q-DIFFERENCE EQUATIONS AND PRISMS

(Congrès joint AMS-SMF-EMS)

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Introduction

The central notion in p-adic Hodge theory is now that of a *prism* due to Bhargav Bhatt and Peter Scholze. On the other hand, q-difference equations have been around for quite a long time. I want to explain the close relation between them.

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\left\{\text{Prismatic vector bundles}\right\} & \xleftrightarrow{C^*} & \left\{q\text{-crystalline vector bundles}\right\} \\
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\left\{q(-1)\text{-difference equations}\right\} & \xleftrightarrow{F^*} & \left\{q\text{-difference equations}\right\}
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Introduction

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The four categories are indeed equivalent. And cohomology upstairs corresponds to solutions/cosolutions downstairs.

Frobenius

A (always commutative) ring B defined over the complex numbers often comes with a complex conjugation $\sigma : B \to B$. A ring A over the finite field $\mathbb{F}_p$ always comes with its frobenius $\phi : B \to B, f \mapsto f^p$. In both cases, this provides a powerful rigidification of the situation. We will concentrate on the second one and fix from now on a prime p.

Definition

A *frobenius F* on a ring B is a ring endomorphism $\phi : B \to B$ whose reduction modulo p is the usual frobenius.

Example

1. If B is defined over $\mathbb{F}_p$, then the above frobenius is the unique frobenius.
2. If $B = \mathbb{Z}$, then the identity is the unique Frobenius.
3. If $B = \mathbb{Z}[q]$, then the frobenii are defined by $\phi(q) = q^p + p\delta(q)$.
4. If $B = \mathbb{Z}[\sqrt{-1}]$, then there exists no frobenius at all when $p = 2$.

Prisms

It is a very strong condition for a morphism to be compatible with the frobenii. Roughly speaking, given a ring $\overline{B}$, a *prism* is a ring B endowed with a frobenius ϕ and an ideal I such that $B/I = \overline{B}$. The correct definition is more involved:

Definition (Bhatt-Scholze)

A (bounded) *prism* is a couple (B, I) where B is a δ-ring, I is an invertible ideal, B is (bounded) complete for the (p, I)-adic topology and $p \in I + \phi(I)B$. It is said to be *perfect* if ϕ is bijective (equivalently $\overline{B} = B/I$ is (integral) *perfectoid*).

Let us comment a bit.

1. A *δ-ring* is always endowed with a frobenius ϕ and this is equivalent when B is p-torsion free (in general $\phi(f) = f^p + p\delta(f)$).
2. I is *invertible* if, locally, $I = (d)$ with $d \in B$ regular (d is then called an *orientation*).
3. *Complete* means that $B = \varprojlim B/(p, I)^n$.
4. *Bounded* means that $\overline{B}$ has bounded p^∞-torsion.

Examples

Example

1. The (perfect) *crystalline* prism $(\mathbb{Z}_p, p)$ in which case $\overline{B} = \mathbb{F}_p$.
2. The *Breuil-Kisin* prism $(\mathbb{Z}_p[[u]], u - p)$ with $\phi(u) = u^p$ in which case $\overline{B} = \mathbb{Z}_p$.
3. The *q-de Rham* prism $(\mathbb{Z}_p[[q-1]], (p)_q)$ with $\phi(q) = q^p$ and $(p)_q = 1 + q + \cdots + q^{p-1}$, in which case $\overline{B} = \mathbb{Z}_p[\zeta]$ with $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$.
4. The (perfect) *Fontaine* prism $(A_{\mathrm{inf}}, \ker\theta)$. Let $\mathbb{C}_p$ be the field of p-adic complex numbers, $\mathcal{O}_{\mathbb{C}_p}$ the closed unit disc and $\mathcal{O}^\flat_{\mathbb{C}_p} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$ its *tilt*. Then, there exists a surjection
$$\theta : A_{\mathrm{inf}} := W(\mathcal{O}^\flat_{\mathbb{C}_p}) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}, \quad \sum p^k[x_k] \to \sum p^k x_k^{(0)}$$
(in which W denotes the Witt vectors). Here we have $\overline{B} = \mathcal{O}_{\mathbb{C}_p}$.

These examples are not unrelated. There exist morphisms of prisms

$$(\mathbb{Z}_p[[u]], u - p) \to (A_{\mathrm{inf}}, \ker(\theta)) \quad \text{and} \quad (\mathbb{Z}_p[[q-1]], (p)_q) \to (A_{\mathrm{inf}}, \ker(\theta))$$

sending respectively u to $[p^\flat]$ and q to $[\zeta^\flat]$.

Prismatic sites

A *site* is a categorical version of (the open subsets of) a topological space. The *prismatic site* Δ of Bhatt-Scholze is the category of bounded prisms with (formally) faithfully flat maps as coverings. More generally, *a* prismatic site is a category fibered over Δ.

Example

1. Fix a base (oriented) prism (R, d). Let A be a an $\overline{R}$-algebra. Then, the prismatic site $\Delta(A/R)$ is the category of morphisms $A \to \overline{B}$ where (B, J) is a prism over (R, d).
2. A *q-PD-pair* is a complete $(p)_q$-torsion free δ-ring B over $\mathbb{Z}_p[[q-1]]$ together with a closed ideal J such that

$$\forall f \in J, \quad \phi(f) - (p)_q \delta(f) \in (p)_q J.$$

   For example, we may consider the δ-pair $(\mathbb{Z}_p[[q-1]], q-1)$.
   Fix a base q-PD-pair $(R, \mathfrak{r})$. Let A be a an $R/\mathfrak{r}$-algebra. Then, the q-crystalline site $q-\mathrm{CRIS}(A/R)$ is the category of morphisms $A \to \overline{B}$ where (B, J) is a (bounded) q-PD-pair over $(R, \mathfrak{r})$.

Cartier descent

A *ringed site* is a site endowed with a sheaf of rings $\mathcal{O}$. A *vector bundle* on a ringed site is a locally finite free $\mathcal{O}$-module. They form a category $\mathrm{Vec}(\mathcal{O})$. For example, $\mathbb{\Delta}$ is endowed with the sheaf of rings $\mathcal{O}_{\mathbb{\Delta}}$ that sends (B, J) to B. Any prismatic site inherits a ringed structure.

Cartier descent in this setting is essentially due to Kumihiko Li:

Theorem (Li)

Let $(R, \mathfrak{r})$ be a q-PD pair, A a complete smooth ring over $R/\mathfrak{r}$ and

$$A' := A \mathbin{\widehat{\otimes}}_{R/\mathfrak{r} \nearrow \overline{\phi}} R/(p)_q.$$

Then there exists an equivalence of categories

$$C^* : \mathrm{Vec}(\mathcal{O}_{\mathbb{\Delta}(A'/R)}) \simeq \mathrm{Vec}(\mathcal{O}_{q-\mathrm{CRIS}(A/R)}).$$

Proof.

[Li21]. □

q-DIFFERENCE EQUATIONS - INFORMAL

Let A be a "ring of functions in one variable x" and $f \in A$. Denote by $\partial(f) := \frac{\mathrm{d}}{\mathrm{d}x}(f)$ the derivative of f, and set

$$\Delta_h(f)(x) := \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \partial_q(f)(x) := \frac{f(qx) - f(x)}{qx - x}.$$

Then,

$$\partial(f) = \lim_{h \to 0} \Delta_h(f) = \lim_{q \to 1} \partial_q(f).$$

Using Δ_h (resp. ∂_q) instead of ∂, we replace usual calculus with *finite difference* (resp. *q-difference*) calculus.

We are interested in q-difference equations here. Actually, a functional (matrix) equation $Y(qx) = M(x)Y(x)$ corresponds bijectively to a *q-difference equation*

$$(\partial_q Y)(x) = G(x)Y(x) \text{ with } G(x) = \frac{M(qx) - M(x)}{qx - x}.$$

We want to formalize this a bit.

We fix q-PD-pair $(R, \mathfrak{r})$ and a complete R-algebra A with a coordinate x (an étale map $R[x] \to A$). For all $m \in \mathbb{N}$, there exists a unique endomorphism

$$A \to A, f \mapsto f^{(q^{p^m})}, \quad f^{(q^{p^m})}(x) = f(q^{p^m}x) \text{ and } f^{(q^{p^m})} \equiv \mathrm{Id} \mod q^{p^m} - 1.$$

Then, there exists a unique R-linear map (a q^{p^m}*-derivation*)

$$\partial_{q^{p^m}} : A \to A, \quad \partial_{q^{p^m}}(x) = 1 \text{ and } \partial_{q^{p^m}}(fg) = \partial_{q^{p^m}}(f)g + f^{(q^{p^m})}\partial_{q^{p^m}}(g).$$

Definition

A $q(-m)$*-difference equation* is a finite projective A-module M endowed with an R-linear map

$$\partial_{q(-m)} : M \to M, \quad \partial_{q(-m)}(fs) = (p^m)_q \partial_{q^{p^m}}(f)s + f^{(q^{p^m})}\partial_{q(-m)}(s).$$

When $m = 0$ (resp. $m = 1$), this is a q-deformation of a module with an integrable connection (resp. a Higgs module). They form a category $\mathrm{MIC}_{q(-m)}(A/R)$.

Frobenius descent

Definition

A $q(-m)$-difference equation is said to be *topologically quasi-nilpotent* if

$$\forall s \in M, \quad \partial^k_{q(-m)}(s) \to 0 \text{ when } k \to \infty.$$

They form a subcategory $\widehat{\mathrm{MIC}}_{q(-m)}(A/R)$ of $\mathrm{MIC}_{q(-m)}(A/R)$.

One can then prove Berthelot's frobenius descent in this setting:

Theorem (Gros-LS-Quirós)

If $A' := A \,\widehat{\otimes}_{R\nearrow\phi} R$, *then there exists an equivalence of categories*

$$F^* : \widehat{\mathrm{MIC}}_{q(-1)}(A'/R) \simeq \widehat{\mathrm{MIC}}_{q(0)}(A/R).$$

Proof.

[GLQ22b]. □

Twisted divided powers

Definition

The ring of *q-divided polynomials of level $-m$* is the free A-module $A\langle\omega\rangle_{q(-m)}$ on generators $\omega^{\{n\}_{q(-m)}}$ with the multiplication rule:

$$\omega^{\{n_1\}_{q(-m)}}\omega^{\{n_2\}_{q(-m)}} = \sum_{0\leqslant i\leqslant \min\{n_1,n_2\}} q^{\frac{p^m i(i-1)}{2}} \binom{n_1+n_2-i}{n_1}_{q^{p^m}} \binom{n_1}{i}_{q^{p^m}} (q-1)^i x^i \omega^{\{n_1+n_2-i\}_{q(-m)}}.$$

Proposition (Gros-LS-Quirós)

There exists (for $m=0$ or $m=1$) a unique natural δ-structure on $A\langle\omega\rangle_{q(-m)}$ such that

$$\xi = (p^m)_q \omega^{\{1\}_{q(-m)}} \quad \Rightarrow \quad \phi(\xi) = (x+\xi)^p - x^p.$$

Proof.

[GLQ22c] in the case $m=0$ and [GLQ22b] when $m=1$. □

Hyperstratifications

Definition

A *q-hyperstratification of level* $-m$ on a finite projective A-module M is an isomorphism (we use the Taylor map $x \mapsto x + (p^m)_q\omega$ on the left)

$$\epsilon : \widehat{A\langle\omega\rangle}_{q(-m)} \otimes'_A M \simeq M \otimes_A \widehat{A\langle\omega\rangle}_{q(-m)}$$

satisfying the *cocycle condition* $p_{13}^*(\epsilon) = p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon)$.

They form a category $\widehat{\mathrm{Strat}}_{q(-m)}(A/R)$.

Proposition (Gros-LS-Quirós)

There exists an equivalence of categories

$$\widehat{\mathrm{Strat}}_{q(-m)}(A/R) \simeq \widehat{\mathrm{MIC}}_{q(-m)}(A/R).$$

Proof.

It is given by $\epsilon(1 \otimes s) \equiv s \otimes 1 + \partial_{q(-m)}(s) \otimes \omega^{\{1\}_{q(-m)}} \mod \omega^{\{>1\}_{q(-m)}}$. □

Prismatic envelope

Theorem (Gros-LS-Quirós)

The prism $\left(\widehat{A\langle\omega\rangle}_{q(-1)}, \omega^{\{>0\}_{q(-1)}}\right)$ *is the prismatic envelope of the kernel* I *of the composite map* $A \otimes_R A \to A \to \overline{A}$*: it is universal among all morphisms* $(A \otimes_R A, I) \to (B, J)$ *to a prism.*

Proof.

[GLQ22b]. □

Corollary

There exists an equivalence of categories

$$\mathrm{Vec}(\mathcal{O}_{\Delta(\overline{A}/R)}) \simeq \widehat{\mathrm{Strat}}_{q(-1)}(A/R).$$

Proof.

It is obtained by evaluating a bundle on the prism $(A, (p)_q)$ which is a covering of $\overline{A}$. □

Conclusion

There exists an exact analog of the previous slide in the case $m = 0$ for the q-crystalline site. We can now summarize our constuctions:

$$
\begin{array}{ccc}
\mathrm{Vec}(\mathcal{O}_{\Delta(\overline{A}'/R)}) & \xleftrightarrow{\;C^*\;} & \mathrm{Vec}(\mathcal{O}_{q-\mathrm{CRIS}(\overline{A}/R))}) \\
\updownarrow & & \updownarrow \\
\widehat{\mathrm{Strat}}_{q(-1)}(A'/R) & \longleftrightarrow & \widehat{\mathrm{Strat}}_{q(0)}(A/R)\} \\
\updownarrow & & \updownarrow \\
\widehat{\mathrm{MIC}}_{q(-1)}(A'/R) & \xleftrightarrow{\;F^*\;} & \widehat{\mathrm{MIC}}_{q(0)}(A/R)
\end{array}
$$

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