Exercise 1 Define the categories $\mathbf{Ord}$, $\mathbf{Mon}$, $\mathbf{Grp}$, $\mathbf{Ring}$, $\mathbf{CRing}$, G-$\mathbf{Set}$ (resp. $\mathbf{Set}$-G), A-$\mathbf{Mod}$ (resp. $\mathbf{Mod}$-A) and k-$\mathbf{Alg}$ of preordered sets, monoids, groups, rings, commutative rings, left (resp. right) G-sets, left (resp. right) A-modules and k-algebras.

Exercise 2 Show that, if G is a monoid, then the set of objects of $\mathbf{Mor}(\mathbf{G})$ is G and that, if X is a preordered set, then the set of objects of $\mathbf{Mor}(\mathbf{X})$ is the graph Γ of the relation.

Exercise 3 Make $\mathcal{C}_{/X}$ explicit when X is an object of a category $\mathcal{C}$.

Exercise 4 What is an isomorphism in $\mathbf{Set}$, in $\mathbf{Top}$, in $\mathbf{Ab}$, etc. ? In $\mathbf{G}$ if G is a monoid ? In $\mathbf{X}$ if X is a preordered set ?

Exercise 5 What is a functor $\mathbf{G} \to \mathbf{H}$ between categories associated to monoids ? What is a functor $\mathbf{X} \to \mathbf{Y}$ between categories associated to preordered sets ?

Exercise 6 What are the analogs of the "free abelian group" functor $X \mapsto \mathbb{Z} \cdot X$ for the categories $\mathbf{Mon}$, $\mathbf{Grp}$, A-$\mathbf{Mod}$ and k-$\mathbf{Alg}$?

Exercise 7 Show that, besides the inclusion functor $\mathbf{Ab} \rightarrowtail \mathbf{Grp}$, there exists an *abelianization* functor $G \mapsto G^{\mathrm{ab}} = G/[G,G]$ in the other direction. Show that the center is *not* functorial in the sense that a group homomorphism $\varphi : G \to H$ does not necessarily induce a morphism of abelian groups $\mathrm{Z}(G) \to \mathrm{Z}(H)$.

Exercise 8 Show that the categories $\mathbb{Z}$-$\mathbf{Mod}$ and $\mathbf{Ab}$ are isomorphic. Same thing with the categories $\mathbb{Z}$-$\mathbf{Alg}$ and $\mathbf{Ring}$, and, more generally, k-$\mathbf{Alg}$ and a full subcategory of ${}_{k\backslash}\mathbf{Ring}$ (the image of k must be in the center).

Exercise 9 Show that the image of a section (resp. a retraction, resp. an inverse) by a functor is a section (resp. a retraction, resp. an inverse).

Exercise 10 Let us denote by $\mathbf{Op}(\mathcal{C}) \subset \mathbf{Mor}(\mathcal{C})$ the subcategory whose objects are morphisms with codomain identical to the domain and morphisms have same component on domain and codomain (objects with operator). Show that, if k is a commutative ring, then $\mathbf{Op}(k\text{-}\mathbf{Mod})$ is isomorphic to $k[t]$-$\mathbf{Mod}$.

Exercise 11 Show that

1. if $\mathcal{C}$ is a small category, then there exists an isomorphism of categories $\mathbf{Hom}(\mathcal{C},\mathcal{D})^{\mathrm{op}} \simeq \mathbf{Hom}(\mathcal{C}^{\mathrm{op}},\mathcal{D}^{\mathrm{op}})$,
2. if $\mathcal{C}$ and $\mathcal{C}'$ are two small categories, then there exists an isomorphism of categories

$$\mathbf{Hom}(\mathcal{C} \times \mathcal{C}', \mathcal{D}) \simeq \mathbf{Hom}(\mathcal{C}, \mathbf{Hom}(\mathcal{C}', \mathcal{D})).$$

Exercise 12 Show that, there exists a functor

$$\mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}, \quad (\mathcal{C},\mathcal{D}) \mapsto \mathbf{Hom}(\mathcal{C},\mathcal{D}).$$

Exercise 13 Show that the category **SSet** of simplicial sets is isomorphic to the category $\widehat{\mathbf{\Delta}}$ of presheaves of sets on $\mathbf{\Delta}$. Analog with semi-simplicial sets and $\widehat{\mathbf{\Delta}}_{\mathrm{inj}}$.

Exercise 14 Show that there exists a cosimplicial object $|\Delta^\bullet|$ of **Top** sending $[n]$ to $|\Delta^n|$ and $u : [n] \to [m]$ to the unique linear map sending e_i to $e_{u(i)}$ if $(e_0, \ldots, e_n)$ denotes the usual basis of $\mathbb{R}^{n+1}$.

Exercise 15 Show that the simplicial set $S_\bullet(X)$ associated to a topological space X corresponds to $h_X \circ |\Delta^\bullet|$ under $\mathbf{SSet} \simeq \widehat{\mathbf{\Delta}}$ and that this provides a functor $\mathbf{Top} \to \widehat{\mathbf{\Delta}}$ sending a topological space X to its associated simplical set $S_\bullet(X)$.

Exercise 16 Show that the forgetful functors $\mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Ab} \to \mathbf{Set}$ are faithful but not fully faithful.

Exercise 17 Show that there exists a fully faithful functor

$$\mathbf{Mon} \rightarrowtail \mathbf{Cat}, \quad G \mapsto \mathbf{G}, \quad (\text{resp. } \mathbf{Ord} \rightarrowtail \mathbf{Cat}, \quad X \mapsto \mathbf{X}).$$

What is the essential image ?

Exercise 18 Show that there exists a fully faithful functor $\mathbf{Cat} \rightarrowtail \widehat{\mathbf{\Delta}}$.

Exercise 19 Show that if $F \simeq F'$, then F is faithful (resp. full, resp. fully faithful, essentially surjective, an equivalence) if and only if F' is.

Exercise 20 Show that if X is a preordered set and Y denotes its ordered quotient, then the categories $\mathbf{X}$ and $\mathbf{Y}$ are equivalent.

Exercise 21 Show that, if A is a ring, then $\mathbf{Mat}(A) := \mathbb{N}$, endowed with $\mathrm{Hom}(m, n) = \mathrm{M}_{n\times m}(A)$ and multiplication of matrices, is a small category. Show that if A is a field k, then $\mathbf{Mat}(k)$ is equivalent, but not isomorphic, to the category of finite dimensional k-vector spaces (which is large).

Exercise 22 Show that, if F is represented by both X and X', then $X \simeq X'$. More precisely, show that if both (X, s) and (X', s') are universal for F, then there exists a unique isomorphism $f : X \simeq X'$ such that $F(f)(s) = s'$.

Exercise 23 Show that usual forgetful functors are representable.

Exercise 24 Show that $M \otimes_k N$ is universal for (the functor that sends P to the set of) bilinear maps $M \times N \to P$.

Exercise 25 Let k be a commutative ring and $f_1, \ldots, f_r \in k[t_1, \ldots, t_n]$. Show that the functor that sends a commutative k-algebra A (make the morphisms explicit) to the set

$$\mathcal{S}(A) := \{(a_1, \ldots, a_n) \in A^n \ / \ f_1(a_1, \ldots, a_n) = \cdots = f_r(a_1, \ldots, a_n) = 0\}$$

of all solutions with values in A, is representable.

Exercise 26 Show that if $\mathcal{C}$ is a small category, then there exists a fully faithful *Yoneda* functor

$$\text{よ} : \mathcal{C} \rightarrowtail \widehat{\mathcal{C}}, \quad X \mapsto h_X .$$

Exercise 27 Write down a diagram for the colimit as in (**??**).

Exercise 28 Show that in an preordered set, a limit (resp. colimit) is a least upper bound or *inf* or *join* (resp. greatest lower bound or *sup* or *meet*). What about cone and cocone ?

Exercise 29 Show that, if $D \to D'$ is a morphism of diagrams of shape I in $\mathcal{C}$ with respective limits X and X', then there exists a unique morphism $X \to X'$ making commutative the diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \uparrow & & \uparrow \\ \underline{X} & \longrightarrow & \underline{X'} \end{array}$$

Exercise 30 Show that there exists "canonical" isomorphisms (and dual)

1. $X \times 1_{\mathcal{C}} \simeq X$ for $X \in \mathcal{C}$,
2. $X \times Y \simeq Y \times X$ for $X, Y \in \mathcal{C}$ and
3. $(X \times Y) \times Z \simeq X \times (Y \times Z)$ for $X, Y, Z \in \mathcal{C}$.

Exercise 31 Show that, if $\mathcal{C}$ is a cartesian category, then there exists a functor

$$\mathcal{C} \to \widehat{\mathbb{\Delta}}(\mathcal{C}) := \mathbf{Hom}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C}), \quad X \mapsto X_\bullet$$

where $X_n := X^{n+1}$ and $u : [m] \to [n]$ is sent to the unique morphism $X^{n+1} \to X^{m+1}$ whose i-th component is the j-th projection with $j := u(i-1)$

Exercise 32 Show that the fibered coproduct in the category of commutative rings is tensor product.

Exercise 33 Show that[a], in a diagram

$$\begin{array}{ccccc} Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ X_2 & \longrightarrow & X_1 & \longrightarrow & X_0, \end{array}$$

if the right hand square is cartesian, then the left hand square is cartesian if and only if the full rectangle is cartesian.

[a] This is an illustration of the more general proposition **??** below.

Exercise 34 Show that if $\mathcal{C}$ is a small category, then $\mathbf{Op}(\mathcal{C})$ is the kernel of the domain and codomain functors $\mathbf{Mor}(\mathcal{C}) \rightrightarrows \mathcal{C}$ in $\mathbf{Cat}$.

Exercise 35 Make explicit specific limits (final object, products, fibered product and kernel) and colimits (initial object, coproduct, fibered coproduct and cokernel) in **Mon**, **Grp**, G-**Set**, A-**Mod** or **Cat**.

Exercise 36 Show that a split monomorphism is a regular monomorphism and that a regular monomorphism is a monomorphism (and dual).

Exercise 37 Show that a morphism $f : X \to Y$ is a monomorphism (resp. an epimorphism) if and only if the induced functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ (resp. ${}_{Y\backslash}\mathcal{C} \to {}_{X\backslash}\mathcal{C}$) is fully faithful.

Exercise 38 Show that[a] if both $f : X \to Y$ and $g : Y \to Z$ are monomorphisms, then so is $g \circ f$ (and dual). Show that, conversely, if $g \circ f$ is a monomorphism, then so is f (and dual).

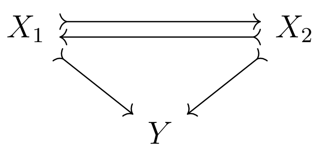
[a] Be careful that this is not the case for regular monomorphisms in general.

Exercise 39 Show that a regular epimorphism which is also a monomorphism is automatically an isomorphism (and dual).

Exercise 40 Show that, if we are given $f, g : X \to Y$ and $i : Y \rightarrowtail Z$ is a monomorphism, then[a] $\ker(i \circ f, i \circ g) = \ker(f, g)$ (and dual). Analog for fibered products (and dual) ?

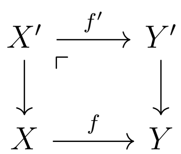
[a] If one of them exists, then so does the other and...

Exercise 41 Show that, if we are given a commutative diagram of monomorphisms



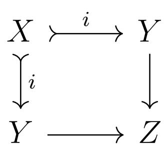
then, the upper arrows are inverse isomorphisms to each other (and dual).

Exercise 42 Show that, in a cartesian diagram



if f is a monomorphism, then f' also is monomorphism (and dual).

Exercise 43 Show that if $i : X \rightarrowtail Y$ is an injective map and the diagram of *sets*



is cocartesian, then it is also cartesian.

Exercise 44 Show that, if $f : X \to Y$ is an epimorphism, then $\operatorname{im}(f) = Y$ (and dual).

Exercise 45 Show that a strict epimorphism is regular (and dual).

Exercise 46 Assume given $f, g : X \to Y$ such that $Y \times Y$ exists. Show that $\ker(f, g)$ exists if and only if there exists a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (f,g)} \\ Y & \xrightarrow{\delta} & Y \times Y \end{array}$$

in which case $Z = \ker(f, g)$.

Exercise 47 Show that **Set**, **Top**, **Ab**, etc. are bicomplete.

Exercise 48 Show that colimits of *sets* are stable under pullback.

Exercise 49 Show that, if F is left exact, then F preserves (regular, strict) monomorphisms (and dual).

Exercise 50 Show that, if $\mathcal{C}$ is a small category and all limits of shape I exist in $\mathcal{D}$, then all limits of shape I also exist in the category $\widehat{\mathcal{C}}(\mathcal{D})$ of presheaves and they are preserved by the functor

$$\widehat{\mathcal{C}}(\mathcal{D}) \to \mathcal{D}, \quad F \mapsto F(X)$$

for fixed $X \in \mathcal{C}$ (and dual).

Exercise 51 Show that a representable functor $F : \mathcal{C} \to \mathbf{Set}$ preserves all limits[a].

[a] There exists no dual statement and the notion of a limit plays a special role.

Exercise 52 Show that a small category I is filtered if and only if

1. $I \neq \emptyset$,
2. $\forall i, j \in I, \exists k \in I, i \to k, j \to k$,
3. $\forall u, v : i \to j, \exists k \in I, c : j \to k \;/\; c \circ u = c \circ v$.

Exercise 53 Show that

1. a set (resp. a small category) is the filtered colimit of its finite subsets (resp. finite subcategories),
2. a category with finite colimits (resp. finite coproducts) and filtered colimits has all colimits (resp. all coproducts).
3. a functor that preserves finite colimits (resp. finite coproducts) and filtered colimits preserves all colimits (resp. coproducts).

Exercise 54 Show that filtered colimits are exact in **Ab**, **Top**, etc. and that they are preserved by the forgetful functors to **Set** (one may show the second assertion first).

Exercise 55 Show that **Ab** satisfies AB4* extra condition: arbitrary products are exact.

Exercise 56 Show that **Ab** satisfies AB6 extra condition: filtered colimits commute with arbitrary products:

$$\prod_{j\in J} \varinjlim_{i_j\in I_j} M_{i_j} \simeq \varinjlim_{\prod_{j\in J} I_j} \prod_{j\in J} M_{i_j}$$

when each I_j is filtered.

Exercise 57 Show that most forgetful and inclusion functors we have already met have an adjoint (and sometimes a coadjoint) and make them explicit.

Exercise 58 Show that the adjoint to the forgetful functor k-**Alg** $\to$ k-**Mod** is the tensor algebra functor $M \mapsto \mathrm{T}(M)$. Same thing with $\mathrm{S}(M)$ when we restrict to commutative algebras.

Exercise 59 Sow that, if $f : A \to B$ is a morphism of rings, then the forgetful functor B-**Mod** $\to$ A-**Mod** has both an adjoint $M \mapsto B \otimes_A M$ and a coadjoint $M \mapsto \mathrm{Hom}_A(B, M)$.

Exercise 60 Show that (for fixed Y) the functor $X \mapsto X \times Y$ from **Set** to itself is adjoint to the functor $Z \mapsto \mathcal{F}(Y, Z)$:

$$\mathcal{F}(X \times Y, Z) \simeq \mathcal{F}(X, \mathcal{F}(Y, Z)).$$

This is called *Currying*. Write down the analogous statements for **Cat** and **Ab**.

Exercise 61 Show that if both F_1 and F_2 are adjoint to G, then $F_1 \simeq F_2$ (and dual).

Exercise 62 Show that if $F_1 : \mathcal{C} \leftrightarrows \mathcal{C}' : G_1$ and $F_2 : \mathcal{C}' \leftrightarrows \mathcal{C}'' : G_2$ then, $F_2 \circ F_1 : \mathcal{C} \leftrightarrows \mathcal{C}'' : G_1 \circ G_2$.

Exercise 63 Describe unit and counit in all the examples studied so far. Deduce in each case faithfulness or full faithfulness of the functors.

Exercise 64 Show that, if a small category $\mathcal{C}$ has (self) coproducts, then all representable functors F on $\mathcal{C}$ have an adjoint.

Exercise 65 Show that any adjunction between two functors F and G extends to an adjunction on diagrams of a given shape I:

$$\mathrm{Hom}(F(D), E) \simeq \mathrm{Hom}(D, G(E)).$$

Exercise 66 Show that $k[t]$ (resp. $k[t]_t$) endowed with $t \mapsto t \otimes 1 + 1 \otimes t$ (resp. $t \mapsto t \otimes t$) is an abelian group of the category opposite to the category of k-algebras (it is called a *bialgebra*).

Exercise 67 Show that, if all limits exist in $\mathcal{C}$, then the same holds in $\mathbf{Ab}(\mathcal{C})$ and they are preserved by the forgetful functor $\mathbf{Ab}(\mathcal{C}) \to \mathcal{C}$.

Exercise 68 Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between cartesian categories. Show that

1. if F preserves finite products, then F induces a functor still written $F : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{C}')$,
2. if moreover, F preserves all limits of $\mathcal{C}$, then it preserves all limits of $\mathbf{Ab}(\mathcal{C})$.

Exercise 69 Show that if $\mathcal{C}$ is a small category, then there exists an isomorphism of categories $\widehat{\mathcal{C}}(\mathbf{Ab}) \simeq \mathbf{Ab}(\widehat{\mathcal{C}})$.

Exercise 70 Show that if $\mathcal{C}$ is a small cartesian category, then there exists a fully faithful functor $\mathbf{Ab}(\mathcal{C}) \rightarrowtail \widehat{\mathcal{C}}(\mathbf{Ab})$.

Exercise 71 Define the category k-$\mathbf{Alg}(\mathcal{C})$ of k-algebras of a cartesian category $\mathcal{C}$ when k is a commutative ring of $\mathcal{C}$.

Exercise 72 Show that $k[t]$ endowed with $t \mapsto t \otimes 1 + 1 \otimes t$ and $t \mapsto t \otimes t$ is a commutative ring of the category opposite to the category of k-algebras.

Exercise 73 Show that a coproduct of projectives is projective (and dual).

Exercise 74 Show that if X is projective, then

1. any epimorphism $Y \twoheadrightarrow X$ is split,
2. if a $Y \rightarrowtail X$ (resp. $X \twoheadrightarrow Y$) is a split monomorphism (resp epimorphism), then Y also is projective (and dual).

Exercise 75 Show that

1. if each X_i is projective, then $(X_i)_{i \in I}$ is projective in $\mathcal{C} := \prod_{i \in I} \mathcal{C}_i$,
2. if each $\mathcal{C}_i$ has enough projectives, so does $\mathcal{C}$.

Exercise 76 An *isogeny* is a homomorphism of abelian groups $f : M \to N$ such that

1. $\forall y \in N, \forall n \in \mathbb{Z} \setminus 0, \exists x \in M, f(nx) = y$,
2. $\forall x \in M, f(x) = 0 \Rightarrow \exists n \in \mathbb{Z} \setminus 0, nx = 0$.

Show that, if W is the collection of all isogenies, then $W^{-1}\mathbf{Ab} \simeq \mathbb{Q}$-$\mathbf{Vec}$ ($:= \mathbb{Q}$-$\mathbf{Mod}$).

Exercise 77 The category of topological spaces up to homotopy has topological spaces as objects but sets of morphisms $[X, Y] := \mathcal{C}(X, Y)/\sim$ and composition induced by usual composition. Show that it is equivalent to $W^{-1}\mathbf{Top}$ if W denotes the collection of homotopy equivalences.

Exercise 78 Show that if $\mathcal{C}$ admits right calculus of fraction with respect to W, then the localization functor $Q : \mathcal{C} \to W^{-1}\mathcal{C}$ is exact.

Exercise 79 Show that, if $\mathcal{P}$ is a preadditive category, then $\mathcal{P}^{\mathrm{op}}$ is preadditive too.

Exercise 80 Show that $h^M : \mathcal{P} \to \mathbf{A}\mathrm{b}$ still preserves all limits and is in particular left exact (and dual).

Exercise 81 Show that

1. if A, B are two rings, then $\mathrm{Hom}_{\mathbf{Ring}}(A, B) \simeq \mathrm{Hom}_+(\mathbf{A}, \mathbf{B})$,
2. If A is a ring, then $\mathbf{A}\text{-}\mathbf{M}\mathrm{od} \simeq A\text{-}\mathbf{M}\mathrm{od}$.

Exercise 82 Prove the additive Yoneda lemma: if $\mathcal{P}$ is a small[a] preadditive category, $M \in \mathcal{P}$ and $F \in \mathcal{P}\text{-}\mathbf{M}\mathrm{od}$, then there exists a natural isomorphism of abelian groups $\mathrm{Hom}(h^M, F) \simeq F(M)$ (and dual).

[a]The statement generalizes to large categories.

Exercise 83 Show that, if $\mathcal{P}$ is an additive category, then $\mathcal{P}^{\mathrm{op}}$ is additive too. Show also that, if I is a small category, then $\mathcal{P}^I$ is additive.

Exercise 84 Show that, if we are given two morphisms $f = M \to N$ and $g : N \to P$ in an additive category, and decompositions $M = \bigoplus_{k=1}^m M_k$ and $N = \bigoplus_{j=1}^n N_j$ and $P = \bigoplus_{i=1}^p N_p$, then $g \circ f = [h_{ik}]$ with $h_{ik} = \sum_j g_{ij} \circ f_{jk}$.

Exercise 85 Show that the forgetful functor $\mathbf{A}\mathrm{b}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism when $\mathcal{P}$ is additive.

Exercise 86 Show that a category $\mathcal{P}$ is additive if and only if there exists a (not unique) cartesian category $\mathcal{C}$ such that $\mathbf{A}\mathrm{b}(\mathcal{C}) \simeq \mathcal{P}$.

Exercise 87 Show that

1. a subcategory $\mathcal{P}'$ of an additive category $\mathcal{P}$ is an additive subcategory if and only if it is stable under finite direct sums: if $M_1, \ldots, M_n \in \mathcal{P}'$, then $\bigoplus_{k=1}^n M_k \in \mathcal{P}'$.
2. a functor $F : \mathcal{P} \to \mathcal{Q}$ between additive categories is additive if and only if it preserves finite direct sums: if $M_1, \ldots, M_n \in \mathcal{P}$, then $F(\bigoplus_{k=1}^n M_k) = \bigoplus_{k=1}^n F(M_k)$.

Exercise 88 Let $\mathcal{Q}$ be an additive category. Show that, if a fully faithful functor $\mathcal{P} \rightarrowtail \mathcal{Q}$ has an adjoint or a coadjoint, then $\mathcal{P}$ also is additive.

Exercise 89 Show that, if $\mathcal{P}$ is a small preadditive category and $\mathcal{Q}$ is additive, then $\mathbf{H}\mathrm{om}_+(\mathcal{P}, \mathcal{Q})$ is additive.

Exercise 90 Show that

1. a sequence $0 \to M \xrightarrow{f} N \to 0$ is left exact if and only if it is right exact[a] if and only if f is an isomorphism,
2. if $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is left exact (resp. short exact), then f is an isomorphism if and only if $g = 0$ (resp. $M'' = 0$) – and dual.

[a]So that there is no ambiguity in saying that such a sequence is exact.

Exercise 91 Show that two sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to N' \to N \to N'' \to 0$ are left (resp. right, resp. short) exact if and only if the (corresponding) sequence $0 \to M' \oplus N' \to M \oplus N \to M'' \oplus N'' \to 0$ is left (resp. right, resp. short) exact.

Exercise 92 Show that any commutative diagram with exact rows

$$\begin{array}{ccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \end{array}$$

may be uniquely completed as described (and dual).

Exercise 93 Show that, in A-**Mod**, any short exact sequence is isomorphic to some

$$0 \to M' \to M \to M/M' \to 0$$

where M' is a submodule of M.

Exercise 94 Show that a diagram

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

is cartesian (resp. cocartesian, resp. both cartesian and cocartesian) if and only if the sequence

$$0 \longrightarrow M' \xrightarrow{\left[\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right]} M \oplus N' \xrightarrow{[f\ \ -g]} N \longrightarrow 0$$

is left exact (resp. right exact, resp. short exact).

Exercise 95 Show that a sequence

$$0 \to M' \to M \to M'' \to 0$$

is left exact (resp. right exact, resp. short exact) if and only if the diagram

$$\begin{array}{ccc} M' & \longrightarrow & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M'' \end{array}$$

is cartesian (resp. cocartesian, resp. both cartesian and cocartesian).

Exercise 96 Show that if A is a commutative ring, M is an A-module and $f, g \in A$ are comaximal (the ideal generated by f and g is the full ring) then the sequence

$$0 \to M \to M_f \times M_g \overset{-}{\to} M_{fg} \to 0$$

is exact.

Exercise 97 A *representation* of a monoid G (or a G-*module*) is a morphism $\rho : G \to \mathrm{End}_k(M)$ where M is a module on some fixed commutative ring k. A morphism of representations is a k-linear map $M \to N$ which is compatible with the actions of G. Show that they form an abelian category $\mathbf{Rep}_k(G)$.

Exercise 98 Let $\mathcal{O}$ be the ring of holomorphic functions on an open subset U of $\mathbb{C}$. A *derivation* on an $\mathcal{O}$-module M is a group homomorphism $\partial_M : M \to M$ such that $\forall s \in M, \partial(fs) = f's + f\partial_M(s)$. An $\mathcal{O}$-linear map $f : M \to N$ between modules with derivations is sait to be *horizontal* if it commutes with the derivations. Show that modules with derivations and horizontal maps form an abelian category.

Exercise 99 Show that an abelian subcategory is indeed an abelian category.

Exercise 100 Show that a functor $F : \mathcal{A} \to \mathcal{B}$ between two preabelian categories is left exact if and only if it is additive and preserves left exact sequences (and dual).

Exercise 101 Let $\mathcal{B}$ be an abelian category. Show that, if a fully faithful functor $\mathcal{A} \rightarrowtail \mathcal{B}$ has an exact adjoint or coadjoint, then $\mathcal{A}$ also is abelian.

Exercise 102 Show that the category of free abelian groups of finite rank (or equivalently $\mathbf{Mat}_{\mathbb{Z}}$) is preabelian but that the inclusion in the category of all abelian groups is not an exact functor.

Exercise 103 Show that the condition for being exact is autodual: $\operatorname{coker} f = \operatorname{im} g$ if and only if $\operatorname{im} f = \ker g$.

Exercise 104 Show that a sequence

$$0 \longrightarrow M' \overset{i}{\longrightarrow} M \overset{p}{\longrightarrow} M'' \longrightarrow 0$$

is left exact (resp. right exact, resp. short exact) if and only if it is exact in M', M (resp. M, M'', resp. M', M, M'').

Exercise 105 Show that a sequence

$$0 \longrightarrow M' \longrightarrow M \overset{p}{\longrightarrow} M'' \longrightarrow 0$$

is short exact if and only if it is left exact and p is an epimorphism (and dual).

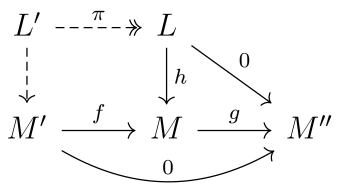
Exercise 106 Show that $M \overset{f}{\to} N \overset{g}{\to} Q \overset{h}{\to} R$ is exact if and only if it splits into right and left exact sequences

$$M \xrightarrow{f} N \to P \to 0 \quad \text{and} \quad 0 \to P \to Q \xrightarrow{h} R.$$

Exercise 107 Let $\mathcal{B}$ be another abelian category and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. Show that the following are equivalent:

1. F is exact,
2. if $M' \to M \to M''$ is exact (in the middle), then $F(M') \to F(M) \to F(M'')$ is exact in the middle,
3. F preserves short exact sequences.

Exercise 108 Show that a sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact in M if and only if $g \circ f = 0$ and, given any $h : L \to M$ such that $g \circ h = 0$, there exists an epimorphism $\pi : L' \twoheadrightarrow L$ such that $h \circ \pi$ factors through f:



Exercise 109 Show that if

$$\begin{array}{ccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \end{array}$$

is a commutative diagram with exact rows, then there exists a natural autodual (long) exact sequence

$$\ker f' \to \ker f \to \ker f'' \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f''.$$

Exercise 110 Prove the four-lemma: if

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 \end{array}$$

is a commutative diagram with exact rows with f_1 an epimorphism and f_2, f_4 monomorphisms, then f_3 is a monomorphism (and dual).

Exercise 111 Prove the five-lemma: if

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow\!\!\downarrow f_1 & & \simeq \downarrow f_2 & & \downarrow f_3 & & \simeq \downarrow f_4 & & \rightarrowtail f_5 \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

is a commutative diagram with exact rows with f_1 an epimorphism and f_2, f_4 isomorphisms and f_5 a monomorphism, then f_3 is an isomorphism.

Exercise 112 Show that if E is an extension of M by N, $f : M' \to M$ and $g : N \to N'$, then $f^* g_* E = g_* f^* E$ (up to isomorphism).

Exercise 113 Show that $\text{Ext}(M, N)$ is an abelian group for the Baer sum.

Exercise 114 Show that (and dual)

1. if $g : N \to N'$ then $g_* : \text{Ext}(M, N) \to \text{Ext}(M, N')$ is a morphism of groups,
2. if E is an extension of M by N then the map

$$\text{Hom}(M', M) \to \text{Ext}(M', N), \quad f \mapsto f^* E$$

is a morphism of groups.

Exercise 115 Show that if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then there exists a long exact sequence (and dual)

$$0 \longrightarrow \text{Hom}(M, N') \longrightarrow \text{Hom}(M, N) \longrightarrow \text{Hom}(M, N'') \longrightarrow \text{Ext}(M, N') \longrightarrow \text{Ext}(M, N) \longrightarrow \text{Ext}(M, N'').$$

Exercise 116

1. Show that the inclusion $\mathbf{1} = \{0\} \rightarrowtail (\mathbb{Z}, \leq)$ induces a functor $\mathcal{P} \rightarrowtail \mathbf{C}(\mathcal{P})$ which is fully faithful and preserves all limits and colimits. We shall identify $\mathcal{P}$ with its image in $\mathbf{C}(\mathcal{P})$ so that $M^0 = M$ and $M^n = 0$ otherwise.
2. Show that the inclusion $[\mathbf{1}] := \{0 < 1\} \rightarrowtail (\mathbb{Z}, \leq)$ induces a functor

$$\mathbf{Mor}(\mathcal{P}) \rightarrowtail \mathbf{C}(\mathcal{P}), \quad (M \xrightarrow{f} N) \mapsto [M \xrightarrow{f} N]$$

(with M in degree 0 and N in degree 1) which again is fully faithful and preserves all limits and colimits.
3. Analog with left, right or short exact sequences ?

Exercise 117 Show that if $K_\bullet$ is a (semi) simplicial object of $\mathcal{P}$ and we set $d_n := \sum_{i=0}^{n-1} (-1)^i d_n^i$, then $K_\bullet$ becomes a chain complex. Show that this provides a functor $\mathbf{\Delta}(\mathcal{P}) \to \mathbf{C}(\mathcal{P})$ (called the Dold-Kan correspondance[a]).

[a]When $\mathcal{P}$ is abelian, we obtain an equivalence between simplicial complexes and chain complexes indexed by $\mathbb{N}$.

Exercise 118 Show that there exists an adjunction between differential objects and complexes.

Exercise 119 Show that morphisms that are homotopic to 0 form a subgroup of $\mathrm{Hom}_{\mathbf{C}(\mathcal{P})}(K^\bullet, L^\bullet)$.

Exercise 120 Show that if $f \sim g : K^\bullet \to L^\bullet$ and $\varphi : L^\bullet \to M^\bullet$ (resp. $\psi : J^\bullet \to K^\bullet$), then $\varphi \circ f \sim \varphi \circ g$ (resp. $f \circ \psi \sim g \circ \psi$).

Exercise 121
1. Show that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a homotopy equivalence if and only if there exists a morphism $g : L^\bullet \to K^\bullet$ such that $g \circ f \sim \mathrm{Id}$ and $f \circ g \sim \mathrm{Id}$.
2. Show that a complex $K^\bullet$ is contractible if and only if $\mathrm{Id}_{K^\bullet} \sim 0_{K^\bullet}$.

Exercise 122 Show that a complex $[M \xrightarrow{\mathrm{Id}} M]$ is always contractible but the complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$

is not (although the sequence is exact).

Exercise 123 Show that any additive functor $F : \mathcal{P} \to \mathcal{P}'$ provides a functor

$$F^\bullet : \mathbf{K}(\mathcal{P}) \to \mathbf{K}(\mathcal{P}').$$

Exercise 124 Show that the mapping cone is indeed a complex.

Exercise 125 Show that, conversely, if $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ is a termwise split extension, then there exists a morphism $h : M^\bullet[-1] \to K^\bullet$ and an isomorphism $L^\bullet \simeq \mathrm{M}(h)$ (of extensions).

Exercise 126 Show that
1. $K^\bullet \xrightarrow{\mathrm{Id}} K^\bullet \to 0 \to K^\bullet[1]$ and
2. $[M \xrightarrow{f} N] \to M \xrightarrow{f} N \to [M \xrightarrow{f} N][1]$

are always distinguished.

Exercise 127 Show that
1. if $0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$ is a termwise split extension, then there exists a distinguished triangle $K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to K^\bullet[1]$,
2. conversely, any distinguished triangle is isomorphic to a triangle coming form a termwise split extension.

Exercise 128 Show that $\mathbf{C}(\mathcal{A})$ also is abelian.

Exercise 129 Show that cohomology provides us with an additive functor

$$\mathrm{H}^n : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$$

which is *not* left or right exact in general.

Exercise 130 Show that if Q is a divisible group, then

$$\mathrm{H}^n_{\mathrm{sing}}(X,Q) \simeq \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{H}^{\mathrm{sing}}_n(X),Q).$$

Exercise 131 Show that, if $H : \mathbf{K}(\mathcal{P}) \to \mathcal{A}$ is a *cohomological* functor and $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ a distinguished triangle, then there exists a natural long exact sequence of complexes

$$\cdots \to H(K^\bullet[n]) \to H(L^\bullet[n]) \to H(M^\bullet[n]) \to H(K^\bullet[n+1]) \to \cdots .$$

Exercise 132 Show that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$ if and only if the morphism $M' \to [M \to M'']$ (resp. $[M' \to M][1] \to M''$) is a quasi-isomorphism.

Exercise 133 Assume $K^\bullet \xrightarrow{f} L^\bullet \to M^\bullet \to K^\bullet[1]$ is a distinguished triangle. Show that f is a quasi-isomorphism if and only if $M^\bullet$ is acyclic (and idem with a short exact sequence).

Exercise 134 Show that, if

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1] \end{array}$$

is a morphism of distinguished triangles and two among u, v and w are quasi-isomorphisms, then so is the third (and idem with a short exact sequences).

Exercise 135 Prove the following :

1. if $M = M' \oplus M''$, then M is injective if and only if both M' and M'' are injective (and dual).
2. if $0 \to M' \to M \to M'' \to 0$ is an exact sequence with M' injective, then M is injective if and only if M'' is (and dual).

Exercise 136 Assume $\mathcal{B}$ is another abelian category and $F : \mathcal{A} \to \mathcal{B}$ is adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$. Show that (and dual)

1. if F is exact then G preserves injectives,
2. if F is faithful exact and $\mathcal{B}$ has enough injectives, then $\mathcal{A}$ too has enough injectives.

Exercise 137 Show that if E is an open covering of a topological space X, M is an abelian group, then there exists a complex $C^\bullet(E,M)$ with $C^n(E,M) := \mathcal{F}(S_n(E),M)$ and a quasi-isomorphism

$$C^\bullet(X,M) \twoheadrightarrow C^\bullet(E,M).$$

Exercise 138 Show that if X is a contractible topological space (resp. and M is an abelian group), then $C_\bullet(X)$ (resp. $C^\bullet(X,M)$) is a left (resp. right) resolution of $\mathbb{Z}$ (resp. M).

Exercise 139 Show that, if $\mathbb{R}^n$ is seen as a smooth manifold, then $\Omega^\bullet(\mathbb{R}^n)$ is a right resolution of $\mathbb{R}$.

Exercise 140 Show that $\mathbf{D}(\mathcal{A})$ is also the localization of $\mathbf{C}(\mathcal{A})$ at quasi-isomorphisms (even it this last category does not admits right or left calculus of fractions).

Exercise 141 Show that $\mathrm{Ext}^n(M,N) = 0$ for $n < 0$ and $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$ if $M, N \in \mathcal{A}$.

Exercise 142 Show that $\mathrm{Ext}^1(M,N) \simeq \mathrm{Ext}(M,N)$ when $\mathcal{A}$ has enough injectives or projectives (although this condition is not necessary).

Exercise 143 Assume $\mathcal{A}$ has enough injectives. Show that $I \in \mathcal{A}$ is injective if and only if $\mathrm{Ext}^n(M,I) = 0$ for all $M \in \mathcal{A}$ and $n \neq 0$ (and dual).

Exercise 144 Show that if M, N are two abelian groups, then $\mathrm{Ext}^n(M,N) = 0$ for $n \neq 0, 1$.

Exercise 145 Show that if M is an abelian group, then

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, M) \simeq M/nM.$$

Exercise 146 Show that, in the category of abelian groups,

$$\mathrm{Ext}^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0,1 \\ 0 \text{ otherwise} \end{cases}$$

with $d = m \wedge n$ (for $m, n > 0$).

Exercise 147 Show that

$$\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0,1 \\ 0 \text{ otherwise} \end{cases}$$

with $d = m \wedge n$ (for $m, n \geq 1$).

Exercise 148 Show that, if G is a group and $k = \mathbb{Z}$, then $\mathrm{H}^1(G,M) \simeq \mathrm{Z}(G,M)/\mathrm{B}(G,M)$ where

$$\mathrm{Z}(G,M) := \{f : G \to M / \forall g, h \in G, f(gh) = f(g) + gf(h)\}$$

is the set of *crossed homomorphisms* and

$$\mathrm{B}(G,M) := \{f : G \to M, g \mapsto gm - m : m \in M\}$$

is the set of *principal homomorphisms*.

Exercise 149 Assume G is a group and $k = \mathbb{Z}$. Compute $\mathrm{H}^1(G, M)$ when

1. the action of G on M is trivial,
2. $M = \mathbb{Z}$ with the non-trivial action of $\mu_2 := \{1, -1\}$.

Exercise 150 Show that

$$\mathrm{R}^n\mathrm{Sol}(M) \simeq \begin{cases} \ker \partial_M & \text{if } n = 0 \\ \operatorname{coker} \partial_M & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 151 Show that an object is injective if and only if it is F-acyclic for all left exact functors F if and only if this is the case when $F = \mathrm{Hom}(M, -)$ for all $M \in \mathcal{A}$ (and dual).

Exercise 152 Show that the condition is automatic when F has an exact adjoint.

Exercise 153 Show that $\widehat{X}$ is a cartesian category with final object given by $1(U) = 1 := \{0\}$ for all open subsets U of X and such that

$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$

for all presheaves of sets $\mathcal{F}$ and $\mathcal{G}$.

Exercise 154 Show that, if $\mathcal{A}$ is a presheaf of rings on X, then a presheaf of $\mathcal{A}$-modules is (the same thing as) a presheaf of sets $\mathcal{M}$ together with a structure of $\mathcal{A}(U)$-module on $\mathcal{M}(U)$ for U open in X such that

1. $\forall s, s' \in \mathcal{M}(U), \quad (s + s')_{|V} = s_{|V} + s'_{|V}$ and
2. $\forall f \in \mathcal{A}(U), \forall s \in \mathcal{M}(U), \quad (fs)_{|V} = f_{|V} s_{|V}$

for V open in U. Show that a morphism of presheaves of $\mathcal{A}$-modules $\alpha : \mathcal{M} \to \mathcal{N}$ is a morphism of presheaves of sets such that $\alpha_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{A}(U)$-linear whenever U is an open subset of X.

Exercise 155 Show that, if $\mathcal{F}$ is a presheaf of sets (resp. $\mathcal{M}$ is a presheaf of $\mathcal{A}$-modules), then there exists a bijection (resp. an isomorphism)

$$\mathrm{Hom}(1_U, \mathcal{F}) \simeq \mathcal{F}(U) \quad \left(\text{resp. } \mathrm{Hom}_{\mathcal{A}}(\widehat{\mathcal{A}}_U, \mathcal{M}) \simeq \mathcal{M}(U)\right)$$

with

$$1_U(V) = \begin{cases} 1 := \{0\} & \text{if } V \subset U, \\ 0 := \emptyset & \text{otherwise} \end{cases} \quad \left(\text{resp. } \widehat{\mathcal{A}}_U(V) = \begin{cases} \mathcal{A}(U) & \text{if } V \subset U, \\ \{0\} & \text{otherwise} \end{cases}\right).$$

Exercise 156 Show that the forgetful functor $\widehat{\mathbb{Z}_X\text{-}\mathbf{Mod}} \to \widehat{X}(\mathbf{Ab})$ is an isomorphism of categories.

Exercise 157 Show that the forgetful functor $\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{X}$ has an adjoint.

Exercise 158 Show that the functor

$$\widehat{X} \to \mathbf{Set}^{\mathrm{Open}(X)}, \quad \mathcal{F} \to (\mathcal{F}(U))_{U \in \mathrm{Open}(X)}$$

1. is faithful and conservative,
2. preserves (and reflects) all limits and colimits.

Exercise 159 Show that $\widehat{\mathcal{A}\text{-}\mathbf{Mod}}$ is an abelian category (satisfying AB6 and AB4*).

Exercise 160 Show that the stalk functor $\mathcal{F} \mapsto \mathcal{F}(x)$ on $\widehat{X}$ is exact and preserves all colimits.

Exercise 161 Show that $\mathcal{O}_X(x)$ is a local ring when X is a manifold.

Exercise 162 Show that, if U is an open subset of X, then the restriction has and adjoint and a coadjoint given respectively by[a]

$$\mathcal{F} \mapsto \left(V \mapsto \begin{cases} \mathcal{F}(V) \text{ if } V \subset U \\ \emptyset \text{ otherwise} \end{cases} \right) \quad \text{and} \quad \mathcal{F} \mapsto (V \mapsto \mathcal{F}(U \cap V)).$$

[a] Use $\{0\}$ in place of $\emptyset$ in the algebraic case.

Exercise 163 Show that, if $R \in J(X)$ and U is an open subset of X, then $R \cap \mathrm{Open}(U) \in J(U)$.

Exercise 164 Show that if $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}(\emptyset) = 1$.

Exercise 165 Show that a sheaf $\mathcal{F}$ is uniquely determined by $\mathcal{F}(V)$ when V runs through a basis $\mathcal{B}$ of open subsets of X.

Exercise 166 Show[a] that a presheaf $\mathcal{F}$ with values in a complete category $\mathcal{D}$ is a sheaf if and only if the presheaf $U \mapsto \mathrm{Hom}(E, \mathcal{F}(U))$ is a sheaf of sets whenever $E \in \mathcal{D}$.

[a] This provides a way to extend the definition of a sheaf even when there are not enough limits in $\mathcal{D}$.

Exercise 167 Show that a presheaf of abelian groups or rings is a sheaf if and only if the underlying presheaf of sets is sheaf.

Exercise 168 Show that if $\mathcal{A}$ is a sheaf of rings on X, then

$$\mathcal{A}\text{-}\mathrm{Mod} \simeq \mathcal{A}\text{-}\mathbf{Mod}(\widetilde{X})$$

(a sheaf of $\mathcal{A}$ modules is essentially the same thing as an $\mathcal{A}$-module in the category of sheaves of sets).

Exercise 169 Show that $\mathcal{O}_X$ is a sheaf on $X := \mathrm{Spec}(A)$ and that $\widetilde{M}$ is an $\mathcal{O}_X$-module when M is an A-module.

Exercise 170 Show that, if $X = \mathrm{Spec}(A)$, then there exists an adjunction

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{N}) \simeq \mathrm{Hom}_A(M, \Gamma(X, \mathcal{N}))$$

if M is an A-module and $\mathcal{N}$ an $\mathcal{O}_X$-module.

Exercise 171 An $\mathcal{A}$-module is said to be *locally free of rank* n if it is locally isomorphic to $\mathcal{A}^n$. Show that the category of vector bundles on a manifold X is equivalent to the category of locally free $\mathcal{O}_X$-modules of finite rank.

Exercise 172 Show that, if $\mathcal{F} \to \widetilde{\mathcal{F}}$ is an epimorphism of presheaves, then $\widetilde{\mathcal{F}} = \check{\mathcal{H}}(\mathcal{F})$.

Exercise 173 Show that, if E is a set, then $E_X \simeq \mathcal{C}_E^X$ when E is endowed with the discrete topology, and that there exists an adjunction

$$\mathrm{Hom}(E_X, \mathcal{F}) \simeq \mathrm{Hom}(E, \Gamma(X, \mathcal{F}))$$

if $\mathcal{F}$ is a sheaf.

Exercise 174 Show that if x is a point, then there exists an isomorphism of categories $\widetilde{x} \simeq \mathbf{Set}$ given by $\mathcal{F} \mapsto E := \mathcal{F}(x)$ (and we shall identify both categories).

Exercise 175 Show that, if $\mathcal{M}$ is an $\mathcal{A}$-module, then there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_U, \mathcal{M}) \simeq \mathcal{M}(U)$$

where $\mathcal{A}_U$ is the sheafification of $\widehat{\mathcal{A}}_U$.

Exercise 176 Show that the forgetful functor $\mathbb{Z}_X$-$\mathbf{Mod} \to \widetilde{X}(\mathbf{Ab})$ is an isomorphism.

Exercise 177 Show that the forgetful functor $\mathcal{A}$-$\mathbf{Mod} \to \widetilde{X}$ has an adjoint $\mathcal{F} \mapsto \mathcal{A} \cdot \mathcal{F}$.

Exercise 178 Show that the functor $\mathcal{F} \to (\mathcal{F}(x))_{x \in X}$ is adjoint to the functor that sends $(E_x)_{x \in X}$ to $\mathcal{E}$ with $\mathcal{E}(U) = \prod_{x \in U} E_x$.

Exercise 179 Show that the functor $\mathcal{F} \to (\mathcal{F}(x))_{x \in X}$ is an equivalence when X is discrete.

Exercise 180 Show that the following are equivalent:

1. $\alpha : \mathcal{F} \to \mathcal{G}$ is an epimorphism,
2. for all U open in X and all $t \in \mathcal{G}(U)$, there exists an $R \in J(U)$ and $s_v \in \mathcal{F}(V)$ for all $V \in R$ such that $t_{|V} = \alpha_V(s_V)$,
3. for all U open in X and all $t \in \mathcal{G}(U)$, there exists an open covering $U = \bigcup_{i \in I} U_i$ and for all $i \in I$, $s_i \in \mathcal{F}(U_i)$ such that $t_{|U_i} = \alpha_{U_i}(s_i)$.

Exercise 181 Show that

1. if $\mathcal{F}$ is a presheaf on X and U is an open subset, then $(\widetilde{\mathcal{F}})_{|U} \simeq \widetilde{\mathcal{F}_{|U}}$,
2. $\mathcal{F}_{|U}$ is automatically a sheaf when $\mathcal{F}$ is a sheaf,
3. the corresponding functor $\widetilde{X} \to \widetilde{U}$ has both an adjoint and a coadjoint.

Exercise 182 Show that if $\mathcal{I}$ is an injective $\mathcal{A}$-module, then $\mathcal{I}_{|U}$ also is injective.

Exercise 183 Show that $\mathrm{H}^n(X, \mathcal{M}^\bullet) \simeq \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{M}^\bullet)$ for all $n \in \mathbb{Z}$.

Exercise 184 Show that, if $X = \coprod_{i \in I} X_i$, then $\mathrm{H}^n(X, \mathcal{M}^\bullet) \simeq \prod_{i \in I} \mathrm{H}^n(X_i, \mathcal{M}^\bullet_{|X_i})$.