p-adic confluence and twisted differential operators
– joint work with Adolfo Quirós –
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Contents

A result from André-Di Vizio

Let us denote by $\mathcal{R}$ the Robba ring over a p-adic field K. Then, the tannakian formalism provides an equivalence (André using Christol-Mebkhout)

$$\{(F)-\textit{differential equations over } \mathcal{R}\} \simeq \mathrm{Rep}(\mathbb{G}_{\mathrm{a}} \times \mathrm{Gal}).$$

André and Di Visio showed that the same methods lead to

$$\{(F)-q-\textit{difference equations over } \mathcal{R}\} \simeq \mathrm{Rep}(\mathbb{G}_{\mathrm{a}} \times \mathrm{Gal}).$$

As a consequence, they obtain an equivalence ([AD04]):

$$\begin{array}{c}\{(F)-\textit{differential equations over } \mathcal{R}\} \\ \Big\updownarrow \simeq \\ \{(F)-q-\textit{difference equations over } \mathcal{R}\}.\end{array}$$

The approach of Pulita

Now, we let X be a curve over the p-adic field K endowed with an endomorphism σ. Then, Pulita builds in [Pul17] a functor

$$\begin{array}{c}\{\textit{Some differential equations over } X\}\\ \Big\downarrow \\ \{\sigma - \textit{functional equations over } X\}\end{array}$$

that extends somehow the correspondence of André-Di Visio. Actually, if x is a local parameter and $\sigma(x) = qx$, then there exists an equivalence

$$\begin{array}{c}\{q - \textit{difference equations}\}\\ \Big\updownarrow \simeq \\ \{\sigma - \textit{Functional equations}\}.\end{array}$$

Our explicit construction

Assume A is the ring of functions on an annulus $r_1 \leq |x| \leq r$. Call a differential A-module M $\eta\dagger$-*convergent* if

$$\forall \eta' < \eta, \forall s \in M, \quad \lim_{k \to \infty} \frac{\eta'^k}{|k!|} |\partial_M^k(s)| = 0.$$

Assume that $q \in K$ is not a root of unity and $|1 - q|r < \eta \leq r_1$.

Theorem (LS-Quirós)

There exists a fully faithful functor (here $\sigma(x) = qx$)

$$\{\eta \dagger -convergent\ (M, \partial_M)\} \to \{\sigma - module\ (M, \sigma_M)\}$$

given by

$$\sigma_M(s) = \sum_{k=0}^{\infty} \frac{(q-1)^k x^k}{k!} \partial_M^k(s) \quad (\sigma_M = q^{x\partial_M}).$$

An example (or two)

Example

1. From the differential equation $y' - cy = 0$, we get the functional equation

$$y(qx) = \exp((q-1)cx)y(x).$$

   They both have the same solution $y = \exp(cx)$.

2. From the differential equation $xy' - ay = 0$, we obtain the functional equation

$$y(qx) = q^a y(x).$$

   They both have the same solution $y = x^a$

3. More generally, any differential equation $y' = a(x)y$ will give rise to a functional equation $f(qx) = b(x)f(x)$ with the same (formal) solution.

Settings

We let K be an ultrametric field, R an affinoid K-algebra and A a *twisted* affinoid R-algebra: an affinoid R-algebra endowed with an endomorphism σ.

Example (Running)

K p-adic field, $R = K$, $A = K\{x/r, r_1/x\}$, $\sigma(x) = qx$, $\frac{r_1}{r} \leq |q| \leq 1$.

A *twisted derivation* $\partial_{A,\sigma}$ on A is an R-linear map such that

$$\forall f, g \in A, \partial_{A,\sigma}(fg) = f\partial_{A,\sigma}(g) + \sigma(g)\partial_{A,\sigma}(f).$$

Example (Running)

$\partial_{A,\sigma}(x^n) = \frac{1 - q^n}{1 - q}x^{n-1}$ (or $\partial_{A,\sigma}(x^n) = \partial_A(x^n) = nx^{n-1}$ if $q = 1$).

Twisted Weyl algebra

The *twisted Weyl algebra* of A (associated to $\partial_{A,\sigma}$) is the non-commutative R-algebra ([LQ18b])

$$D_\sigma := \left\{ \sum_{0<k<<\infty} f_k \partial_\sigma^k, \quad f_k \in A \right\}$$

with the commutation rule $\partial_\sigma \circ f = \partial_{A,\sigma}(f) + \sigma(f)\partial_\sigma$.

Convention: If D is an A-algebra, a D-module M will always be assumed to be *finite* over A.

A *σ-differential A-module* is a D_σ-module.

We will also need to introduce the ring $D_\sigma^{(\infty)}$ of twisted differential operators of *infinite* level (Grothendieck) ([LQ18a]) by comparison with D_σ which is a ring of twisted differential operators of level *zero* (Berthelot).

$\eta\dagger$-convergence

If we assume that $\partial_{A,\sigma}$ is associated to a *twisted coordinate* x (see below), then there is a canonical map $D_\sigma \to D_\sigma^{(\infty)}$.

We will define *η-convergence* and *$\eta\dagger$-convergence* for a $D_\sigma^{(\infty)}$-module.

Our main result is the fact that the category of $\eta\dagger$-convergent $D_\sigma^{(\infty)}$-modules is essentially *independent* of the choice of σ. In particular, we may choose $\sigma = \mathrm{Id}_A$ as well (and drop σ from the notations).

Example (Running)

If q not a root of unity and and $|1-q|r < \eta \leq r_1$, then there exists an equivalence between $\eta\dagger$-convergent q-difference equations and $\eta\dagger$-convergent differential equations.

Twisted operators of finite level

We will then introduce the rings $D_\sigma^{(\eta)}$ and $D_\sigma^{\eta\dagger}$ of twisted differential operators of (multiplicative) level η and $\eta\dagger$.

We will see that the category of $\eta\dagger$-convergent σ-differential A-modules is equivalent to the category of $D_\sigma^{\eta\dagger}$-modules (this is not true with η instead of $\eta\dagger$).

We will show that when x is both a *twisted coordinate* (see below) for two different endomorphisms σ and τ of A, then there exists a canonical (deformation) isomorphisms

$$D_\sigma^{(\eta)} \simeq D_\tau^{(\eta)} \quad \text{and} \quad D_\sigma^{\eta\dagger} \simeq D_\tau^{\eta\dagger}.$$

This will provide us with the equivalence between $\eta\dagger$-convergent differential modules with respect to σ and τ.

Twisted coordinate

We let

$$P := A\widehat{\otimes}_R A \quad \text{and} \quad I := \ker(P \to A, \quad f \otimes g \mapsto fg).$$

We still denote by σ the endomorphism $\sigma \otimes_R \mathrm{Id}_A$ of P and let

$$I^{(n+1)} := I\sigma(I)\cdots\sigma^n(I) \subset P, \quad P_{(n)} := P/I^{(n+1)}, \quad \widehat{P}_\sigma := \varprojlim P_{(n)}.$$

We call $x \in A$ a *twisted coordinate* if P_n is free on $1, \xi, \ldots \xi^{(n)}$ where $\xi : 1 \otimes x - x \otimes 1$ and $\xi^{(k+1)} := \xi\sigma(\xi)\cdots\sigma^k(\xi)$.

There always exists a universal *twisted derivation*

$$d_\sigma : A \to \Omega^1_\sigma := I/I^{(2)}, \quad f \mapsto \overline{1 \otimes f - f \otimes 1}.$$

If x is a twisted coordinate, then Ω^1_σ is free on $\mathrm{d}x$ and there exists a unique twisted derivation on A such that $\partial_{A,\sigma}(x) = 1$.

Twisted differential operators

The ring of *twisted differential operators (of infinite level)* ([LQ18a]) is

$$D_\sigma^{(\infty)} := \bigcup_n \mathrm{Hom}_A(P_{(n)}, A) \subset \mathrm{End}_R(A).$$

Ring multiplication may also be obtained by duality from comultiplication

$$\delta : P \to P \widehat{\otimes}'_A P, \quad f \otimes g \mapsto (f \otimes 1) \otimes' (1 \otimes g),$$

where A acts on the right (resp. left) hand side through the *canonical* map (resp. the *Taylor* map)

$$A \to P, f \mapsto f \otimes 1 \quad (\text{resp. } \theta : A \to P, f \mapsto 1 \otimes f).$$

Concretely

We fix from now on a twisted coordinate x on A. We have

$$\widehat{P}_\sigma = A[[\xi]]_\sigma := \varprojlim A[\xi]/\xi^{(n)} \quad \text{and}$$

$$\mathrm{D}_\sigma^{(\infty)} = \left\{ \sum_{0 \le k << \infty} f_k \partial_\sigma^{[k]}, \quad f_k \in A \right\},$$

where $(\partial_\sigma^{[k]})_{k\in\mathbb{N}}$ denotes the dual basis to $(\xi^{(n)})_{n\in\mathbb{N}}$.

Example (Running)

If q is not a root of unity (or $q = 1$), then $\mathrm{D}_\sigma^{(\infty)} \simeq \mathrm{D}_\sigma$ with

$$\partial_\sigma^{[k]} = \frac{(1-q)^k}{\prod_{i=1}^k (1-q^i)} \partial_\sigma^k \quad (\text{and } \partial^{[k]} = \frac{1}{k!}\partial^k \text{ if } q = 1).$$

η-convergence

If M is a $\mathrm{D}_\sigma^{(\infty)}$-module and $\eta \in \mathbb{R}$, we call M *η-convergent* (resp. *$\eta\dagger$-convergent*) if

$$\forall s \in M, \quad \|\partial_\sigma^{[k]}(s)\|\eta^k \to 0 \text{ when } k \to +\infty$$

(resp. M is η'-convergent whenever $\eta' < \eta$).

Example

$\mathcal{X} = \mathrm{Spf}(A)$: smooth affine formal K°-scheme with an étale coordinate x, $\mathcal{M}$: coherent $\mathcal{O}_{\mathcal{X}_K}$-module and $M := \Gamma(\mathcal{X}_K, \mathcal{M})$. A connection on $\mathcal{M}$ is convergent (in the sense of rigid cohomology) if and only if it is $1\dagger$-convergent on M with respect to $\sigma = \mathrm{Id}_A$.

Example (Running)

The ring A itself is $r_1\dagger$-convergent (the Robba ring is $1\dagger$-convergent).

The twisted Taylor map

The *x-radius* of σ is $\rho := \|x - \sigma(x)\|$ (for a fixed contractive norm).

Example (Runnnig)

We have $\rho = |1 - q|r$.

If A is η-convergent with $\eta \geq \rho$, then there exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\theta} & P & \longrightarrow & \widehat{P}_\sigma \\ \downarrow{\scriptstyle \theta_\eta} & & & & \| \\ A\{\xi/\eta\} & & \hookrightarrow & & A[[\xi]]_\sigma \end{array}$$

with $A \xrightarrow{\theta_\eta} A\{\xi/\eta\} := \left\{\sum_{n\geq 0} f_n \xi^n, \|f_n\|\eta^n \to 0\right\}$

$$f \longmapsto \sum_{k=0}^{\infty} \partial_\sigma^{[k]}(f)\xi^{(k)}.$$

Twisted differential operators again

We set $D_\sigma^{(\eta)} := \mathrm{Hom}_{A-\mathrm{cont}}(A\{\xi/\eta\}, A)$ and $\mathrm{D}_\sigma^{(\eta\dagger)} := \varinjlim_{\eta'<\eta} \mathrm{D}_\sigma^{(\eta')}$. They are endowed with a ring structure that comes by duality from

$$\delta_\eta : A\{\xi/\eta\} \to A\{\xi/\eta\}\widehat{\otimes}'_A A\{\xi/\eta\}, \quad \xi \mapsto 1 \otimes' \xi - \xi \otimes' 1,$$

where A acts on the right (resp. left) hand side through the canonical map (resp. the twisted Taylor map). Concretely

$$\mathrm{D}_\sigma^{(\eta)} = \left\{ \sum_{k=0}^{\infty} f_k \partial_\sigma^{[k]}, \quad \exists C > 0, \forall k \in \mathbb{N}, \|f_k\| \leq C\eta^k \right\} \quad \text{and}$$

$$\mathrm{D}_\sigma^{(\eta\dagger)} = \left\{ \sum_{k=0}^{\infty} f_k \partial_\sigma^{[k]}, \quad \exists \eta' < \eta, \frac{\|f_k\|}{\eta'^k} \to 0 \right\}$$

Example

If $\mathcal{X} = \mathrm{Spf}(A)$ as above and $\sigma = \mathrm{Id}_A$, we have $\Gamma(\mathcal{X}, \mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}) = \mathrm{D}_\sigma^{(1\dagger)}$.

Deformation

Theorem

Assume x is a twisted coordinate for both σ and τ, that A is η-convergent (for σ or τ or both) and that $\eta \geq \rho(\sigma), \rho(\tau)$. Then there exists a canonical isomorphism

$$\mathrm{D}_\sigma^{(\eta)} \simeq \mathrm{D}_\tau^{(\eta)} \quad (\text{resp. } \mathrm{D}_\sigma^{(\eta\dagger)} \simeq \mathrm{D}_\tau^{(\eta\dagger)}).$$

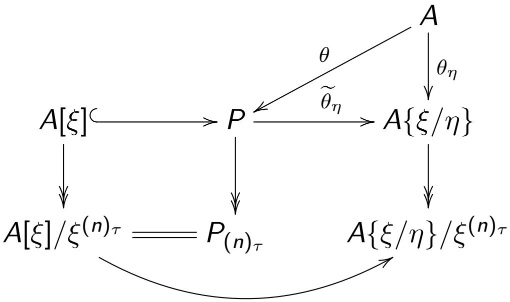
Before we give a proof, notice that this isomorphism is explicit in the sense that

$$\partial_\sigma = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} \left(\sigma(x) - \tau^i(x) \right) \right) \partial_\tau^{[k]}$$

for example.

Proof.

The point is to show that the twisted Taylor map is the same for σ and τ. If θ_η denotes the twisted Taylor map of radius η with respect to σ, then there exists a commutative diagram



where $\widetilde{\theta}_\eta$ is the A-linearization of θ_η. The explicit formulas are obtained by duality from the base change $\xi^{(n)_\tau} \leftrightarrow \xi^{(n)_\sigma}$ in $A\{\xi/\eta\}$. □

Confluence

One can check that the categories of $\mathrm{D}_{\sigma}^{\eta\dagger}$-modules and $\eta\dagger$-convergent $\mathrm{D}_{\sigma}^{(\infty)}$-modules are isomorphic (this is wrong without the $\dagger$). It follows that:

Corollary

The categories of $\eta\dagger$-convergent $\mathrm{D}_{\tau}^{(\infty)}$-modules and $\eta\dagger$-convergent $\mathrm{D}_{\sigma}^{(\infty)}$-modules are canonically equivalent.

Finally, σ itself is a twisted differential operator of order one:

$$\sigma = 1 - (x - \sigma(x))\partial_{A,\sigma}.$$

One obtains a functor from $\eta\dagger$-convergent $\mathrm{D}_{\tau}^{(\infty)}$-modules to σ-modules. In the case $\tau = \mathrm{Id}_A$, we recover the original statement.

Yves André and Lucia Di Vizio. “q-difference equations and p-adic local monodromy”. In: *Astérisque* 296 (2004). Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I, pages 55–111.

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Extra slide: fomulas

Back to the running example. Remember that

$$\sigma_M(s) = \theta_\eta(s)(qx - x) := \sum_{k=0}^{\infty} \frac{(q-1)^k x^k}{k!} \partial_M^k(s).$$

It just means that $\sigma = \sum_{k=0}^{\infty} \frac{(q-1)^k x^k}{k!} \partial^k \in D^{(\eta)}$.

One can also show that

$$\sigma_M(s) = (q^{x\partial_M})(s) := \sum_{k=0}^{\infty} \log(q)^k (x\partial_M)^k(s)$$

because

$$\sum_{k=0}^{\infty} \log(q)^k (x\partial)^k(x^n) = \sum_{k=0}^{\infty} \log(q)^k n^k x^n = q^n x^n.$$