Université de Rennes 2025-2026

Homological algebra
Homework (due December 10th)

You can compose in english or in french. You can freely (no need to make a precise reference) use any result (exercises included) obtained prior to the statement of the exercise in the (online) course.

1. Show that a complex $[M \stackrel{\mathrm{Id}}{\to} M]$ is always contractible but the complex of abelian groups
$$\cdots \to 0 \to \mathbb{Z} \stackrel{2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$
is not.

Solution: For the complex $[M \stackrel{\mathrm{Id}}{\to} M]$, we have by definition $d_0 = \mathrm{Id}_M$ and $d_n = 0$ otherwise. Moreover, if we denote by f the identity of this complex, then we have $f_0 = f_1 = \mathrm{Id}_M$ and $f_n = 0$ otherwise. Now, we set $s_1 := \mathrm{Id}_M$ and $s_n = 0$ otherwise. Since $f_0 = s_1 \circ d_0$ and $f_1 = d_0 \circ s_1$, we have
$$\forall n \in \mathbb{Z}, \quad f_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n.$$
It means that there exists a homotopy $\mathrm{Id}_M \simeq 0_M$ showing that M is trivial in the homotopy category or, equivalently, contractible.

Now, if the complex $\cdots \to 0 \to \mathbb{Z} \stackrel{2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$ were contractible, there would then exist a homotopy between the identily and the zero map. Since the only morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the zero map, there would exist a morphism $s : \mathbb{Z} \to \mathbb{Z}$ such that $\mathrm{Id}_{\mathbb{Z}} = d \circ s$ where d is multiplication by 2. This cannot happen.

2. Show that, in an abelian category, if
$$\begin{array}{ccccccc}
K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1]
\end{array}$$
is a morphism of distinguished triangles and two among u, v and w are quasi-isomorphisms, then so is the third (and idem with a short exact sequences).

Solution: In both cases, there exists a morphism of long exact sequences
$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathrm{H}^n(K^\bullet) & \longrightarrow & \mathrm{H}^n(L^\bullet) & \longrightarrow & \mathrm{H}^n(M^\bullet) & \longrightarrow & \mathrm{H}^{n+1}(K^\bullet) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{H}^n(K'^\bullet) & \longrightarrow & \mathrm{H}^n(L'^\bullet) & \longrightarrow & \mathrm{H}^n(M'^\bullet) & \longrightarrow & \mathrm{H}^{n+1}(K'^\bullet) & \longrightarrow & \cdots
\end{array}$$

Any vertical morphism which is not an isomorphism by hypothesis is surrounded by two isomorphisms. It then follows from the five-lemma that this is actually also an isomorphism.

3. Assume that a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$. Show that (and dual)
   1. if F is exact then G preserves injectives,
   2. if F is faithful exact and $\mathcal{B}$ has enough injectives, then $\mathcal{A}$ too has enough injectives.

Solution: We have an adjunction $\mathrm{Hom}(FM, I) \simeq \mathrm{Hom}(M, GI)$ showing that $h_{GI} = h_I \circ F$ is exact when I is injective and F is exact. It means that GI will be injective too.

If $M \in \mathcal{A}$, then there exists a monomorphism $FM \rightarrowtail I$ with I injective. Since G has an adjoint, it preserves all limits and we have a monomorphism $GFM \rightarrowtail GI$. If F is faithful, then the unit also is a monomorphism $M \rightarrowtail GFM$. By composition, we obtain a monomorphism $M \rightarrowtail GI$. To conclude, it is sufficient to recall what we just proved: GI is injective since F is exact.

By duality, if G is exact then F preserves projectives and if G is also faithful and $\mathcal{A}$ has enough projectives, then $\mathcal{B}$ too has enoug projectives.

4. Show that $\mathrm{Ext}^1(M, N) \simeq \mathrm{Ext}(M, N)$ in an abelian category $\mathcal{A}$ with enough injectives or projectives.

Solution: By duality, we may assume that there are enough injectives. Then, there exists a short exact sequence

$$0 \to N \to I \overset{\pi}{\to} C \to 0$$

with I injective. We know that $\mathrm{Ext}^1(M, I) = \mathrm{Ext}(M, I) = 0$. There exists therefore two right exact sequences

$$\mathrm{Hom}(M, I) \overset{\pi*}{\to} \mathrm{Hom}(M, C) \to \mathrm{Ext}(M, N) \to 0$$

and

$$\mathrm{Hom}(M, I) \overset{\pi*}{\to} \mathrm{Hom}(M, C) \to \mathrm{Ext}^1(M, N) \to 0$$

implying the expected identification.

There also exists an unconditional proof by showing directly that the map

$$\mathrm{Ext}(M, N) \to \mathrm{Ext}^1(M, N), \quad E \mapsto \left(\begin{array}{ccc} [N \to E][1] & & \\ {\scriptstyle \text{q-is}}\downarrow & \searrow & \\ M & & N[1] \end{array} \right)$$

is an isomorphism of groups. Details are left to the reader but surjectivity is proved as follows: a quasi-isomorphism $K^\bullet \to M$ provides an extension $0 \to B^0 \to Z^0 \to M \to 0$, a morphism $K^\bullet \to N[1]$ provides a morphism $\phi : B^0 \to N$ and we can then set $E = \varphi_* Z^0$.

It is also possible to mix both approaches by sending E to the image of Id_M under the connecting map $\mathrm{Hom}(M, M) \to \mathrm{Ext}^1(M, N)$ coming from E.

5. Assume G is a group and $k = \mathbb{Z}$. Compute $\mathrm{H}^1(G, M)$ when

   1. the action of G on M is trivial,
   2. $M = \mathbb{Z}$ with the non-trivial action of $\mu_2 := \{1, -1\}$.

Solution: If the action of G on M is trivial, then

$$\mathrm{Z}(G, M) = \{f : G \to M / \forall g, h \in G, f(gh) = f(g) + f(h)\} = \mathrm{Hom}_{\mathrm{Grp}}(G, M)$$

$$\text{and} \quad \mathrm{B}(G, M) := \{f : G \to M, g \mapsto 0\} = 0$$

so that

$$\mathrm{H}^1(G, M) \simeq \mathbb{Z}(G, M)/B(G, M) = \mathrm{Hom}_{\mathbf{Grp}}(G, M).$$

If μ_2 acts non-trivially on $\mathbb{Z}$, then a crossed morphism is a map $f : \mu_2 \to \mathbb{Z}$ satisfying

$$\begin{array}{ll} f(1) = f(1) + f(1) & f(-1) = f(1) + f(-1), \\ f(-1) = f(-1) - f(1), & f(1) = f(-1) - f(-1). \end{array}$$

This blows down to $f(1) = 0$ and there exists therefore an isomorphism

$$\mathrm{Z}(\mu_2, \mathbb{Z}) \simeq \mathbb{Z}, \quad f \mapsto f(-1).$$

A crossed homomorphism is principal if and only if there exists $n \in \mathbb{Z}$ such that $f(-1) = -2n$ so that

$$\mathrm{B}(\mu_2, \mathbb{Z}) \simeq 2\mathbb{Z}, \quad f \mapsto f(-1)$$

and finally $\mathrm{H}^1(\mu_2, \mathbb{Z}) \simeq \mathrm{Z}(\mu_2, \mathbb{Z})/\mathrm{B}(\mu_2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

6. Show that

$$\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0, 1 \\ 0 \text{ otherwise} \end{cases}$$

with $d = m \wedge n$ (for $m, n \geq 1$).

Solution: Recall first that $\mathrm{Tor}_k(\mathbb{Z}, M) = 0$ for $k \neq 0$ and $\mathrm{Tor}_0(\mathbb{Z}, M) = \mathbb{Z} \otimes M \simeq M$. Then the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

provides an exact sequence

$$0 \to \mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \to \mathrm{Tor}_0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to 0$$

and $\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = 0$ for $k \neq 0, 1$. It is therefore sufficient to recall the exact sequence

$$0 \to \mathbb{Z}/d\mathbb{Z} \xrightarrow{n'} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \to 0$$

where $n = dn'$. Alternatively, one can compute directly the homology of the complex

$$\mathbb{Z}/n\mathbb{Z} \otimes^L \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \otimes [\mathbb{Z} \xrightarrow{m} \mathbb{Z}][1] \simeq [\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}][1].$$