.$$

It provides an isomorphism $\mathbb{M} \simeq \varprojlim \mathbb{M}/\mathbb{M}t^n$ showing that $\mathbb{M}$ is complete for the t-adic topology. For example, $1 = \sum_{n=0}^{\infty} e_n t^n$.

We shall denote by $\mathbb{M}^{\otimes 2}$ the set of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$-matrices with integer coefficients and finite columns (first index is finite). We have

$$\mathbb{M}^{\otimes 2} \simeq \mathbb{Z}_\bullet^{\mathbb{N}\times\mathbb{N}} \simeq (\mathbb{Z}_\bullet^{\mathbb{N}})^{\mathbb{N}} \simeq \mathbb{M}^{\mathbb{N}}$$

in two different ways. This is therefore a (left) $\mathbb{M}$-module with two equivalent structures of right $\mathbb{M}$-modules.

Definition

The internal tensor product of two (left) $\mathbb{M}$-modules E and F is the (left) $\mathbb{M}$-module

$$E \otimes_{\mathbb{Z}_\bullet} F := (\mathbb{M}^{\otimes 2} \otimes_{\mathbb{M}} E) \otimes_{\mathbb{M}} F$$

obtained by using successively both right $\mathbb{M}$-module structures of $\mathbb{M}^{\otimes 2}$.

Proposition

Internal tensor product endows $\mathbb{M}$-modules with a closed symmetric monoidal structure with unit $\mathbb{Z}_\bullet$.

In particular, there exists a bifunctor $\mathcal{H}\mathrm{om}_{\mathbb{Z}_\bullet}$ and a natural isomorphism

$$\mathcal{H}\mathrm{om}_{\mathbb{Z}_\bullet}(E \otimes_{\mathbb{Z}_\bullet} F, G) \simeq \mathcal{H}\mathrm{om}_{\mathbb{Z}_\bullet}(E, \mathcal{H}\mathrm{om}_{\mathbb{Z}_\bullet}(F, G)).$$

We shall denote by $E^\vee := \mathcal{H}\mathrm{om}_{\mathbb{Z}_\bullet}(E, \mathbb{Z}_\bullet)$ the dual of E.

Proposition

$\mathbb{M}^\vee$ is (isomorphic to) the projective twosided ideal of finite matrices and $\mathbb{M}^{\vee\vee} \simeq \mathbb{M}$ – so that $\mathbb{M}$ is reflexive.

Proposition

Finitely presented $\mathbb{M}$-modules are stable under finite colimit, countable limit, extension and internal tensor product. Any $\mathbb{M}$-module is a filtered colimit of finitely presented $\mathbb{M}$-modules.

Proposition

For an $\mathbb{M}$-module E, the following conditions are equivalent:

1. *E is finitely presented (resp. finitely generated projective),*
2. *$E \simeq \mathbb{M}/\mathbb{M}u$ (resp. $\mathbb{M}/\mathbb{M}e$ with $e^2 = e$),*
3. *there exists an exact sequence $\mathbb{Z}_\bullet^I \to \mathbb{Z}_\bullet^J \to E \to 0$ with countable I, J (resp. an isomorphism $\mathbb{Z}_\bullet^I \simeq E$ with countable I).*

Example

1. $\mathbb{M}$, $\mathbb{Z}_\bullet$ and $\mathbb{M}^{\otimes 2}$ are finitely generated projective,
2. $\mathbb{M}^\vee$ is projective but not finitely generated,
3. $\mathbb{Z}_{p\bullet} := \mathbb{M}/\mathbb{M}(t-p)$ is finitely presented but not projective (p a prime),
4. $\mathbb{Z}[1/n]_\bullet := \mathbb{M}^\vee/\mathbb{M}^\vee(1-nt)$ and $\mathbb{Q}_{p\bullet} := \varinjlim_{p\cdot} \mathbb{Z}_{p\bullet}$ are neither.

Using an argument of Clausen and Scholze, one shows:

Lemma

If E is a finitely presented $\mathbb{M}$-module, then

$$E \simeq \mathbb{Z}_\bullet^D \oplus \mathcal{E}\mathrm{xt}_{\mathbb{Z}_\bullet}(\mathbb{Z}_\bullet \otimes_{\mathbb{Z}} T, \mathbb{Z}_\bullet)$$

where T is a countable abelian group with $T^\vee := \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0$.

Theorem

Any finitely generated ideal of $\mathbb{M}$ is principal projective.

Proof.

Equivalently: any finitely presented $\mathbb{M}$-module admits a length 2 projective resolution. With the notations of the lemma, if $0 \to \mathbb{Z} \cdot J \to \mathbb{Z} \cdot I \to T \to 0$ is exact, the same holds for $0 \to \mathbb{Z}_\bullet^I \to \mathbb{Z}_\bullet^J \to \mathcal{E}\mathrm{xt}_{\mathbb{Z}_\bullet}(\mathbb{Z}_\bullet \otimes_{\mathbb{Z}} T, \mathbb{Z}_\bullet) \to 0$. □

Corollary

$\mathbb{M}$ is a (left) coherent ring.

Linearly topologized abelian groups

Definition

A topological abelian group V is said to be linearly topologized if its open subgroups V' form a basis of neighborhoods of zero and then complete if $V \simeq \varprojlim V/V''$.

Example

1. Any module over an adic ring (such as $\mathbb{Z}[[X]]$ or $\mathbb{Z}_p$),
2. Any finitely presented module over a Huber ring (such as $\mathbb{Q}_p$ or $\mathbb{Q}_p\{X\}$).

When V is a topological abelian group, we shall write $V_\bullet := \{(s_n)_{n\in\mathbb{N}}, s_n \to 0\}$.

Example

1. We fall back onto our former $\mathbb{Z}_\bullet$ when $\mathbb{Z}$ has the discrete topology,
2. We have $\mathbb{M} \simeq (\mathbb{Z}^{\mathbb{N}})_\bullet$ (product topology) and $\mathbb{M}^\vee \simeq (\mathbb{Z}\cdot\mathbb{N})_\bullet$ (discrete topology),
3. $\mathbb{Z}_{p\bullet} \simeq \mathbb{M}/\mathbb{M}(t-p)$, $\mathbb{Z}[1/n]_\bullet \simeq \mathbb{M}^\vee/\mathbb{M}^\vee(1-nt)$ and $\mathbb{Q}_{p\bullet} \simeq \varinjlim_{p\cdot} \mathbb{Z}_{p\bullet}$.

Proposition

If V is a complete linearly topologized abelian group, then $V_\bullet$ is canonically an $\mathbb{M}$-module and the functor $V \mapsto V_\bullet$

1. *admits an adjoint $E \to \widehat{E(*)}$ (with $E(*) := E^{t=0}$),*
2. *is monoïdal: there exists a natural $\mathbb{M}$-linear map $V_\bullet \otimes_{\mathbb{Z}_\bullet} W_\bullet \to (V \widehat{\otimes}_{\mathbb{Z}} W)_\bullet$,*
3. *is fully faithful on metrizable abelian groups,*
4. *is strong monoidal on discrete abelian groups: $V_\bullet \otimes_{\mathbb{Z}_\bullet} W_\bullet \simeq (V \otimes_{\mathbb{Z}} W)_\bullet$, in which case it has a coadjoint $E \mapsto E(*)$ (automatically monoidal) that itself has a coadjoint.*

Point (3) follows from the next fundamental lemma. Recall first that there exists the notion of a linearly $\mathbb{M}$-topologized $\mathbb{M}$-module: basis of open neighborhoods of zero made of $\mathbb{M}$-submodules. Then,

Lemma

Any $\mathbb{M}$-linear map $E \to F$ between linearly $\mathbb{M}$-topologized $\mathbb{M}$-modules with E metrizable and F complete is automatically continuous.

It means that continuity is a built-in feature for $\mathbb{M}$-modules.

The surprise !

Theorem (Gabriel-Popescu)

If $\mathcal{A}$ is a bicomplete abelian category with exact filtered colimits and finitely presented projective generator G, then $\mathcal{A}$ is equivalent to the category of A-modules with $A = \mathrm{End}(G)$.

Theorem (Clausen-Scholze)

(Light) solid abelian groups form a bicomplete abelian category with exact filtered colimits and finitely presented projective generator $\underline{\mathbb{Z}}^{\mathbb{N}}$.

Corollary

The category of solid abelian groups is equivalent to the category of $\mathbb{M}$-modules. □

As a consequence, all the formal properties of $\mathbb{M}$-modules are also valid for solid abelian groups!

$\mathbb{M}$-rings (solid rings !)

Definition

An $\mathbb{M}$-ring A (resp. an A-module E) is an $\mathbb{M}$-module endowed with an associative unital morphism $A \otimes_{\mathbb{Z}_\bullet} A \to A$ (resp. $A \otimes_{\mathbb{Z}_\bullet} E \to E$).

Formally, A is monoid of the monoidal category of $\mathbb{M}$-modules and E is an A-object (this corresponds to solid rings and modules).

Proposition

1. *If R (resp. V) is a complete linearly topologized ring (resp. R-module), then $R_\bullet$ (resp. $V_\bullet$) is canonically an $\mathbb{M}$-ring (resp. $R_\bullet$-module),*
2. *restricted to metrizable rings (resp. modules), we obtain a fully faithful functor,*
3. *if R is discrete, then the category of $R_\bullet$-modules is equivalent to that of usual $R \otimes_{\mathbb{Z}} \mathbb{M}$-modules.*

Example

$\mathbb{Z}_\bullet$ is an $\mathbb{M}$-ring and a $\mathbb{Z}_\bullet$-module is the same thing as a usual $\mathbb{M}$-module.

Example

1. A $\mathbb{Z}[X]_\bullet$-module is a usual $\mathbb{M}[X]$-module (a usual $\mathbb{M}$-module endowed with an endomorphism),
2. a $\mathbb{Z}[[X]]_\bullet$-module is a usual $\mathbb{M}[[X]]$-module and (full embeddings !)

$$\mathbb{Z}[[X]]\text{-}\mathrm{Mod}^\wedge \subset \mathbb{Z}[[X]]_\bullet\text{-}\mathrm{Mod} \subset \mathbb{Z}[X]_\bullet\text{-}\mathrm{Mod} = \mathbb{M}[X]\text{-}\mathrm{Mod}$$

   ($\wedge$: complete metrizable linearly topologized + continuous linear maps),
3. a $\mathbb{Z}_{p\bullet}$-module is a usual $\mathbb{M}_p$-module (where $V_p := \varprojlim V/p^nV$) and (full embeddings !) $\mathbb{Z}_p\text{-}\mathrm{Mod}^\wedge \subset \mathbb{Z}_{p\bullet}\text{-}\mathrm{Mod} \subset \mathbb{Z}_\bullet\text{-}\mathrm{Mod} = \mathbb{M}\text{-}\mathrm{Mod}$,
4. a $\mathbb{Q}_\bullet$-module is a usual $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{M}$-module (do not confuse with the ring $\mathbb{M}_{\mathbb{Q}}$ of column-finite matrices with rational coefficients),
5. a $\mathbb{Q}_{p\bullet}$-module is a usual $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{M}_p$-module and (full embeddings !) $\mathbb{Q}_p\text{-}\mathrm{Ban} \subset \mathbb{Q}_{p\bullet}\text{-}\mathrm{Mod} \subset \mathbb{Z}_\bullet\text{-}\mathrm{Mod} = \mathbb{M}\text{-}\mathrm{Mod}$ ($\mathbb{Q}_p\text{-}\mathrm{Ban} = \{p\text{-adic Banach spaces}\}$),
6. a $\mathbb{Z}_p\{X\}_\bullet$-module is a usual $\mathbb{M}_p\{X\}$-module (we denote by $A\{X\} := \varprojlim_V A[X]/V[X]$, V open subgroup, the ring of convergent functions on the closed unit disc).

The previous results rely on the remarkable stability of internal tensor product:

Example

1. $\mathbb{Z}[[X]]_\bullet \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Z}[[Y]]_\bullet \simeq \mathbb{Z}[[X,Y]]_\bullet$,
2. $\mathbb{Z}_{p\bullet} \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Z}[[X]]_\bullet \simeq \mathbb{Z}_p[[X]]_\bullet$,
3. $\mathbb{Z}_{p\bullet} \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Z}_{p\bullet} \simeq \mathbb{Z}_{p\bullet}$ and $\mathbb{Z}_{p\bullet} \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Z}_{\ell\bullet} = 0$ when $\ell \neq p$,
4. $\mathbb{Q}_{p\bullet} \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Z}[[X]]_\bullet \simeq \mathbb{Q}_p[[X]]^{\mathrm{bd}}_\bullet$ and $\mathbb{Q}_{p\bullet} \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} \mathbb{Q}_{p\bullet} \simeq \mathbb{Q}_{p\bullet}$ (where $\mathbb{Q}_p[[X]]^{\mathrm{bd}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p[[X]]$ is the ring of bounded functions on the disc),
5. $E_\bullet \otimes^{\mathrm{L}}_{\mathbb{Z}_\bullet} F_\bullet \simeq (E \widehat{\otimes}_{\mathbb{Q}_p} F)_\bullet$ if E and F are two $\mathbb{Q}_p$-Banach spaces,
6. $\mathbb{Z}[[X]]_\bullet \otimes^{\mathrm{L}}_{\mathbb{Z}[X]_\bullet} \mathbb{Z}[[X]]_\bullet \simeq \mathbb{Z}[[X]]_\bullet$.

Theorem (Mann)

If R is finitely generated (over $\mathbb{Z}$), $I \subset R$ is an ideal and E, F are two (derived) I-complete $R_\bullet$-modules, then $E \otimes^{\mathrm{L}}_{R_\bullet} F$ also is (derived) I-complete.

– THANK YOU –

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