Université de Rennes 2025-2026

Homological algebra
Homework (due December 10th)

You can compose in english or in french. You can freely (no need to make a precise reference) use any result (exercises included) obtained prior to the statement of the exercise in the (online) course.

1. Show that a complex $[M \stackrel{\mathrm{Id}}{\to} M]$ is always contractible but the complex of abelian groups
$$\cdots \to 0 \to \mathbb{Z} \stackrel{2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$
is not.

2. Show that, in an abelian category, if
$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow u[1] \\ K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1] \end{array}$$
is a morphism of distinguished triangles and two among u, v and w are quasi-isomorphisms, then so is the third (and idem with a short exact sequences).

3. Assume that a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$. Show that (and dual)
   1. if F is exact then G preserves injectives,
   2. if F is faithful exact and $\mathcal{B}$ has enough injectives, then $\mathcal{A}$ too has enough injectives.

4. Show that $\mathrm{Ext}^1(M, N) \simeq \mathrm{Ext}(M, N)$ in an abelian category $\mathcal{A}$ with enough injectives or projectives.

5. Assume G is a group and $k = \mathbb{Z}$. Compute $\mathrm{H}^1(G, M)$ when
   1. the action of G on M is trivial,
   2. $M = \mathbb{Z}$ with the non-trivial action of $\mu_2 := \{1, -1\}$.

6. Show that
$$\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0, 1 \\ 0 \text{ otherwise} \end{cases}$$
with $d = m \wedge n$ (for $m, n \geq 1$).