APPLICATIONS OF RIGID COHOMOLOGY TO ARITHMETIC GEOMETRY

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Claudine, Lola et Colin

qui m' ont donné ce bonheur de vivre

sans lequel aucun travail créatif ne me semble possible

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ABSTRACT

Rigid cohomology is a new cohomological theory which is due to P. Berthelot. We use it to prove a trace formula which can be applied to exponential sums and to study the de Rham cohomology of a curve over a complete ultrametric field of characteristic zero.

Let X be a separated scheme of finite type over a finite field $\mathbf{F}_q$, K a complete ultrametric field of characteristic zero whose residue field contains $\mathbf{F}_q$ and $\mathcal{F}$ an overconvergent F-isocrystal on X/K. Given any rational point x on X, there is a finite dimensional vector space $\mathcal{F}(x)$, called the fiber of $\mathcal{F}$ at x, and an automorphism $\Phi(x)$ of $\mathcal{F}(x)$ called the Frobenius of $\mathcal{F}(x)$. Given any integer i, there is a vector space $H^i_c(\mathcal{F})$, called the i-th rigid cohomology space with proper support of $\mathcal{F}$, and an automorphism Φ^i_c of $H^i_c(\mathcal{F})$ called the Frobenius of $H^i_c(\mathcal{F})$. We prove that, for all i, Φ^i_c is a nuclear operator and that

$$\sum_{x\in X(\mathbf{F}_q)} \mathrm{tr}\,\Phi(x) = \sum_{i\in \mathbf{Z}} (-1)^i\,\mathrm{tr}\,\Phi^i_c.$$

Let C be a non singular projective curve over a complete ultrametric field K of characteristic zero with perfect residue field and $\mathfrak{X}$ a flat formal scheme over the valuation ring of K whose generic fibre is isomorphic to the analytification of C and whose special fibre X is reduced. We show that $H^1_{rig}(X)$ is a subspace of $H^1_{DR}(C)$ and that, if $\tilde{X}$ is the normalization of X, $H^1_{rig}(\tilde{X})$ is a quotient of $H^1_{rig}(X)$. We define the weight filtration on $H^1_{DR}(C)$ as the shortest filtration for which $\mathrm{Fil}^1 = H^1_{rig}(X)$ and $\mathrm{Gr}^1 = H^1_{rig}(\tilde{X})$. We show that if we consider the orthogonal filtration with respect to the Poincaré pairing, then $\mathrm{Fil}^2 = \mathrm{Fil}^1 \cap \mathrm{Fil}^{-1}_{\perp}$ and $\mathrm{Fil}^{-1}_{\perp} = \mathrm{Fil}^0_{\perp} \cup \mathrm{Fil}^2$ and that the filtration is autodual if and only if X has only ordinary multiple points (with normal tangents) as singularities.

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I whish to thank Steven Sperber for introducing me to p-adic number theory. I am deeply indebted to him since it is his questions about Berthelot's article on Dwork Theory and Rigid cohomology which lead me to the trace formula which appears here.

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TABLE OF CONTENTS

INTRODUCTION

Rigid cohomology is a cohomological theory for separated varieties of finite type over a field k which is due to Berthelot. The category of 'lisse' coefficients for this theory are called overconvergent F-isocrystals. The cohomological spaces are vector spaces over a complete ultrametric field of characteristic zero having k as residue field.

The Trace Formula for Overconvergent F-isocrystals

Inspired by a result of Tsen, Artin made a conjecture concerning the p-divisibility of the number of rational points of an hypersurface in $\mathbf{A}^n_{\mathbf{F}_q}$. This conjecture was first proven by Chevalley and further improved by Warning and Ax to give the classical theorem that the number of rational points is divisible by q.

Following the work of Ax, Katz conjectured that the Newton polygon of a smooth complete intersection in the projective space over a perfect field of positive characteristic p lies above its Hodge polygon. This was already known to Dwork in the case of a projective hypersurface whose degree is prime to p and Katz was able to prove it for the first slopes. Mazur and Ogus showed that this conjecture is actually true for any proper smooth variety.

In the meantime, Sperber and Adolphson applied the method of Katz to obtain p-adic estimates for exponential sums. Before that, the question of p-divisibility of exponential sums had not received much interest since Stickelberger's study of Gauß sums. We can only mention Dwork's application of his study of Bessel function to Kloosterman sums.

In the proof of his theorem, Ax uses Dwork Theory to obtain p-adic estimates. In the work of Sperber and Adolphson, these methods, and in particular p-adic trace formulas, play a major role. Pre-cohomological p-adic trace formulas have been established by Dwork for the affine space, by Reich for the complement of an hypersurface and, as an intermediate step, by Monsky for a smooth affine variety. Cohomological p-adic trace formulas have been obtained by Berthelot and Etesse for proper smooth varieties and by Monsky for smooth affine varieties.

An exponential sum on a (separated) $\mathbf{F}_q$-scheme of finite type X is a sum

$$S := S(X, \chi_i, h_i, \psi_j, f_j) := \sum_{x \in X(\mathbf{F}_q)} \prod_{i=1}^{r} \chi_i(h_i(x)).\prod_{j=1}^{s} \psi_j(f_j(x))$$

where for $i \in \{1, \ldots, r\}$, χ_i is a multiplicative character of $\mathbf{F}_q$ and h_i is a nowhere vanishing regular function on X and for $j \in \{1, \ldots, s\}$, ψ_j is an additive character of $\mathbf{F}_q$ and f_j is a regular function on X. A pre-cohomological p-adic trace formula is an equality S = tr u where u is an endomorphism of a vector space E over a complete ultrametric field of characteristic zero whose residue field contains $\mathbf{F}_q$. Generally, E will not have finite dimension and it is necessary to specify what is meant by the trace of u.

If $(\mathcal{F}, \Phi)$ is an overconvergent F-isocrystal on X, the fiber of $\mathcal{F}$ at a rational point x on X is a finite dimensional K-vector space $\mathcal{F}(x)$ together with an automorphism $\Phi(x)$ called its Frobenius. To the above data χ_i, h_i, ψ_j, f_j, Berthelot associates an overconvergent F-isocrystal $\mathcal{L}$ on X and we have

$$S(X, \chi_i, h_i, \psi_j, f_j) = \sum_{x \in X(\mathbf{F}_q)} \mathrm{Tr}\, \Phi(x).$$

By analogy, we define the exponential sum associated to an overconvergent F-isocrystal is $S(X, \mathcal{F}) := \sum_{x \in X(\mathbf{F}_q)} \mathrm{Tr}\, \Phi(x)$. To prove a cohomological trace formula for exponential sums, it is therefore sufficient to prove a trace formula for overconvergent F-isocrystals.

The i-th space of rigid cohomology with compact support of an overconvergent F-isocrystal $\mathcal{F}$ is a K-vector space $H^i_{rig,c}(X, \mathcal{F})$ together with an automorphism Φ^i_c called its Frobenius. The trace formula for overconvergent F-isocrystals states that for all $i \in \mathbf{N}$, ϕ^i_c is a nuclear operator and that

$$S(X, \mathcal{F}) := \sum_{i \in N} (-1)^i \operatorname{tr} \phi_c^i.$$

We will give here a complete proof of this formula but no applications to exponential sums.

The Weight Filtration on the de Rham Cohomology of a Curve. Orthogonality Theorem:

If A is an abelian variety over the fraction field K of an Henselian discrete valuation ring $\mathcal{V}$ and ℓ is a prime different from the characteristic p of the residue field k of $\mathcal{V}$, Grothendieck defined a filtration (weight filtration)

$$T_\ell(A(\bar{K})^t \subset T_\ell(A(\bar{K})^f \subset T_\ell(A(\bar{K})$$

on the Tate Module of A.

If A' is the dual abelian variety of A, the Poincaré Bundle on A × A' induces a perfect alternating pairing

$$T_\ell(A(\bar{K}) \times T_\ell(A'(\bar{K})) \longrightarrow \mathbf{Z}_\ell(1).$$

Grothendieck showed that, with respect to this pairing, we have

$$T_\ell(A(\bar{K}))^f \cap (T_\ell(A'(\bar{K}))^f)^\perp = T_\ell(A(\bar{K}))^t.$$

If A is the Albanese Variety of a smooth projective variety V over K, the Tate Module of A is dual to $H^1_{et}(V_{\bar{K}}, \mathbf{Z}_\ell)$ and therefore induces a filtration (weight filtration) on the first space of ℓ-adic cohomology of V.

When ℓ = p is the characteristic of k, a similar discussion can be carried out provided we make the further assumptions that $\mathcal{V}$ is complete and that A has (semi-) stable reduction. Also, we have to work with Barsotti-Tate groups instead of Tate Modules.

Grothendieck showed that there is a filtration (weight filtration)

$$T_p(A)^t \subset T_p(A)^f \subset T_p(A)$$

on the Barsotti-Tate group of A and that, with respect to the pairing

$$T_p(A) \times T_p(A') \longrightarrow T_p(G_m)$$

induced by the Poincaré Bundle, we have $T_p(A)^t = T_p(A')^{f,\perp}$. Note that the above orthogonality formula for Tate Modules (case $\ell \neq p$) also takes this form when A has stable reduction.

Assume now that K is a local field (i.e. a complete discretely valued field of mixed characteristic with perfect residue field). Fontaine and Messing proved that an abelian variety A over K is de Rham (i.e. one can recover its de Rham cohomology from its p-adic cohomology). In their first proof, after a reduction to the stable case, they considered the filtration

$$H^1_{DR}(A)^t \subset H^1_{DR}(A)^f \subset H^1_{DR}(A),$$

(weight filtration) which is obtained by applying Dieudonné-Fontaine Theory to the weight filtration on $T_p(A)$. The theorem then followed from the fact that A is Hodge (i.e. one can recover its Hodge cohomology from its p-adic cohomology).

If V is a smooth projective variety and A the Albanese Variety of V, there is a natural isomorphism $H^1_{DR}(V) \cong H^1_{DR}(A)$. It therefore follows from the theorem of Fontaine and Messing that, when $i \leq 1$, $H^i_{DR}(V)$ can be recovered from $H^i_{et}(V, Q_p)$ (the case $i = 0$ is trivial). Fontaine conjectured that this is still true when $i > 1$. Faltings recently gave a proof of the conjecture that any smooth proper variety is Hodge. One may therefore think that, with the right notion of weight filtration on higher de Rham cohomology, one could prove Fontaine's conjecture. To be able to define the weight filtration on H^i for all i, it is necessary to first improve our understanding of the filtration when $i \leq 1$.

In my *thèse de troisième cycle*, I showed that, using rigid cohomology, one can give a purely cohomological interpretation to the weight filtration on the H^1_{DR} of an abelian variety with stable reduction over K. Unfortunately, the construction made an essential use of the group structure of A. It seems therefore natural to ask for a direct construction (i.e. without using its Albanese Variety) of the weight filtration on the H^1_{DR} of an arbitrary variety.

We will define here the weight filtration on the H^1_{DR} of a non singular projective curve over a complete ultrametric field of characteristic zero with perfect residue field and prove an orthogonality theorem for this filtration with respect to the Poincaré pairing.

Unfortunately, many questions about this filtration will not be answered here: What is the relation between the weight filtration, the Hodge filtration and the Analytic Conjugate Filtration on $H^1_{DR}(C)$? Is the weight filtration independent of the choice of a formal model for C? When the valuation is discrete, the residue field k of positive characteristic and the Albanese variety A of C has semi stable reduction, is the natural isomorphism $H^1_{DR}(C) \cong H^1_{DR}(A)$ compatible with the filtrations? When k has positive characteristic, does the Frobenius act on the graded space for the weight filtration? Can we recover the weight filtration from a filtration on $H^1_{et}(C, \mathbf{Q}_p)$? What would be a good definition for the weight filtration on the H^2_{DR} of a surface over K?, . . .

Conventions

We fix a field k and when k has positive characteristic p, a power $q = p^s$ such that $\mathbf{F}_q \subset k$. We fix a complete ultrametric field K with valuation ring $\mathcal{V}$ and maximal ideal $\mathfrak{m}$, having k as residue field. Schemes over k and $\mathbf{F}_q$ are separated and of finite type. (Formal) schemes over $\mathcal{V}$ are (topologically) finitely presented. Schemes over K are locally of finite type. A curve is a scheme of dimension at most 1 over a field. Real numbers are always in $|K^\times| \otimes_{\mathbf{Z}} \mathbf{Q}$.

Outline

Section 1 is devoted to the study of overconvergent isocrystals on a separated scheme of finite type over k. All the results of this section are due to Berthelot.

In section 2 (where K has characteristic zero), we give the construction of the cohomology spaces associated to an overconvergent F-isocrystal and state some of their properties. Again, all the results of this section are due to Berthelot.

The definition of an overconvergent F-isocrystal given in section 3 (where K has characteristic zero and k contains $\mathbf{F}_q$) is slightly different from Berthelot's since we are only interested in schemes over $\mathbf{F}_q$ but want to be able to consider cohomology spaces over K. We define the exponential sum and the L-function associated to an overconvergent F-isocrystal and discuss the relation with the exponential sum associated to a character.

In section 4 (where K has characteristic zero and k contains $\mathbf{F}_q$), we prove that the Frobenius endomorphism of the rigid cohomology spaces with compact support of an overconvergent F-isocrystals are bijective, we state the trace formula and prove its multiplicative corollary.

Section 5 (where K has characteristic zero and k contains $\mathbf{F}_q$) contains a pre-cohomological Trace Formula which can be viewed as dual to the classical ones. Many ideas in the proof are inspired by Monsky's and Berthelot' s work.

In section 6 (where K has characteristic zero and k contains $\mathbf{F}_q$), we prove the trace formula by reducing it to the result of section 5. It is also possible to prove this formula by reducing it to Reich's trace formula.

Section 7 (in which K does not appear) is totally independent of the previous ones. We define and compare some invariants of a curve.

In section 8 (where K has characteristic zero), we compute the dimension of the rigid cohomology spaces of a proper curve. Apart from some basic facts from rigid cohomology (e.g. section 1 and 2), this section makes only use of the results of section 7.

In section 9, which can be read just after section 2, we study the notion of support in rigid analytic geometry and (when K has characteristic zero) its relation to rigid cohomology. We prove some technical results which will be used in the following section.

In Section 10 (where K has characteristic zero), we study the relations between the de Rham cohomology of the generic fibre of a generically smooth proper formal scheme $\mathfrak{X}$ and various cohomology spaces associated to its special fibre X. In particular, when $\mathfrak{X}$ is projective, we describe a pairing between two Gysin sequences relating the de Rham cohomology of the generic fibre of $\mathfrak{X}$, the rigid cohomology of a smooth dense open subset U of the special fibre of $\mathfrak{X}$ and the de Rham cohomology of the tube of the complement of U.

In section 11 (where K has characteristic zero), we prove Poincaré duality for (smooth) curves in rigid cohomology. Poincaré duality in rigid cohomology is known for (smooth) proper schemes when K is discrete and k perfect since it can then be deduced from Poincaré duality in crystalline cohomology. Berthelot can also prove it for (smooth) affine schemes of dimension at most 3. Since a smooth curve is a disjoint union of affine and proper schemes, the main theorem of this section is not really new. The idea in the present proof is to deduce the theorem from an analogous result in analytic de Rham cohomology.

In section 12, we define when K has characteristic zero, the weight filtration on the first de Rham cohomology space of a smooth projective curve over K and prove an orthogonality theorem similar to Grothendieck's.

In section 13, we introduce the notion of nuclear sheaf on the affine K-space which allows us to use cohomological arguments in order to (re-) prove some basic results about nuclear operator on a K-vector space.

In section 14, we introduce the notion of distinguished formal $\mathcal{V}$-scheme which allows us to use rigid analytic geometry in order to remove Noetherian hypothesis from several theorems about formal schemes.

Notations and Terminology

We will write ":=" instead of "$=_{def}$" to mean that a letter will represent a given object. We will say that a property is satisfied if $\eta \xrightarrow{<} 1$ when there exists an η_o such that this property is satisfied whenever $\eta_o \leq \eta < 1$.

We denote the n-th dimensional affine (resp. projective) space over a base S by $\mathbf{A}^n_S$ (resp. $\mathbf{P}^n_S$.). If $1 < \eta < \lambda$, we write $\mathbf{B}^n_\lambda$ (resp. $\mathbf{B}^{n,-}_\eta$, resp. $\mathbf{C}^n_{\eta,\lambda}$) for the closed ball of radius λ (resp. the open ball of radius η, resp. the closed annulus of radii η and λ) and dimension n over K.

An open subset or a covering of a rigid analytic space will always be meant to be admissible. A neighborhood of a point is an (admissible) open which contains this point. Also, we will simply say that a rigid analytic space is compact when it has a finite affinoid covering.

REVIEW

COHOMOLOGY

OF

AN OVERCONVERGENT ISOCRYSTAL

(1) OVERCONVERGENT ISOCRYSTALS

This section is devoted to the study of overconvergent isocrystals on a separated scheme of finite type over k. All the results of this section are due to Berthelot and will appear in [Berthelot 89?] (see also [Berthelot 86]).

(1.1) Tube of Radius η of a k-Scheme in a Formal $\mathcal{V}$-Scheme

(1.1.1) If $\mathcal{A}$ is a topologically finitely presented $\mathcal{V}$-algebra, then $A := \mathcal{A} \otimes_{\mathcal{V}} K$ is an affinoid K-algebra and $P_K :=$ Spm A is an affinoid space called the generic fibre of $P :=$ Spf $\mathcal{A}$. Also, the obvious map $\mathcal{A} \longrightarrow A$ induces a morphism sp: $P_K \longrightarrow P$ of ringed spaces (or, more precisely, ringed sites) called the specialization map.

(1.1.2) Let P be a formal affine $\mathcal{V}$-scheme and X a closed subscheme of P defined over k. Let $\{f_1, \ldots, f_r\}$ be a set of generators for the ideal of X modulo $\mathfrak{m}$ and $\eta \le 1$. Then the open subset

$$]X[_{P,\eta} := \{x \in P_K, \forall\, i \in \{1, \ldots, r\}, |f_i(x)| < \eta\}$$

of P_K is called the *tube of radius* η of X in P. If $\eta < 1$, the tube $]X[_{P,\eta}$ does depend on the choice of the defining equations but two different choices will however give the same thing if $\eta \xrightarrow{<} 1$.

(1.1.3) If P is a formal $\mathcal{V}$-scheme, one can paste the generic fibres of the affine open subsets of P in order to obtain a rigid analytic space P_K called the generic fibre of P. One can also paste the specialization maps corresponding to the affine open subsets of P in order to obtain the specialization map sp: $P_K \longrightarrow P$.

(1.1.4) Let P be a formal $\mathcal{V}$-scheme and X a closed subscheme of P defined over k. If $P = \cup P_i$ is a finite affine open cover of P, the trace of $]X \cap P_i[_{P_i,\eta}$ on $P_{i,K} \cap P_{j,K}$ is identical to the trace of $]X \cap P_j[_{P_j,\eta}$ if $\eta \xrightarrow{<} 1$ or $\eta = 1$. One can therefore paste the tubes $]X \cap P_i[_{P_i,\eta}$ to get the *tube* $]X[_{P,\eta}$ *of radius* η of X in

P. Two different choices of coverings and defining equations will give the same tubes if $\eta \xrightarrow{<} 1$ or $\eta = 1$.

(1.1.5) Definition Let P be a formal $\mathcal{V}$-scheme and X a subscheme of P defined over k. Let $\overline{X}$ be the closure of X in P and ∞_X the complement of X in $\overline{X}$. The *tube of radius* η of X in P is $]X[_{P,\eta} :=]\overline{X}[_\eta \setminus]\infty_X[_\eta$. It is an open subset of P_K. We will write $]X[_P :=]X[_{P,1}$ and call it the *tube* of X in P. Replacing strict inequalities by large inequalities in the above construction, one also defines the *closed tube* $[X]_{P,\eta}$ *of radius* $\eta < 1$ of X in P.

(1.1.6) Let P be a formal $\mathcal{V}$-scheme and X a subscheme of P defined over k. Proposition 1.3.3 in [Berthelot 89?] states that if X is connected and P smooth in a neighborhood of X, then $]X[_P$ too is connected. Proposition 1.1.5 in loc. cit. states that if P is flat, then the specialization map $P_K \longrightarrow P$ is surjective onto the closed points. Also, the tubes $]X[_\eta$ for $\eta < 1$ form an increasing (admissible) covering of $]X[_P$.

(1.2) Strict Neighborhood of a k-Scheme into a Formal $\mathcal{V}$-Scheme

(1.2.1) Definition Let P be a formal $\mathcal{V}$-scheme and X a subscheme of P defined over k. Let $\overline{X}$ be the closure of X in P and ∞_X the complement of X in $\overline{X}$. A *strict neighborhood of* X *in* P is an open subset V of $]\overline{X}[_P$ such that given any compact open subset W of $]\overline{X}[_P$, there exists $\eta < 1$ such that $W \subset V \cup]\infty_X[_\eta$.

(1.2.2) Definition If P is a formal $\mathcal{V}$-scheme and X a subscheme of P defined over k, then the set of all strict neighborhoods of X in P is directed (by inclusion). A *fundamental system of strict neighborhoods of* X *in* P is a cofinal subset.

(1.2.3) Example If $Y \hookrightarrow \mathbf{A}^n_{\mathcal{V}}$ is a closed immersion of an affine $\mathcal{V}$-scheme into the affine n-space, then the set $\{\mathbf{B}^n_\lambda \cap Y^{an}_K\}_{\lambda>1}$ is a fundamental system of strict neighborhoods of Y_k in $\hat{\mathbf{P}}^n_{\mathcal{V}}$ for the obvious embedding $Y_k \hookrightarrow \hat{Y} \hookrightarrow \hat{\mathbf{P}}^n_{\mathcal{V}}$.

(1.2.4) Definition An embedding $X \hookrightarrow P$ of a k-scheme into a formal $\mathcal{V}$-scheme is *admissible* if the following conditions are satisfied:

i) The closure $\overline{X}$ of X in P is proper over k.

ii) P is smooth in a neighborhood of X.

The scheme X is *admissible* if there exists such an embedding.

(1.3) Overconvergent Functions on a k-Scheme

(1.3.1) Definition Let $X \hookrightarrow P$ be an embedding into a formal scheme and V a strict neighborhood (1.2.1) of X in P. The *sheaf of overconvergent functions on* V is the sheaf of rings

$$\mathcal{O}_V^\dagger := \varinjlim j_{\lambda *}\mathcal{O}_{V_\lambda}$$

where $\{V_\lambda\}_{\lambda \in \Lambda}$ is a fundamental system of strict neighborhoods (1.2.2) of X inside V and for all $\lambda \in \Lambda$, $j_\lambda \colon V_\lambda \hookrightarrow V$ is the inclusion map. If $\overline{X}$ is the closure of X in P, we will write $\mathcal{O}^\dagger_{X \subset P} := \mathcal{O}^\dagger_{]\overline{X}[}$.

(1.3.2) Definition Let $X \hookrightarrow P$ and $Y \hookrightarrow Q$ be two embeddings in formal schemes. A *morphism* $(f \subset u) \colon Y \subset Q \longrightarrow X \subset P$ is a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \cap & & \cap \\ Q & \xrightarrow{u} & P. \end{array}$$

(1.3.3) Given a morphism of k-schemes $f \colon Y \longrightarrow X$ and two embeddings $X \hookrightarrow P$, $Y \hookrightarrow Q$, we can diagonally embed Y in $Q' := P \times Q$ and consider the first projection $p_1 : Q' \twoheadrightarrow P$. We get a morphism of embeddings $(f \subset p_1) : Y \subset Q' \longrightarrow X \subset P$. Note that $Y \hookrightarrow Q'$ is admissible if $X \hookrightarrow P$ and $Y \hookrightarrow Q$ are.

(1.3.4) A morphism $(f \subset u) : Y \subset Q \longrightarrow X \subset P$ of embeddings gives rise to a morphism

$]\overline{f}[_u :]\overline{Y}[_Q \longrightarrow]\overline{X}[_P$ of rigid analytic spaces. The pull back along this morphism preserves strict neighborhoods and it follows that there is a ring homomorphism $\mathcal{O}^{\dagger}_{X\subset P} \longrightarrow]\overline{f}[_{u*}\mathcal{O}^{\dagger}_{Y\subset Q}$ which makes $]\overline{f}[_u$ into a morphism of ringed spaces. When there is no chance of confusion, we will write u instead of $]\overline{f}[_u$. In particular, if $\mathcal{F}$ is a sheaf of $\mathcal{O}^{\dagger}_{X\subset P}$-module, then $u^*\mathcal{F} := u^{-1}\mathcal{F} \otimes_{u^{-1}\mathcal{O}^{\dagger}_{X\subset P}} \mathcal{O}^{\dagger}_{Y\subset Q}$.

(1.4) Overconvergent Isocrystals on a k-Scheme

Let $X \hookrightarrow P$ be an embedding into a formal scheme and $\overline{X}$ the closure of X in P. For $n \in \mathbf{N}$, let $P^n := P \times \ldots \times P$ and for $i \leq n$,

$$p_{1,\ldots,\hat{i},\ldots,n} : P^n \twoheadrightarrow P^{n-1}$$

be the map which omits the i-th factor. Also, let $\Delta: P \longrightarrow P^2$ be the diagonal immersion. We embed X diagonally in P^n and consider the morphisms

$$(\mathrm{Id}_X, p_{1,\ldots,\hat{i},\ldots,n}): X \subset P^n \longrightarrow X \subset P^{n-1} \quad \text{and} \quad (\mathrm{Id}_X, \Delta): X \subset P \longrightarrow X \subset P^2.$$

(1.4.1) Definition An *overconvergent isocrystal on* $X \subset P$ is a locally free $\mathcal{O}^{\dagger}_{X\subset P}$-module of finite rank $\mathcal{F}$, together with an isomorphism $\varepsilon: p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ called its *Taylor isomorphism*, subject to the conditions:

i) $\Delta^*(\varepsilon) = \mathrm{Id}_{\mathcal{F}}$ on $]\overline{X}[_P$

ii) $p_{12}^*(\varepsilon) \circ p_{23}^*(\varepsilon) = p_{13}^*(\varepsilon)$ on $]\overline{X}[_{P^3}$

(1.4.2) Definition Let $\mathcal{F}$ and $\mathcal{G}$ be two overconvergent isocrystals on $X \subset P$, with Taylor isomorphisms ε and η respectively. A *morphism* $\Phi: \mathcal{F} \longrightarrow \mathcal{G}$ of overconvergent isocrystals is an homomorphism of $\mathcal{O}^{\dagger}_{X\subset P}$-modules such that $\eta \circ p_1^*(\Phi) = p_2^*(\Phi) \circ \varepsilon$.

(1.4.3) Overconvergent isocrystal on $X \subset P$ form an abelian category $\mathrm{Isoc}^{\dagger}(X \subset P)$ with an internal Hom, $\mathcal{H}om$ and left adjoint $\otimes$.

(1.5) Pull Back of an Overconvergent Isocrystal

(1.5.1) Let $(f \subset u) : Y \subset Q \longrightarrow X \subset P$ be a morphism of embeddings (1.3.2) and $\mathcal{F}$ an overconvergent isocrystal (1.4.1) on $X \subset P$ with Taylor isomorphism ε. Then $u^*\mathcal{F}$ is an overconvergent isocrystal on $Y \subset P$ with Taylor isomorphism $(u \times u)^*(\varepsilon)$. If $v: Q \longrightarrow P$ is such that $(f,v) : Y \subset Q \longrightarrow X \subset P$ is another morphism of embeddings, then the pull back of ε along $(u,v): Q \longrightarrow P \times P$ is an isomorphism $(u,v)^*(\varepsilon): u^*\mathcal{F} \xrightarrow{\sim} v^*\mathcal{F}$ of overconvergent isocrystals on $Y \subset P$.

(1.5.2) Definition and Notations Let $(f \subset u) : Y \subset Q \longrightarrow X \subset P$ be a morphism of embeddings into formal schemes, it follows from (1.5.1) that the overconvergent isocrystal $u^*\mathcal{F}$ is essentially independent of u. It is called the *pull back of* $\mathcal{F}$ *along* f and written $f^*\mathcal{F}$ or $\mathcal{F}_f$ or even just $\mathcal{F}$ if no confusion can arise. Also, if $\Phi: \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of overconvergent isocrystals on $X \subset P$, we will write $f^*\Phi$ instead of $u^*\Phi$. In the future, when no reference to u is necessary, we will say that $f: Y \subset Q \longrightarrow X \subset P$ is a morphism of embeddings.

(1.5.3) If $f: Y \subset Q \longrightarrow X \subset P$ is a morphism of embeddings, then the functor $f^*: \mathrm{Isoc}^\dagger(X \subset P) \longrightarrow \mathrm{Isoc}^\dagger(Y \subset Q)$ is exact and additive.

(1.5.4) THEOREM (Berthelot) Let $u: Q \longrightarrow P$ be a morphism of formal $\mathcal{V}$-schemes such that $(\mathrm{Id}_X,u): X \subset Q \longrightarrow X \subset P$ is a morphism of admissible (1.2.4) embeddings. If u is smooth in a neighborhood of X, then we have an equivalence of category

$$\mathrm{Id}_X^*: \mathrm{Isoc}^\dagger(X \subset P) \xrightarrow{\sim} \mathrm{Isoc}^\dagger(X \subset Q).$$

(1.5.5) Corollary If $X \hookrightarrow P$ is an admissible embedding, the category $\mathrm{Isoc}^\dagger(X \subset P)$ is essentially independent of P. In particular, we can drop the reference to P and just write $\mathrm{Isoc}^\dagger(X)$. There is a well defined overconvergent isocrystal $\mathcal{O}_X^\dagger$ on X called the trivial isocrystal.

(1.5.6) If X is a scheme over k which is not necessarily admissible, the category Isoc'(X/K) of overconvergent isocrystals on X/K is defined as follows:

Let $\overline{X}$ be a compactification of X, $\overline{X} = \cup Y_i$ a finite open cover and for all i, $Y_i \hookrightarrow Q_i$ a closed embedding into a formal scheme which is smooth in a neighborhood of $X_i := X \cap Y_i$. An overconvergent isocrystal on X is a family of overconvergent isocrystals $\mathcal{F}_i$ on $X_i \subset Q_i$, together with a family of compatible isomorphisms between their restrictions to $X_i \cap X_j \subset Q_i \times Q_j$. This again can be shown to depend only on X.

One can also define the pull back map f^*: Isoc'(X/K) $\longrightarrow$ Isoc'(Y/K) along a morphism f: Y $\longrightarrow$ X.

(2) RIGID COHOMOLOGY

In this section, we assume that K has characteristic zero. We give the construction of the cohomology spaces associated to an overconvergent F-isocrystal and state some of their properties. All the results of this section are due to Berthelot and will appear in [Berthelot 89?] (see also [Berthelot 86]).

(2.1) Modules with Overconvergent Integrable Connections

Let $X \hookrightarrow P$ be an embedding into a formal scheme, V a smooth strict neighborhood (1.2.1) of X in P, $S := V \setminus]X[_P$ and $\{V_\lambda\}_{\lambda\in\Lambda}$ a fundamental system of stricts neighborhoods (1.2.2) of X in V. Let $j: V \hookrightarrow]\overline{X}[_P$, $i: S \hookrightarrow V$ and, for all V_λ with $\lambda \in \Lambda$, $j_\lambda: V_\lambda \hookrightarrow V$ be the inclusion maps. We write $j^\dagger := \varinjlim j_{\lambda *} j_\lambda^*$.

(2.1.1) Let $\mathcal{F}$ be a coherent $\mathcal{O}^\dagger_X$-module. Then, if V is small enough, there exists a coherent $\mathcal{O}_V$-module $\mathcal{M}$ on V such that if we set $\mathcal{M}^\dagger := j^\dagger\mathcal{M}$, then $j_*\mathcal{M}^\dagger \cong \mathcal{F}$ (and $\mathcal{M}^\dagger \cong \mathcal{F}_{|V}$). Let $\Phi: \mathcal{F} \longrightarrow \mathcal{G}$ be an homomorphism and $\mathcal{N}$ a coherent $\mathcal{O}_V$-module such that $j_*\mathcal{N}^\dagger \cong \mathcal{G}$. Then there exists, if V is small enough, a morphism $\phi: \mathcal{M} \longrightarrow \mathcal{N}$ such that if we set $\phi^\dagger := j^\dagger\phi$, then $\Phi = j_*\phi^\dagger$. Finally, if $(f \subset u) : Y \subset Q \longrightarrow X \subset P$ is a morphism of embeddings (1.3.2), then $u^*\mathcal{F} = (u_K^*\mathcal{M})^\dagger$.

(2.1.2) Definition Let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module with an integrable connection ∇. Assume that there exists a strict neighborhood V' of X in $P \times P$ such that $V' \subset V \times V$ and $\Delta(V) \subset V'$ and an isomorphism $\eta: p_2^*\mathcal{M} \xrightarrow{\sim} p_1^*\mathcal{M}$ on V' such that, if $\mathcal{I}$ is the ideal of $\Delta(V)$ in V', then η induces the stratification

$$\eta^n: \mathcal{O}_{V'}/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_V} \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{O}_{V'}/\mathcal{I}^{n+1}$$

of $\mathcal{M}$. Then ∇ is said *overconvergent*.

(2.1.3) Let $\mathcal{F}$ be an overconvergent isocrystal (1.4.1) on X and $\mathcal{M}$ a coherent $\mathcal{O}_V$-module such that $j_*\mathcal{M}^\dagger \cong \mathcal{F}$. Let $\Delta: P \hookrightarrow P \times P$ be the diagonal embedding. If V is small enough, there exists a strict neighborhood V' of X in P × P such that $V' \subset V \times V$ and $\Delta(V) \subset V'$ and an isomorphism $e: p_2^*\mathcal{M} \xrightarrow{\sim} p_1^*\mathcal{M}$ on V' such that, if $j': V' \hookrightarrow]\overline{X}[_{P^2}$ is the inclusion map, then $\varepsilon = j'_*e^\dagger$. The isomorphism e induces a stratification on $\mathcal{M}$ which is therefore endowed with an overconvergent integrable connection. Conversely, any coherent $\mathcal{O}_V$-module $\mathcal{M}$ with an overconvergent integrable connection comes from an overconvergent isocrystal $\mathcal{F}$ on $X \subset P$. Finally, let us remark that morphisms of overconvergent isocrystals correspond to horizontal homomorphisms.

(2.2) Rigid Cohomology and Rigid Cohomology with Proper Support

Let $X \hookrightarrow P$ be an embedding into a formal scheme, V a smooth strict neighborhood of X in P and $S := V \setminus]X[_P$. Let $\mathcal{F}$ be an overconvergent isocrystal on $X \subset P$ and $\mathcal{M}$ a coherent $\mathcal{O}_V$-module with an (overconvergent) integrable connection such that $\mathcal{M}^\dagger \cong \mathcal{F}_{|V}$.

(2.2.1) Notations $\mathbf{H}^i_{]X[_P}(V, -)$ is the i-th derived functor of

$$\Gamma_{]X[_P}(V, -) := \operatorname{Ker}[\Gamma(V, -) \longrightarrow \Gamma(V \setminus]X[_P, -)]$$

on the category of abelian sheaves.

(2.2.2) Definition The i-th *space of rigid cohomology of* $\mathcal{F}$ is

$$H^i_{rig}(X \subset P, \mathcal{F}) := \mathbf{H}^i(V, \mathcal{M}^\dagger \otimes_{\mathcal{O}_V} \Omega^\bullet_V).$$

The i-th *space of rigid cohomology with proper support of* $\mathcal{F}$ is

$$H^i_{rig,c}(X \subset P, \mathcal{F}) := \mathbf{H}^i_{]X[_P}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V).$$

(2.2.3) Notations For $\mathcal{F} = \mathcal{O}_X^{\dagger}$, we just write $H^i_{rig}(X \subset P)$ and $H^i_{rig,c}(X \subset P)$.

(2.2.4) Remark Since the inclusion i: $V \setminus]X[_P \hookrightarrow V$ is quasi-Stein (as in [Kiehl 67, Inv. Math]), we have for a coherent $\mathcal{O}_V$-module $\mathcal{M}$,

$$H^i_{]X[_P}(V, \mathcal{M}) \cong H^i(V, [\mathcal{M} \longrightarrow i_* i^* \mathcal{M})$$

and it follows that

$$H^i_{rig,c}(X, \mathcal{F}) \cong H^i(V, [\mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V \longrightarrow i_* i^* \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V]).$$

(2.3) Functoriality of Rigid Cohomology

(2.3.1) It is clear that rigid cohomology and rigid cohomology with compact support are both functorial in $\mathcal{F}$. Also, rigid cohomology with coefficient in an overconvergent isocrystal is functorial in $X \subset P$. Finally, note that rigid cohomology with proper support is covariant with respect to open immersions and contravariant with respect to proper morphisms.

(2.3.2) THEOREM (Berthelot) Let u: $Q \longrightarrow P$ be a morphism of formal $\mathcal{V}$-schemes such that (Id_X, u): $X \subset Q \longrightarrow X \subset P$ is a morphism of admissible (1.2.4) embeddings and $\mathcal{F}$ an overconvergent isocrystal on X. If u is smooth in a neighborhood of X, we have the isomorphisms

$$H^i_{rig}(Id_X, \mathcal{F}) : H^i_{rig}(X \subset P, \mathcal{F}) \xrightarrow{\sim} H^i_{rig}(X \subset Q, \mathcal{F}) \quad \text{and}$$

$$H^i_{rig,c}(Id_X, \mathcal{F}) : H^i_{rig,c}(X \subset P, \mathcal{F}) \xrightarrow{\sim} H^i_{rig,c}(X \subset Q, \mathcal{F}).$$

(2.3.3) Corollary If $X \hookrightarrow P$ is an admissible embedding and $\mathcal{F}$ an overconvergent isocrystal on X, the spaces $H^i_{rig}(X \subset Q, \mathcal{F})$ and $H^i_{rig,c}(X \subset Q, \mathcal{F})$ are essentially independent of P. In particular, we can drop the reference to P and write $H^i_{rig}(X/K, \mathcal{F})$ and $H^i_{rig,c}(X/K, \mathcal{F})$. When there is no chance of confusion, we will drop the reference to K or to X.

(2.3.4) If X is a scheme over k which is not necessarily admissible, one can also define the functors $H^i_{rig}(X/K, -)$ and $H^i_{rig,c}(X/K, -)$ on the category $\mathrm{Isoc}^\dagger(X/K)$. Because of this, in the sequel we will not require the k-schemes to be admissible.

(2.4) Some Properties of Rigid Cohomology.

Let X be a k-scheme and $\mathcal{F}$ an overconvergent isocrystal on X/K.

(2.4.1) If $X = X_1 \amalg X_2$, then,

$$H^i_{rig}(X, \mathcal{F}) = H^i_{rig}(X_1, \mathcal{F}) \oplus H^i_{rig}(X_2, \mathcal{F}) \quad \text{and} \quad H^i_{rig,c}(X, \mathcal{F}) = H^i_{rig,c}(X_1, \mathcal{F}) \oplus H^i_{rig,c}(X_2, \mathcal{F}).$$

(2.4.2) We have $H^i_{rig}(X, \mathcal{F}) = H^i_{rig}(X_{red}, \mathcal{F})$ and $H^i_{rig,c}(X, \mathcal{F}) = H^i_{rig,c}(X_{red}, \mathcal{F})$.

(2.4.3) If K' is a finite extension of K with residue field k', then

$$H^i_{rig}(X_{k'}/K', \mathcal{F}) = H^i_{rig}(X/K, \mathcal{F}) \otimes_K K' \quad \text{and} \quad H^i_{rig,c}(X_{k'}/K', \mathcal{F}) = H^i_{rig,c}(X/K, \mathcal{F}) \otimes_K K'$$

(2.4.4) There is a canonical map,

$$H^i_{rig,c}(X, \mathcal{F}) \longrightarrow H^i_{rig}(X, \mathcal{F}).$$

It is an isomorphism when X is proper in which case we identify these two spaces.

(2.4.5) If Z is a closed subscheme of X and U its open complement, there is a Gysin sequence

$$\cdots \longrightarrow H^i_{rig,c}(U, \mathcal{F}) \longrightarrow H^i_{rig,c}(X, \mathcal{F}) \longrightarrow H^i_{rig,c}(Z, \mathcal{F}) \longrightarrow H^{i+1}_{rig,c}(X, \mathcal{F}) \longrightarrow \cdots .$$

(2.4.6) If Y is a proper smooth (formal) scheme over $\mathcal{V}$, with special fibre X, there are natural isomorphisms

$$H^i_{rig}(X) \cong H^i_{DR}(Y_K).$$

(2.4.7) If dim X = n, then $H^i_{rig,c}(X, \mathcal{F}) = 0$ for $i > 2n$. Using the Gysin sequence, this non trivial result can easily be deduced by induction from the results of section 5.

PART I

THE TRACE FORMULA

FOR

OVERCONVERGENT F-ISOCRYSTALS

(3) OVERCONVERGENT F-ISOCRYSTALS AND EXPONENTIAL SUMS

In this section, we assume that K has characteristic zero and that k contains $\mathbf{F}_q$ with $q = p^s$. We give a definition of an overconvergent F-isocrystal which slightly differs from Berthelot's: We are only interested in schemes over $\mathbf{F}_q$ but want to be able to consider cohomology spaces over K. We define the exponential sum and the L-function associated to an overconvergent F-isocrystal and discuss the relation with the exponential sum associated to a character.

(3.1) Overconvergent F-Isocrystal.

Let X be an $\mathbf{F}_q$-scheme.

(3.1.1) Notations F_X is the s-th power of the absolute Frobenius on X. We set $X_k := X \otimes_{\mathbf{F}_q} k$ and $F_{X_k} := F_X \otimes_{\mathbf{F}_q} k$. If $\mathcal{F}$ is an overconvergent isocrystal (1.4.1) on X_k, then $\mathcal{F}^{(q)} := F_{X_k}^* \mathcal{F}$ and, by induction for each positive integer r, $\mathcal{F}^{(q^{r+1})} = (\mathcal{F}^{(q^r)})^{(q)}$. If $\alpha\colon \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of overconvergent isocrystals, then $\alpha^{(q)} := F_{X_k}^* \alpha$ and by induction for each positive integer r, $\alpha^{(q^{r+1})} = (\alpha^{(q^r)})^{(q)}$.

(3.1.2) Definition An *overconvergent* F-*isocrystal on* X is an overconvergent isocrystal $\mathcal{F}$ on X_k, together with an isomorphism of overconvergent isocrystals $\Phi\colon \mathcal{F}^{(q)} \xrightarrow{\sim} \mathcal{F}$ called its *Frobenius isomorphism.*

(3.1.3) Definition Let $\mathcal{F}$ and $\mathcal{G}$ be two overconvergent F-isocrystals on X with Frobenius isomorphisms Φ and Ψ, respectively. A *morphism* $\alpha\colon \mathcal{F} \longrightarrow \mathcal{G}$ *of overconvergent* F-*isocrystals* is a morphism of overconvergent isocrystals such that $\alpha \circ \Phi = \Psi \circ \alpha^{(q)}$.

(3.1.4) Definition If $f : Y \longrightarrow X$ be a morphism of $\mathbf{F}_q$-schemes and $\mathcal{F}$ an overconvergent F-isocrystal on X with Frobenius isomorphisms Φ. The *pull back* $f^* \mathcal{F}$ *of* $\mathcal{F}$ *along* f is the overconvergent isocrystal $f_k^* \mathcal{F}$ together with the Frobenius isomorphism $f_k^* \Phi$. Similarly, we define the pull back of a morphism.

(3.1.5) Overconvergent F-isocrystals on X form an abelian category F-iso†(X) with an internal Hom, $\mathcal{H}om$ and left adjoint $\otimes$. If $f : Y \longrightarrow X$ be a morphism of $\mathbf{F}_q$-schemes, the functor $f^* :$ F-iso†(X) $\longrightarrow$ F-iso†(Y) is additive and exact. The trivial overconvergent F-isocrystal is $\mathcal{O}^\dagger$ together with the trivial endomorphism as Frobenius.

(3.2) Overconvergent F-Isocrystals on a Point

(3.2.1) An overconvergent F-isocrystal (3.1.2) on Spec $\mathbf{F}_q$ may be (and will be) viewed as a finite dimensional K-vector space H together with an automorphism ϕ.

(3.2.2) If k contains $\mathbf{F}_{q^r}$ with $r > 1$, then $\mathbf{F}_{q^r} \otimes_{\mathbf{F}_q} k$ is a canonically isomorphic over k to the product of r copies of k via $\alpha\otimes 1 \longmapsto (\alpha, \alpha^q, \ldots, \alpha^{q^{r-1}})$. In terms of schemes, if we set $x :=$ Spec $\mathbf{F}_{q^r}$ and $x_k := x \otimes_{\mathbf{F}_q} k$, we have an isomorphism $x_k \cong \coprod_0^{r-1} x^{(i)}$ with each $x^{(i)}$ isomorphic to Spec k. Under this isomorphism, the Frobenius endomorphism of x_k corresponds to the endomorphism of $\coprod_0^{r-1} x^{(i)}$ which sends $x^{(i)}$ to $x^{(i-1 \bmod r)}$.

(3.2.3) It follows from (3.2.2) that, when k contains $\mathbf{F}_{q^r}$, an overconvergent F-isocrystal on Spec $\mathbf{F}_{q^r}$ can be (and will be) viewed as an ordered product $H := \prod_0^{r-1} H^{(i)}$ of finite dimensional K-vector spaces. The pull back of H along the Frobenius endomorphism is $\prod_0^{r-1} H^{(i+1 \bmod r)}$. The Frobenius isomorphism on H is therefore a product ϕ of isomorphisms $\phi^{(i)}: H^{(i)} \xrightarrow{\sim} H^{(i-1 \bmod r)}$. In particular, all the $H^{(i)}$'s have the same dimension.

(3.3) Exponential Sums and L-Functions Attached to an Overconvergent F-Isocrystals

Let $\mathcal{F}$ be an overconvergent F-isocrystal (3.1.2) on an $\mathbf{F}_q$-scheme X with Frobenius isomorphism Φ.

(3.3.1) Notations If $x \hookrightarrow X$ is a closed point, then $\mathcal{F}(x) := x^*\mathcal{F}$ and $\Phi(x) := x^*\Phi$. Note that when x is rational, then $\Phi(x)$ is an automorphism of $\mathcal{F}(x)$.

(3.3.2) Definition The *exponential sum attached to* $\mathcal{F}$ is

$$S(X, \mathcal{F}) := \sum_{x \in X(\mathbf{F}_q)} \mathrm{tr}(\Phi(x)).$$

(3.3.3) Notations If $f: Y \longrightarrow X$ a morphism, then $S(Y, f, \mathcal{F}) := S(Y, f^* \mathcal{F})$. When there is no chance of confusion, we will omit to mention f or Y.

If k contains $\mathbf{F}_{q^r}$, then $\mathcal{F}$ also defines an overconvergent isocrystal $\mathcal{F}_r$ on $X_r := X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r}$ with Frobenius isomorphism $\Phi_r := \Phi \circ \Phi^{(q)} \circ \ldots \circ \Phi^{(q^{r-1})}$. We set $S_r(Y, f, \mathcal{F}) := S(Y_r, f_r, \mathcal{F}_r)$.

(3.3.4) Definition If k is algebraically closed, the *L-function attached to* $\mathcal{F}$ is the power series

$$L(X, \mathcal{F}, t) := \exp\left(\sum_{r=1}^{\infty} S_r(X, \mathcal{F})\, t^r/r\right).$$

(3.3.5) Clearly, $S(X, \mathcal{O}^\dagger)$ is just the number N(X) of rational points of X. It follows that $L(X, \mathcal{O}^\dagger, t)$ is the zeta function Z(X, t) of X.

(3.3.6) Notations (k algebraically closed) If $f: Y \longrightarrow X$ is a morphism, then $L(Y, f, \mathcal{F}, t) := L(Y, f^* \mathcal{F}, t)$. When there is no chance of confusion, we will omit to mention Y or f.

(3.4) Primitive F-Isocrystals on a Commutative Group Scheme

Let G be a commutative group scheme over $\mathbf{F}_q$ and

$$m, p_1, p_2: G \times G \longrightarrow G$$

the addition and the projections.

(3.4.1) Definition A rank one overconvergent F-isocrystal (3.1.2) $\mathcal{L}$ on G is *primitive* if there exists an isomorphism of overconvergent F-isocrystals $m^* \mathcal{L} \cong p_1^* \mathcal{L} \otimes p_2^* \mathcal{L}$ on $G \times G$.

(3.4.2) Definition Let $\mathcal{L}$ be a primitive overconvergent F-isocrystal on G with Frobenius isomorphism Φ. Then the map

$$\chi : G(\mathbf{F}_q) \longrightarrow K^\times$$
$$x \longmapsto \mathrm{tr}\, \Phi(x)$$

is a character of $G(\mathbf{F}_q)$ called the *character associated with* $\mathcal{L}$.

(3.4.3) Let $\mathcal{L}$ be a primitive overconvergent F-isocrystal on G and χ the character associated with $\mathcal{L}$. If $f : X \longrightarrow G$ is a morphism of $\mathbf{F}_q$-schemes, then

$$S(X, f, \mathcal{L}) = \sum_{x \in X(\mathbf{F}_q)} \chi(f(x))$$

is just the classical exponential sum $S(X, f, \chi)$ attached to χ and f on X.

(3.4.4) The map $\mathcal{L} \longmapsto \chi$ which associates a character to a primitive overconvergent F-isocrystal on G is a group homomorphism. It is shown in [Berthelot 83] that when G is the additive group $\mathbf{G}_{a,\mathbf{F}_q}$ or the multiplicative group $\mathbf{G}_{m,\mathbf{F}}$, this homomorphism is surjective. More precisely, for any character ψ of $\mathbf{F}_q$ (resp. χ of $\mathbf{F}_q^\times$), a primitive overconvergent F-isocrystal $\mathcal{L}_\psi$ on $\mathbf{G}_{a,\mathbf{F}_q}$ (resp. $\mathcal{K}_\chi$ on $\mathbf{G}_{m,\mathbf{F}_q}$) is explicitly given.

(3.5) L-Function Attached to a Primitive Overconvergent F-Isocrystal.

Let G be a commutative group scheme over $\mathbf{F}_q$ and $\mathcal{L}$ a primitive (3.4.1) overconvergent

F-isocrystal (3.1.2) on G with Frobenius isomorphism Φ and associated character χ (3.4.2).

(3.5.1) Let t: $G \longrightarrow G$ the map obtained by composing the addition mapm: $G \times \ldots \times G \longrightarrow G$ with $(Id, F_G, \ldots, F_G^{r-1})$: $G \longrightarrow G \times \ldots \times G$. If $x \in G(\mathbf{F}_{q^r})$, we have

$$(Id, F_G, \ldots, F_G^{r-1}) \circ F_G \circ x = (F_G, \ldots, F_G^{r-1}, Id) \circ x.$$

Since G is commutative, we see that $F_G \circ t \circ x = t \circ F_G \circ x = t \circ x$. It follows that F_G leaves t(x) stable and therefore $t(x) \in G(\mathbf{F}_q)$.

(3.5.2) Definition With the notations of (3.5.1), the map

$$\begin{aligned} tr: G(\mathbf{F}_{q^r}) &\longrightarrow G(\mathbf{F}_q) \\ x &\longmapsto t(x) \end{aligned}$$

is called the *trace map*.

(3.5.3) Remark *If* k *contains* $\mathbf{F}_{q^r}$, *then the character* χ_r *associated with* $\mathcal{L}_r$ *is* $\chi \circ tr$.

Proof. By definition, if $x \in G(\mathbf{F}_{q^r})$, we have $\chi_r(x) = tr\Phi . tr\Phi^{(q)} \ldots tr\Phi^{(q^{r-1})}$. We keep the notations of (3.5.1) and let $p_1, p_2, \ldots, p_r$: $G \times \ldots \times G \longrightarrow G$ be the projections. By definition, we have $(\chi \circ tr)(x) = tr\, \Phi(t(x)) = tr\, (t^* \Phi)(x)$. Since $\mathcal{L}$ is primitive, $m^* \mathcal{L} \cong p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes \ldots \otimes p_r^* \mathcal{L}$ and therefore $t^* \mathcal{L} \cong \mathcal{L} \otimes \mathcal{L}^{(q)} \otimes \ldots \otimes \mathcal{L}^{(q^{r-1})}$. It follows that $tr\, (t^* \Phi)(x) = tr\Phi .\ tr\Phi^{(q)} \ldots tr\Phi^{(q^{r-1})}$.□

(3.5.4) If $f : X \longrightarrow G$ is a morphism of $\mathbf{F}_q$-schemes and k contains $\mathbf{F}_{q^r}$, it follows from the previous lemma that, for each integer r, we have $S_r(X, f, \mathcal{L}) = S_r(X, f, \chi)$. In particular, if k is algebraically closed, $L(X, f, \mathcal{L}, t)$ is just the classical L-function $L(X, f, \chi, t)$ attached to χ and f on X.

(4) THE TRACE FORMULA

In this section, we assume that K has characteristic zero and that k contains $\mathbf{F}_q$. We prove that the Frobenius endomorphism of the rigid cohomology spaces with compact support of an overconvergent F-isocrystal are bijective and we discuss the trace formula.

(4.1) Proposition *Let X be an $\mathbf{F}_q$-scheme and $\mathcal{F}$ an overconvergent isocrystal (1.4.1) on X_k. Then, for all* $i \in \mathbf{N}$, $H^i_{rig,c}(F_{X_k}, \mathcal{F})$ *is an isomorphism.*

Proof. Let us first assume that there exists an embedding $X_k \hookrightarrow \hat{\mathbf{P}}^N_{\mathcal{V}}$ such that none of the coordinate hyperplanes meets X. Since the standard Frobenius F: $(t_0, \ldots, t_N) \longmapsto (t_0^q, \ldots, t_N^q)$ of $\mathbf{P}^N_K$ is a Galois covering outside the coordinates hyperplanes, there exists a strict neighborhood (1.2.1) V of X_k in $\hat{\mathbf{P}}^N_{\mathcal{V}}$ such that the Frobenius induces a Galois covering F: $V^{(q)} \longrightarrow V$, and a coherent $\mathcal{O}_V$-module $\mathcal{M}$ with an (overconvergent) integrable connection such that $\mathcal{M}^\dagger \cong \mathcal{F}_{|V}$.

Any F-automorphism σ of $\hat{\mathbf{P}}^N_{\mathcal{V}}$ induces an automorphism σ^* of the de Rham complex $F^*\mathcal{M} \otimes \Omega^\bullet_{V^{(q)}}$ of $F^*\mathcal{M}$. Clearly, the endomorphism $F_*(\Sigma\, \sigma^*)$ of $F_*F^*\mathcal{M} \otimes \Omega^\bullet_V$ factors uniquely through the pull back

$$F^*: \mathcal{M} \otimes \Omega^\bullet_V \longrightarrow F_*F^*\mathcal{M} \otimes \Omega^\bullet_V$$

along F to give the *trace map* Tr: $F_*F^*\mathcal{M} \otimes \Omega^\bullet_V \longrightarrow \mathcal{M} \otimes \Omega^\bullet_V$. This trace map induces an homomorphism

$$\mathrm{tr}: H^i_{rig,c}(X_k, F^*\mathcal{F}) := H^i_{]X_k[}(V^{(q)}, F^*\mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V) \longrightarrow H^i_{rig,c}(X_k, \mathcal{F}) := H^i_{]X_k[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V).$$

Clearly, if we compose Tr on the left with F^*, we get multiplication by q^N on $\mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V$. Thus, we see that the endomorphism $\psi := (1/q^N)$ tr of $H^i_{rig,c}(X_k, \mathcal{F})$ is a right inverse for $H^i_{rig,c}(F_{X_k}, \mathcal{F})$.

By definition, if we compose F^* with Tr, we get the endomorphism $F_*(\Sigma\, \sigma^*)$ of $F_*F^*\mathcal{M} \otimes \Omega^\bullet_V$. This endomorphism induces an endomorphism of $H^i_{rig,c}(X_k, \mathcal{F})$ with is nothing but the sums of the automorphisms induced by the F-automorphisms of $\hat{P}^N_{\mathcal{V}}$. Since an F-automorphisms of $\hat{P}^N_{\mathcal{V}}$ is smooth and induces the identity on X_k, it follows from Theorem (2.3.2) that it induces the identity on $H^i_{rig,c}(X_k, \mathcal{F})$. This shows that the endomorphism of $H^i_{rig,c}(X_k, \mathcal{F})$ induced by $F_*(\Sigma\, \sigma^*)$ is just multiplication by q^N and therefore that ψ is also a left inverse for $H^i_{rig,c}(F_{X_k}, \mathcal{F})$.

In general, we proceed by induction on the dimension of X. If X is 0-dimensional, then there clearly exists an embedding as above. In higher dimension, there always exists a dense open subset U of X and an embedding of U in $\hat{P}^N_{\mathcal{V}}$ as above. We can then use our induction hypothesis and the Gysin sequence

$$\cdots \to H^i_{rig,c}(U_k, \mathcal{F}) \to H^i_{rig,c}(X_k, \mathcal{F}) \to H^i_{rig,c}(X_k \setminus U_k, \mathcal{F}) \to H^{i+1}_{rig,c}(X_k, \mathcal{F}) \to \cdots . \square$$

(4.2) The Frobenius Automorphism on Rigid Cohomology with Proper Support

Let $\mathcal{F}$ be an overconvergent F-isocrystal (3.1.2) with Frobenius isomorphism Φ on an $\mathbf{F}_q$-scheme X.

(4.2.1) Notations For $i \in \mathbf{N}$, $H^i_{rig,c}(X, \mathcal{F}) := H^i_{rig,c}(X_k, \mathcal{F})$. If $f\colon Y \longrightarrow X$ is a morphism of $\mathbf{F}_q$-schemes, then $H^i_{rig,c}(f, \mathcal{F}) := H^i_{rig,c}(f_k, \mathcal{F})$. If $\alpha\colon \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of overconvergent isocrystals, then $H^i_{rig,c}(X, \alpha) := H^i_{rig,c}(X_k, \alpha)$. Also, if $X_k \hookrightarrow P$ is an embedding in a formal scheme, $]X[\, = \,]X_k[$.

(4.2.2) Definition For all $i \in \mathbf{N}$, the composite isomorphism

$$\phi^i_c := H^i_{rig,c}(X, \Phi) \circ H^i_{rig,c}(F_X, \mathcal{F})$$

is the *Frobenius automorphism* of $H^i_{rig,c}(X, \mathcal{F})$.

(4.2.3) *If* $\mathbf{F}_{q^r}$ *is contained in k, then for all* $i \in \mathbf{N}$, *the Frobenius automorphism* ϕ^i_c of $H^i_{rig,c}(X_r, \mathcal{F}_r)$ *is the* r-th *power of* ϕ_c^i.

Proof. By definition, we have

$$\phi_{c,r}^i = H^i_{rig,c}(X_r, \Phi_r) \circ H^i_{rig,c}(F_{X_r}, \mathcal{F}_r) =$$

$$H^i_{rig,c}(X, \Phi) \circ \ldots \circ H^i_{rig,c}(X, \Phi^{(q^{r-1})}) \circ H^i_{rig,c}(F_X, \mathcal{F}^{(q^{r-1})}) \circ \ldots \circ H^i_{rig,c}(F_X, \mathcal{F}).$$

But, for all integer $k > 1$, $\Phi^{(q^{k-1})}$ is the pull back of $\Phi^{(q^{k-2})}$ along F_X so that

$$H^i_{rig,c}(X, \Phi^{(q^{k-1})}) \circ H^i_{rig,c}(F_X, \mathcal{F}^{(q^{k-1})}) = H^i_{rig,c}(F_X, \mathcal{F}^{(q^{k-2})}) \circ H^i_{rig,c}(X, \Phi^{(q^{k-2})}).$$

Applying this successively to $k = r, \ldots, 2$, gives the asserted equality.□

(4.3) The Trace Formula

Let $\mathcal{F}$ be an overconvergent F-isocrystal with Frobenius isomorphism Φ on an $\mathbf{F}_q$-scheme X.

(4.3.1) Statement of the Theorem *For all* $i \in \mathbf{N}$, *the Frobenius automorphism (4.2.2)* ϕ_c^i *of* $H^i_{rig,c}(X, \mathcal{F})$ *is nuclear (13.1.1) and we have*

$$S(X, \mathcal{F}) := \sum_{i\in \mathbf{N}} (-1)^i \operatorname{tr} \phi_c^i.$$

(4.3.2) Corollary *If* k *is algebraically closed, we have*

$$L(X, \mathcal{F}, t) = \prod_{i\in \mathbf{N}} \det(1-t\phi_c^i)^{(-1)^{i+1}}.$$

Proof. According to (4.2.3), the trace formula for $\mathcal{F}_r$ reads

$$S(X_r, \mathcal{F}_r) = \sum_{i\in N} (-1)^i \operatorname{tr}(\phi_c^i)^r$$

and it follows that

$$L(X, \mathcal{F}, t) := \exp[\sum_{r=1}^{\infty} S_r(X, \mathcal{F})\, t^r/r] = \exp\{\sum_{r=1}^{\infty}[\sum_{i\in N} (-1)^i \operatorname{tr}(\phi_c^i)^r]\, t^r/r\}.$$

On the other hand, since K has characteristic zero, we have seen in (13.2.2) that

$$\det(1-t\phi_c^i) = \exp[-\sum_{r=1}^{\infty} \operatorname{tr}(\phi_c^i)^r\, t^r/r]$$

and it follows that

$$\prod_{i\in N} \det(1-t\phi_c^i)^{(-1)^{i+1}} = \prod_{i\in N} \{\exp[-\sum_{r=1}^{\infty} \operatorname{tr}(\phi_c^i)^r t^r/r]\}^{(-1)^{i+1}} = \exp[\sum_{r=1}^{\infty}(\sum_{i\in N}(-1)^i \operatorname{tr}(\phi_c^i)^r)\, t^r/r].\square$$

(4.4) Examples

The trace formula stated in (4.3.1) is already known in several particular cases. We will prove it in complete generality in Section 6.

(4.4.1) If k contains $\mathbf{F}_{q^r}$ and $H := \prod_{i=0}^{r-1} H^{(i)}$ is an overconvergent F-isocrystal on $x := \operatorname{Spec} \mathbf{F}_{q^r}$, then $H^i_{rig,c}(x, H) = 0$ if $i \neq 0$ and $H^0_{rig,c}(x, H) = \bigoplus_{i=0}^{r-1} H^{(i)}$ with Frobenius automorphism (4.2.2) ϕ permuting cyclically $H^{(0)} \ldots$ and $H^{(r-1)}$. The trace formula obviously holds for x when $r = 1$. If $r > 1$, we get $\operatorname{tr}\phi = 0$ and here again, we see that the trace Formula holds for x.

(4.4.2) Let X be an open subset of $\mathbf{A}^1_{\mathbf{F}_q}$. Let f be a regular function on X and ψ an additive character of $\mathbf{F}_q$. Also, let $h_1, \ldots, h_r$ be r invertible functions on X and $\chi_1, \ldots, \chi_r$, r multiplicative characters of $\mathbf{F}_q$. Let $\mathcal{L}_\psi$ (resp. $\mathcal{K}_{\chi_1}$, resp. .. $\mathcal{K}_{\chi_r}$) be the primitive overconvergent F-isocrystal associated by Berthelot to ψ on

$\mathbf{G}_{a,\mathbf{F}_q}$ (resp. χ_1 on $\mathbf{G}_{m,\mathbf{F}_q}$, . . . , resp. χ_r on $\mathbf{G}_{m,\mathbf{F}_q}$) and $\mathcal{L} := f^*\mathcal{L}_\psi \otimes h_1^*\mathcal{K}_{\chi_1} \otimes \ldots \otimes h_r^*\mathcal{K}_{\chi_r}$. It follows from [Berthelot 84] and [Robba] that the trace formula holds for $\mathcal{L}$ when $K = \mathbf{C}_p$.

(4.4.3) Let X be a proper and smooth scheme over $\mathbf{F}_q$. If K is a finite extension of $\mathbf{Q}_p$, it follows from [Berthelot-Ogus] and [Berthelot 73] that the trace formula holds for the trivial isocrystal on X. Using [Etesse], the same argument should work for any (over-) convergent F-isocrystals on X which arises from non degenerate F-crystals on $X/W(\mathbf{F}_q)$.

(4.4.4) Let X be a smooth affine scheme of dimension at most three over $\mathbf{F}_q$ and assume that K is discretely valued with perfect residue field. Berthelot can prove that Poincaré duality holds for X. It therefore follows from [Monsky] that the trace formula holds for the trivial isocrystal on X.

(5) A PRE-COHOMOLOGICAL RESULT

In this section, we assume that K has characteristic zero and that k contains $\mathbf{F}_q$. We prove a trace formula which can be viewed as dual to the classical ones. Many ideas in the proof are inspired from Monsky's and Berthelot' s work. This trace formula involves some cohomology spaces but we do not want to call it a cohomological trace formula since these spaces are not finite dimensional and not canonically associated to the data.

We fix coordinates $t_1, \ldots, t_N$ for $\mathbf{A}^N_K$ and call standard Frobenius of $\mathbf{A}^N_K$ the map which sends t_i to t_i^q.

(5.1) Setting

(5.1.1) We fix the following

i) A smooth affine $\mathbf{F}_q$-scheme X of dimension n,

ii) A smooth affine $\mathcal{V}$-scheme Y such that $X \otimes_{\mathbf{F}_q} k = Y \otimes_{\mathcal{V}} k$,

iii) A closed immersion $i: Y \hookrightarrow \mathbf{A}^N_{\mathcal{V}}$,

iv) A strict neighborhood (1.2.1) V of X_k in $\hat{\mathbf{P}}^N_{\mathcal{V}}$ for the obvious embedding $X_k \hookrightarrow \hat{Y} \hookrightarrow \hat{\mathbf{P}}^N_{\mathcal{V}}$,

v) A coherent $\mathcal{O}_V$-module $\mathcal{M}$,

vi) If $F: V^{(q)} \longrightarrow V$ is the map induced by the standard Frobenius of $\mathbf{A}^N_K$, a strict neighborhood W of X_k in $\hat{\mathbf{P}}^N_{\mathcal{V}}$ contained in V and in $V^{(q)}$,

vi) An homomorphism $\phi : (F^*\mathcal{M})_{|W} \longrightarrow \mathcal{M}_{|W}$.

(5.1.2) The pre-cohomological result we want to prove is the following: i) The composite map

$$\varphi: H^n_{]X[}(W, \mathcal{M}) \cong H^n_{]X[}(V, \mathcal{M}) \xrightarrow{F^*} H^n_{]X[}(V^{(q)}, F^*\mathcal{M}) \cong H^n_{]X[}(W, F^*\mathcal{M}) \xrightarrow{\Phi} H^n_{]X[}(W, \mathcal{M})$$

is nuclear (13.1.1) and ii) If X does not have any rational point, then $\operatorname{tr} \varphi = 0$.

(5.1.3) If $\lambda > 1$, we write

$$\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\} :=$$

$$\{\xi = \sum_{\mu_1,\ldots,\mu_n > 0} a_{\mu_1,\ldots,\mu_n} t_1^{\mu_1} \ldots t_n^{\mu_n},\ a_{\mu_1,\ldots,\mu_n} \in \mathcal{V} \,/\, \lambda^{\mu_1+\ldots+\mu_n} |a_{\mu_1,\ldots,\mu_n}| \to 0 \text{ as } \mu_1+\ldots+\mu_n \to \infty\}).$$

We will need the following corollary to Bosch's version of Artin approximation theorem ([Bosch], Theorem (2.2)]: *For* $i \in \{1, \ldots, m\}$, *let* $F_i \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T_1, \ldots, T_k]$ *with* $\lambda > 1$. *If there exists* $g_1, \ldots, g_k \in \mathcal{V}\{t_1, \ldots, t_n\}$ *such that, for all* $i \in \{1, \ldots, m\}$, $F_i(g_1, \ldots, g_k) = 0$, *then there exists for* λ *sufficiently close to 1*, $g_1', \ldots, g_k' \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ *such that for all* $i \in \{1, \ldots, m\}$, $F_i(g_1', \ldots, g_k') = 0$.

(5.2) Proposition *Let* $\mathcal{L}$ *be a free sheaf of finite rank on* $\mathbf{B}_\lambda^n$. *Then,*

i) *If* $1 < \eta < \lambda$, *then* $H^k_{\mathbf{B}^n_\eta,-}(\mathbf{B}^n_\lambda, \mathcal{L}) = 0$ *for* $k \neq n$.

ii) $H^n_{\mathbf{B}^n_\eta,-}(\mathbf{B}^n_\lambda, \mathcal{O})$ *is isomorphic to the topological direct factor*

$$\{\xi = \sum_{\mu_1,\ldots,\mu_n < 0} a_{\mu_1,\ldots,\mu_n} t_1^{\mu_1} \ldots t_n^{\mu_n},\ a_{\mu_1,\ldots,\mu_n} \in K \,/\, \eta^{\mu_1+\ldots+\mu_n} |a_{\mu_1,\ldots,\mu_n}| \to 0 \text{ as } \mu_1+\ldots+\mu_n \to -\infty\}$$

of $\Gamma(\mathbf{C}^n_{\eta,\lambda}, \mathcal{O})$. *This space will always be endowed with the induced topology.*

iii) *If* $\mathcal{L} \longrightarrow \mathcal{L}'$ *is an homomorphism of free sheaves on* $\mathbf{B}^n_\lambda$, *then the induced map*

$$H^n_{\mathbf{B}^n_\eta,-}(\mathbf{B}^n_\lambda, \mathcal{L}) \longrightarrow H^n_{\mathbf{B}^n_\eta,-}(\mathbf{B}^n_\lambda, \mathcal{L}')$$

is continuous.

iv) *For* $1 < \rho < \eta < \lambda$, *the restriction map* $r: H^n_{\mathbf{B}^n_\rho,-}(\mathbf{B}^n_\lambda, \mathcal{L}) \longrightarrow H^n_{\mathbf{B}^n_\eta,-}(\mathbf{B}^n_\lambda, \mathcal{L})$ *is completely continuous.*

Proof. In i), we may assume that $\mathcal{L} = \mathcal{O}$. We will prove simultaneously i) and ii). The analytic variety

$B^n_\lambda \setminus B^{n,-}_\eta$ can be covered with the products $B^{i-1}_\lambda \times C_{\eta,\lambda} \times B^{n-i}_\lambda$ with $1 \le i \le n$, which are affinoid and therefore do not have higher coherent cohomology. It follows that $H^k_{B^{n,-}_\eta}(B^n_\lambda, \mathcal{O})$ is the k-th cohomology space of the augmented complex

$$K(n) := \Gamma(B^n_\lambda, \mathcal{O}) \longrightarrow \check{C}(\{B^{i-1}_\lambda \times C_{\eta,\lambda} \times B^{n-i}_\lambda\}, \mathcal{O}).$$

Since K(n) is clearly the n-th completed tensor power of K(1), we might actually restrict ourselves to the case $n = 1$. We have

$$K(1) = \Gamma(B_\lambda, \mathcal{O}) \longrightarrow \Gamma(C_{\eta,\lambda}, \mathcal{O}),$$

and we see that $H^0_{B^-_\eta}(B_\lambda, \mathcal{O}) = 0$. Moreover, since the subspace $\{\xi = \sum_{\mu<0} a_\mu t^\mu \; a_\mu \in K / \eta^\mu |a_\mu| \to 0 \text{ as } \mu \to -\infty\}$ of $\Gamma(C_{\eta,\lambda}, \mathcal{O})$ is a topological supplement for $\Gamma(B_\lambda, \mathcal{O})$, it is isomorphic to $H^1_{B^-_\eta}(B_\lambda, \mathcal{O})$.

The assertion iii) follows from i) since $C^n_{\eta,\lambda}$ being affinoid, an homomorphism $\mathcal{L} \longrightarrow \mathcal{L}'$ induces a continuous map $\Gamma(C^n_{\eta,\lambda}, \mathcal{L}) \longrightarrow \Gamma(C^n_{\eta,\lambda}, \mathcal{L}')$.

To prove iv), we may again assume that $\mathcal{L} = \mathcal{O}$. We have to show that r is the limit of finite rank maps. If $N \in \mathbf{N}$, let r_N: $H^n_{B^{n,-}_\rho}(B^n_\lambda, \mathcal{O}) \longrightarrow H^n_{B^{n,-}_\eta}(B^n_\lambda, \mathcal{O})$ be the continuous linear map such that

$$r_N(t_1^{\mu_1} \ldots t_n^{\mu_n}) = \begin{cases} t_1^{\mu_1} \ldots t_n^{\mu_n} \text{ if } \mu_1 + \ldots + \mu_n \ge -N \\ 0 \text{ otherwise.} \end{cases}$$

Let us denote with a subscript η the norm of $H^n_{B^{n,-}_\eta}(B^n_\lambda, \mathcal{O})$. We have $|t_1^{\mu_1} \ldots t_n^{\mu_n}|_\eta = \eta^{\mu_1 + \cdots + \mu_n}$ and therefore $\|t_1^{\mu_1} \ldots t_n^{\mu_n}\|_\eta = (\eta/\rho)^{\mu_1 + \cdots + \mu_n} \|t_1^{\mu_1} \ldots t_n^{\mu_n}\|_\eta$. In particular, if $\mu_1 + \ldots + \mu_n < -N$, we have since $\eta/\rho > 1$, $\|t_1^{\mu_1} \ldots t_n^{\mu_n}\|_\eta < (\rho/\eta)^N \|t_1^{\mu_1} \ldots t_n^{\mu_n}\|_\rho$. It follows that $\|r - r_N\| < (\rho/\eta)^N$. Since $\rho/\eta < 1$, we see that r is the limit of the maps r_N which clearly have finite rank.□

(5.3) Proposition *The composite map*

$$\varphi\colon H^n_{]X[}(W,\mathcal{M}) \cong H^n_{]X[}(V,\mathcal{M}) \xrightarrow{F^*} H^n_{]X[}(V^{(q)}, F^*\mathcal{M}) \cong H^n_{]X[}(W, F^*\mathcal{M}) \xrightarrow{\Phi} H^n_{]X[}(W,\mathcal{M})$$

is nuclear.

Proof. As mentioned in Example (1.2.3), we may assume that $W = \mathbf{B}^n_\lambda \cap Y_K$. Let $i_K\colon W \hookrightarrow \mathbf{B}^n_\lambda$ be the closed immersion induced by i and $\mathcal{L}_1 \longrightarrow \mathcal{L}_0 \longrightarrow i_*\mathcal{M}$ be a free presentation. Since F is flat,

$$\mathcal{L}_1^{(q)} \longrightarrow \mathcal{L}_0^{(q)} \longrightarrow F^* i_*\mathcal{M} \cong i_{K,*}[(F^*\mathcal{M})_{|W}]$$

is also a free presentation and $i_{K,*}\phi$ can be extended to a morphism

$$\begin{array}{ccccc} \mathcal{L}_1^{(q)} & \longrightarrow & \mathcal{L}_0^{(q)} & \longrightarrow & i_*[(F^*\mathcal{M})_{|W}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}_1 & \longrightarrow & \mathcal{L}_0 & \longrightarrow & i_*\mathcal{M}. \end{array}$$

Since $H^n_{]X[}(W,\mathcal{M}) = H^n_{\mathbf{B}}(\mathbf{B}_\lambda, i_{K,*}\mathcal{M})$, we may assume that Y is the affine space and that $\mathcal{M}$ is free. Let φ_η be the composite map

$$\varphi_\eta := H^n_{\mathbf{B}^-_\eta}(\mathbf{B}_\lambda,\mathcal{M}) \xrightarrow{F^*} H^n_{\mathbf{B}^-_{\eta^{1/q}}}(\mathbf{B}_{\lambda^{1/q}}, F^*\mathcal{M}) \xrightarrow{\Phi} H^n_{\mathbf{B}^-_{\eta^{1/q}}}(\mathbf{B}_{\lambda^{1/q}},\mathcal{M}) \xrightarrow{\text{res}} H^n_{\mathbf{B}^-_\eta}(\mathbf{B}_\lambda,\mathcal{M}).$$

We have seen in part iii) and iv) of Proposition (5.2) that res is completely continuous and ϕ continuous. One easily deduces from loc. cit. that F^* too is continuous. It follows that φ_η is completely continuous and therefore by [Gruson] (See also [Serre]) that it is nuclear. It follows from part i) of Proposition (5.2) that $H^n_{\mathbf{B}}(\mathbf{B}_\lambda,\mathcal{M}) = \bigcap_{\eta<\lambda} H^n_{\mathbf{B}^-_\eta}(\mathbf{B}_\lambda,\mathcal{M})$ and it is clear that, for all η, φ is the restriction of the endomorphism φ_η of $H^n_{\mathbf{B}^-_\eta}(\mathbf{B}_\lambda,\mathcal{M})$. Our assertion is therefore a consequence of Lemma (13.3.2).□

(5.4) Lemma *If X does not have any rational point, then the equation* $1 = \Sigma\,(F^*(T_i) - T_i)\,S_i$ *has a solution in* $\Gamma(W, \mathcal{O}_X^\dagger)$. *One might need to make a finite extension of K if the valuation is not discrete.*

Proof. We write X := Spec A and let I be the ideal of A generated by all the expressions $f^q - f$ with $f \in A$. Let x be a closed point on X corresponding to a maximal ideal $\mathfrak{p}$ of A. The point x is rational if and only if the Frobenius endomorphism of $k(x) = A/\mathfrak{p}$ is the identity, or equivalently, if for all $f \in A$, we have $f^q - f \in \mathfrak{p}$, that is, if $I \subset \mathfrak{p}$. In particular, if X does not have any rational point, then I is not contained in any maximal ideal and must therefore be equal to A. It follows that the equation $1 = \Sigma\,(T_i^q - T_i)\,S_i$ has a solution in A.

If the valuation is discrete, the equation $1 = \Sigma\,(F^*(T_i) - T_i)\,S_i$ must therefore also have a solution in $\Gamma(\hat{Y}, \mathcal{O})$. When the valuation is not discrete, we can use Lemma (8.2) below and we see that after a finite extension of K, this equation will have a solution in $\Gamma(\hat{Y}, \mathcal{O})$. It therefore follows from Artin-Bosch approximation theorem that this equation already has a solution in $\Gamma(W, \mathcal{O}_X^\dagger)$. More precisely, if $h_1, \ldots, h_r$ are some generators of the ideal of Y in $A^N_{\mathcal{V}}$, then we can write in $\mathcal{V}\{t_1, \ldots, t_n\}$, $1 = \Sigma\,(F^*(f_i) - f_i)\,g_i + \Sigma\,u_j h_j$ and (5.1.3) tells us that we can take the f_i's, g_i's and u_j's in $K\{t_1/\lambda, \ldots, t_n/\lambda\}$ for λ sufficiently close to 1. □

(5.5) Proposition *Assume that X has no rational points. Then,* $\mathrm{tr}\,\varphi = 0$.

Proof. Let us write $A^\dagger := \Gamma(W, \mathcal{O}_X^\dagger)$ and $M = \mathbf{H}^n_{]X[}(W, \mathcal{M})$. The obvious map $\mathcal{O}_V \times \mathcal{M} \longrightarrow \mathcal{M}$ induces a map $A^\dagger \times M \longrightarrow M$ which makes M into an $A^\dagger$-module. If $f \in A^\dagger$, we have a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ker}\,f & \longrightarrow & M & \xrightarrow{f} & M & \longrightarrow & M/f & \longrightarrow 0 \\
 & \downarrow 0 & & \downarrow \Phi\circ F^*\circ f & & \downarrow f\circ\Phi\circ F^* & & \downarrow 0 & \\
0 \longrightarrow & \mathrm{Ker}\,f & \longrightarrow & M & \xrightarrow{f} & M & \longrightarrow & M/f & \longrightarrow 0
\end{array}$$

We have seen in Proposition (5.3) that $f\circ\Phi\circ F^*$ is nuclear and it follows from Proposition (13.4) that $\Phi\circ F^*\circ f$ also is nuclear and that $\mathrm{tr}\,(f\circ\Phi\circ F^*) = \mathrm{tr}\,(\Phi\circ F^*\circ f)$. Now, again by Proposition (5.3), we know that

$(F^*(f) - f) \circ \Phi \circ F^*$ is nuclear and it follows that $\mathrm{tr}\ (F^*(f) - f) \circ \Phi \circ F^* = 0$.

We can make a finite extension of K if necessary and assume that there exists f_i's, g_i's $\in A^\dagger$ such that $1 = \Sigma\ (F^*(f_i) - f_i)\ g_i$. Thus, if we set $g_i \Phi = \Phi_i$, we have

$$\mathrm{tr}\ \varphi := \mathrm{tr}\ (\Phi \circ F^*) = \mathrm{tr}\ (\Sigma\ (F^*(f_i) - f_i)\ g_i \circ \Phi \circ F^*) = \Sigma\ \mathrm{tr}\ ((F^*(f_i) - f_i) \circ \Phi_i \circ F^*) = 0. \Box$$

(6) PROOF OF THE TRACE FORMULA

In this section, we still assume that K has characteristic zero and that k contains $\mathbf{F}_q$. We prove the trace formula by reducing it to the result of the previous section. It is also possible to prove this formula by reducing it to Reich's trace formula.

(6.1) Lemma i) *Let* $\lambda > 1$ *and* $I \subset \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T]$ *a finitely generated ideal. If* $I\mathcal{V}\{t_1, \ldots, t_n\}[T]$ *contains a monic polynomial so does* $I\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T]$ *for* λ *sufficiently close to 1.*

ii) *Let* $\mathcal{B}$ *be a* $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$*-algebra which is topologically of finite presentation. If the adic completion* $\hat{\mathcal{B}}$ *of* $\mathcal{B}$ *is finite over* $\mathcal{V}\{t_1, \ldots, t_n\}$, *then* $\mathcal{B}$ *is already finite over* $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ *for* λ *sufficiently close to 1.*

Proof. i) Let $P \in I\mathcal{V}\{t_1, \ldots, t_n\}[T]$ be a monic polynomial and $F_1, \ldots, F_k \in I$, $H_1, \ldots, H_k \in \mathcal{V}\{t_1, \ldots, t_n\}[T]$ such that $P = \sum_{i=1}^{k} G_i H_i$.

Let us write $P := g_0 + \ldots + g_{m-1}T^{m-1} + T^m$ and, for all $i \in \{1, \ldots, k\}$, $F_i := \sum_{r \in \mathbf{N}} f_{i,r} T^r$ and $H_i := \sum_{r \in \mathbf{N}} h_{i,r} T^r$. Thus, we have for all $r \in \{1, \ldots, m-1\}$, $g_r = \sum_{r=s+t} f_{i,s} h_{i,t}$. It therefore follows from the corollary of Artin approximation theorem stated in (5.1.3), that for λ sufficiently close to 1, there exists for all $r \in \{1, \ldots, m-1\}$, $g'_r \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ and for all $r \in \{1, \ldots, m-1\}$, $i \in \{1, \ldots, k\}$, $h'_{i,r} \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ such that $g'_r = \Sigma f_{i,s} h'_{i,r}$.

If we set $P' = g'_0 + \ldots + g'_{m-1}T^{m-1} + T^m$ and $H'_i := \Sigma h'_{i,r} T^r \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T]$, we see that $P' = \sum_{i=1}^{k} G_i H'_i \in I\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T]$.

ii) Since $\mathcal{B}$ is topologically of finite presentation, it is sufficient to show that any $b \in \mathcal{B}$ is integral over $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ for λ sufficiently close to 1.

Let $b \in \mathcal{B}$, π: $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T] \longrightarrow \mathcal{B}$ the $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$-linear map which sends T to b and $I := \operatorname{Ker} \pi$. Since b is integral over $\mathcal{V}\{t_1, \ldots, t_n\}$ then, necessarily $I\mathcal{V}\{t_1, \ldots, t_n\}[T]$ contains a monic polynomial. We assumed $\mathcal{B}$ topologically of finite presentation, and therefore I is finitely generated. It follows from i) that $I\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}[T]$ contains a monic polynomial for λ sufficiently close to 1, which implies that b is integral over $\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$.□

(6.2) Proposition *Let f: $Y \longrightarrow Z$ be a morphism of smooth affine $\mathcal{V}$-schemes which induces a finite map on the special fibres. Let $Z \hookrightarrow \mathbf{A}^n_{\mathcal{V}}$ (resp. $Y \hookrightarrow \mathbf{A}^N_{\mathcal{V}}$)be a closed immersion and $\{V_\lambda\}_{\lambda \in \Lambda}$ a fundamental system of strict neighborhoods (1.2.2) for the obvious embedding $Z_k \hookrightarrow \hat{Z} \hookrightarrow \hat{\mathbf{P}}^n_{\mathcal{V}}$. Then, the set $\{W_\lambda := f_K^{-1}(V_\lambda)\}_{\lambda \in \Lambda}$ is a fundamental system of strict neighborhoods for the obvious embedding $Y_k \hookrightarrow \hat{Y} \hookrightarrow \hat{\mathbf{P}}^N_{\mathcal{V}}$. Moreover, for λ sufficiently close to 1, the induced maps f_K: $W_\lambda \longrightarrow V_\lambda$ are finite.*

Proof. As mentioned in Example (1.2.3), the set $\{\mathbf{B}^n_\lambda \cap Z_K\}_{\lambda>1}$ is a fundamental system of strict neighborhoods of Z_k *in* $\hat{\mathbf{P}}^n_{\mathcal{V}}$. We may therefore assume without loss of generality that $Z = \mathbf{A}^n_{\mathcal{V}}$ and that, for all $\lambda > 1$, $V_\lambda = \mathbf{B}^n_\lambda$. Since f_k is finite, so is the completion of f and it follow from part ii) of Lemma (6.1) that, for λ sufficiently close to 1, f induces a finite map

$$Y_\lambda := Y \otimes_{\mathcal{V}\{t_1,\ldots,t_n\}} \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\} \longrightarrow \operatorname{Spf}\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}.$$

After tensoring with K, we see that the induced maps $f_K^{-1}(\mathbf{B}^n_\lambda) \longrightarrow \mathbf{B}^n_\lambda$ are finite for λ sufficiently close to 1.

Since the intersections $Y_K \cap \mathbf{B}^N_\mu$ form a fundamental system of strict neighborhoods of Y_k in $\hat{\mathbf{P}}^N_{\mathcal{V}}$, we only have to show that given any $\mu > 1$, then $f_K^{-1}(\mathbf{B}^n_\lambda) \subset \mathbf{B}^N_\mu$ for λ sufficiently close to 1.

Let $s_1, \ldots, s_N$ be the coordinates on $\mathbf{A}^N_{\mathcal{V}}$. Since (for λ sufficiently close to 1) Y_λ is finite over $\operatorname{Spf}\mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$, there exists $f_1, \ldots, f_m \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ such that if $y \in f_K^{-1}(\mathbf{B}^n_\lambda)$ with $\pi(y) = x$, then

$$s_1(y)^m + f_1(x)s_1(y)^{m-1} + \ldots + f_m(x) = 0.$$

and it follows from the ultrametric inequality that

$$|s_1(y)| = \operatorname{Max}_{1\le j\le m} |f_j(x)|^{1/j} \le \operatorname{Max}_{1\le j\le m} \|f_j\|_\lambda^{1/j}$$

where $\|\text{-}\|_\lambda$ denotes the (spectral) norm on $\mathbf{B}^n_\lambda$. For all $j \in \{1, \ldots, m\}$, we have $f_j \in \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}$ and in particular, $\|f_j\|_1 \le 1$. Since $\|f_j\|_\lambda \longrightarrow \|f_j\|_1$ as $\lambda \longrightarrow 1$, we may assume that $\|f_j\|_\lambda \le \mu^j$ and it follows that $|s_1(y)| \le \mu$. We can proceed the same way with $s_2, \ldots, s_N$.□

(6.3) Lemma *Let* Y *be a smooth affine* $\mathcal{V}$*-scheme with* n*-dimensional fibres. Let* i: $Y \hookrightarrow \mathbf{A}^N_{\mathcal{V}}$ *be a closed immersion and* V *a strict neighborhood for the obvious embedding* $Y_k \hookrightarrow \hat{Y} \hookrightarrow \hat{\mathbf{P}}^N_{\mathcal{V}}$. *If* $\mathcal{M}$ *is a locally free* $\mathcal{O}_V$*-module of finite rank, then* $\mathbf{H}^k_{]Y_k[}(V, \mathcal{M}) = 0$ *for* $k \neq n$.

Proof. i) Case $k > n$. By Noether Normalization Lemma, there exists a finite map $Y_k \longrightarrow \mathbf{A}^n_k$ and we can lift it to a map π: $Y \longrightarrow \mathbf{A}^n_{\mathcal{V}}$. Using Lemma (6.2), we may assume that $V := \pi^{-1}(\mathbf{B}^n_\lambda) \cap X_K$ and that the induced map π_K: $V \longrightarrow \mathbf{B}^n_\lambda$ is finite. The sheaf $\pi_{K,*}\mathcal{M}$ is coherent and we have $\mathbf{H}^k_{]Y_k[}(V, \mathcal{M}) = \mathbf{H}^k_{\mathbf{B}}(\mathbf{B}^n_\lambda, \pi_{K,*}\mathcal{M})$. The analytic variety $\mathbf{B}^n_\lambda \setminus \mathbf{B}^n$ can be covered by the products $\mathbf{B}^{i-1}_\lambda \times (\mathbf{B}^1_\lambda \setminus \mathbf{B}) \times \mathbf{B}^{n-i}_\lambda$ with $1 \le i \le n$, which are quasi-Stein (as in [Kiehl 67, Inv. Math]) and therefore do not have higher coherent cohomology. It follows that $\mathrm{H}^k_{\mathbf{B}^n}(\mathbf{B}^n_\lambda, \pi_{K,*}\mathcal{M})$ is the k-th cohomology space of the augmented complex

$$\Gamma(\mathbf{B}^n_\lambda, \pi_{K,*}\mathcal{M}) \longrightarrow C(\{\mathbf{B}^{i-1}_\lambda \times (\mathbf{B}^1_\lambda \setminus \mathbf{B}) \times \mathbf{B}^{n-i}_\lambda\}, \pi_{K,*}\mathcal{M}).$$

which has length n.

ii) Case $k < n$. We may assume that $V := \mathbf{B}^N_\lambda \cap X_K$ and then, if i_K: $V \hookrightarrow \mathbf{B}^N_\lambda$ is the closed immersion induced by i, we have $\mathbf{H}^k_{]Y_k[}(V, \mathcal{M}) = \mathbf{H}^k_{\mathbf{B}^N}(\mathbf{B}^N_\lambda, i_{K,*}\mathcal{M})$. One can easily build a resolution

$$0 \longrightarrow \mathcal{L}_{N-n} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow i_{K,*}\mathcal{M}$$

of $i_{K,*}\mathcal{M}$ with $\mathcal{L}_0, \ldots, \mathcal{L}_{N-n-1}$ (locally) free. Since Y and $\mathbf{A}^N_{\mathcal{V}}$ are smooth, we can cover $\mathbf{B}^N_\lambda$ by open subsets U such that the restriction of i_K to U is defined by a regular sequence. We may also assume that the restriction of $\mathcal{M}$ to $U \cap X_K$ is free. The Koszul complex $\mathcal{K}.$ on some basis of $\mathcal{M}$ is a resolution of the restriction of $i_{K,*}\mathcal{M}$ to U. Since $\mathcal{K}.$ has length N-n, we see that if s > 0, then

$$\mathcal{E}xt^s_{\mathcal{O}}(-, \mathcal{L}_{N-n})_{|U} = \mathcal{E}xt^s_{\mathcal{O}}(-, (\mathcal{L}_{N-n})_{|U}) = \mathcal{E}xt^{N-n+s}_{\mathcal{O}}(-, (i_{K,*}\mathcal{M})_{|U}) = 0.$$

It follows that $\mathcal{L}_{N-n}$ too is (locally) free.

We are therefore reduced to the case $Y = \mathbf{A}^n_{\mathcal{V}}$. Since $\Gamma_{\mathbf{B}^n}(\mathbf{B}^n_\lambda, \mathcal{M}) = \varprojlim \Gamma_{\mathbf{B}^{n,-}}(\mathbf{B}^n_\lambda, \mathcal{M})$, the assertion is a consequence of part i) of Proposition (5.2).□

(6.4) Lemma *Let X be a scheme over $\mathbf{F}_q$, Z a closed subscheme of X and U its open complement. Assume that the trace formula holds for U and Z. Then, it holds for X.*

Proof. Let $\mathcal{F}$ be an overconvergent F-isocrystal (3.1.2) on X. As mentioned in (2.4.5), there is a Gysin sequence

$$\cdots \longrightarrow H^i_{rig,c}(U, \mathcal{F}) \longrightarrow H^i_{rig,c}(X, \mathcal{F}) \longrightarrow H^i_{rig,c}(Z, \mathcal{F}) \longrightarrow H^{i+1}_{rig,c}(X, \mathcal{F}) \longrightarrow \cdots.$$

For all $i \in \mathbf{N}$, let $\phi^i_c(X)$, $\phi^i_c(Z)$ and $\phi^i_c(U)$ be the Frobenius automorphisms (4.2.2) of $H^i_{rig,c}(X, \mathcal{F})$, $H^i_{rig,c}(U, \mathcal{F})$ and $H^i_{rig,c}(Z, \mathcal{F})$, respectively. If we assume that, for all $i \in \mathbf{N}$, $\phi^i_c(Z)$ and $\phi^i_c(U)$ are nuclear (13.1.1), then it follows from proposition (13.4) that $\phi^i_c(X)$ too is and that the alternating sum

$$\mathrm{tr}\, \phi^0_c(U) - \mathrm{tr}\, \phi^0_c(X) + \mathrm{tr}\, \phi^0_c(Z) - \mathrm{tr}\, \phi^1_c(U) + \ldots$$

is zero. This means that

$$\sum_{i\in\mathbf{N}} (-1)^i \operatorname{tr} \phi_c^i(X) = \sum_{i\in\mathbf{N}} (-1)^i \operatorname{tr} \phi_c^i(U) + \sum_{i\in\mathbf{N}} (-1)^i \operatorname{tr} \phi_c^i(Z).$$

On the other hand, if Φ is the Frobenius isomorphism of $\mathcal{F}$, we have

$$S(X, \mathcal{F}) := \sum_{x\in X(\mathbf{F}_q)} \operatorname{tr}(\Phi(x)) = \sum_{x\in U(\mathbf{F}_q)} \operatorname{tr}(\Phi(x)) + \sum_{x\in Z(\mathbf{F}_q)} \operatorname{tr}(\Phi(x)) = S(U, \mathcal{F}) + S(Z, \mathcal{F}).\square$$

(6.5) Conclusion of the Proof

Using (2.4.2), we may assume that X is reduced. Then, we proceed by induction on the dimension n of X. The zero dimensional case follows from property (2.4.1) and Example (4.4.1). In general, Lemma (6.4) together with the induction hypothesis tells us that we can always replace X by any dense open subset. Since $\mathbf{F}_q$ is perfect and X reduced, X is generically smooth. We may therefore assume that X is smooth. Then, using property (2.4.1), we may assume that X is connected, and therefore irreducible. Also, we can replace X by any affine open subset. Finally, since the rational points form a rare (:= nowhere dense) subset, we may assume that X does not have any rational point.

It follows from Théorème 6 and Remarque 2 c) p 588 of [Elkik] that there exists a smooth affine scheme Y over $\mathcal{V}$ such that $X \otimes_{\mathbf{F}_q} k = Y \otimes_{\mathcal{V}} k$. If $i: Y \hookrightarrow \mathbf{A}^N_{\mathcal{V}}$ is a closed immersion, we can use the obvious embedding $X_k \hookrightarrow \hat{Y} \hookrightarrow \hat{\mathbf{P}}^N_{\mathcal{V}}$ to compute $H^i_{rig,c}(X, \mathcal{F})$. There exists a strict neighborhood V of X in $\hat{\mathbf{P}}^N_{\mathcal{V}}$ and a coherent $\mathcal{O}_V$-module $\mathcal{M}$ with an (overconvergent) integrable connection such that $\mathcal{M}^\dagger \cong \mathcal{F}_{|V}$ and we have

$$H^i_{rig,c}(X, \mathcal{F}) := \mathbf{H}^i_{]X[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^\bullet_V).$$

Since, for all $i \in \mathbf{N}$, $\mathcal{M} \otimes_{\mathcal{O}_V} \Omega^i_V$ is locally free, we know from Lemma (6.3) that $\mathbf{H}^k_{]X[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^i_V) = 0$ for $k \neq n$. It follows that $H^i_{rig,c}(X, \mathcal{F})$ is the (-n+i)-th cohomology space of the complex

$$\cdots \longrightarrow \mathbf{H}^n_{]X[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^i_V) \longrightarrow \mathbf{H}^n_{]X[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^{i+1}_V) \longrightarrow \cdots.$$

If $F: V^{(q)} \longrightarrow V$ is the map induced by the standard Frobenius of A^N_K, there exists a strict neighborhood W of X_k in $\hat{P}^N_{\mathcal{V}}$ contained in V and $V^{(q)}$ and an horizontal isomorphism

$$\phi: (F^*\mathcal{M})_{|W} \xrightarrow{\sim} \mathcal{M}_{|W}$$

which induces the Frobenius isomorphism of $\mathcal{F}$. The Frobenius automorphism of $H^i_{rig,c}(X, \mathcal{F})$ is therefore induced in cohomology by the composite endomorphisms

$$\begin{array}{lll}
\mathbf{H}^n_{]X[}(W, \mathcal{M}_{|W} \otimes_{\mathcal{O}_W} \Omega^i_W) & & \\
\uparrow\wr & & \\
\mathbf{H}^n_{]X[}(V, \mathcal{M} \otimes_{\mathcal{O}_V} \Omega^i_V) \xrightarrow{F} & \mathbf{H}^n_{]X[}(V^{(q)}, F^*\mathcal{M} \otimes_{\mathcal{O}_{V^{(q)}}} \Omega^i_{V^{(q)}}) & \\
& \downarrow\wr & \\
& \mathbf{H}^n_{]X[}(W, (F^*\mathcal{M})_{|W} \otimes_{\mathcal{O}_W} \Omega^i_W) \xrightarrow{\Phi\otimes \mathrm{Id}} & \mathbf{H}^n_{]X[}(W, \mathcal{M}_{|W} \otimes_{\mathcal{O}_W} \Omega^i_W).
\end{array}$$

Our assertion therefore follows from Proposition (13.4) and the pre-cohomological result of Section 5.□

PART II

THE WEIGHT FILTRATION

ON THE DE RHAM COHOMOLOGY

OF A CURVE.

ORTHOGONALITY THEOREM

(7) GEOMETRIC INVARIANTS OF A CURVE

This section is totally independent of the previous ones. We define and compare some invariants of a curve. The field K does not play any role in this section. We call normalization of a ring A (resp. scheme X) the normalization of A_{red} (resp. X_{red}).

(7.1) Number of Branches at a Point on a curve

Let x be a closed point on a k-scheme X.

(7.1.1) Definition The *(geometric) degree* deg x of x is the separable degree $[k(x):k]_{sep}$ of the residue field k(x) of x over k.

(7.1.2) Let k' be an extension of k and $\pi : X_{k'} \longrightarrow X$ the projection. If l is a sufficiently big finite extension of k', we have

$$l^{\deg x} \cong (k(x) \otimes_k l)_{red} \cong \prod_{\pi(x')=x} (k(x') \otimes_{k'} l)_{red} \cong \prod_{\pi(x')=x} l^{\deg x'}$$

and it follows that

$$\deg x = \sum_{\pi(x')=x} \deg x'.$$

(7.1.3) Definition Assume that X is a curve with smooth normalization $\tilde{X}$ (e.g. k perfect) and let $x_1, \ldots, x_m$ be the points on $\tilde{X}$ over x. The *(geometric) number of branches* at x is $m_x := \sum_{i=1}^{m} \deg x_i$.

(7.1.4) With the assumptions of (7.1.3), if k' any extension of k and $\pi : X_{k'} \longrightarrow X$ the projection, the smoothness of $\tilde{X}$ implies that $\tilde{X}_{k'} := \tilde{X} \otimes_k k'$ is smooth and is therefore the normalization of $X_{k'}$. It follows that

$$m_x = \sum_{\pi(x')=x} m_{x'}.$$

(7.1.5) A curve X always has smooth normalization after a finite extension of k: If $\bar{k}$ an algebraic closure of k, the normalization $\tilde{X}_{\bar{k}}$ of $X_{\bar{k}}$ is smooth. After a finite extension of k, there exists a curve Y such that $\tilde{X}_{\bar{k}} \cong Y_{\bar{k}}$ and a birational map $\pi: Y \longrightarrow X$. Since Y is normal, it is necessarily the normalization of X.

(7.1.6) Definition If k' is a sufficiently big extension of k so that $X_{k'}$ has a smooth normalization and $\pi : X_{k'} \longrightarrow X$ is the projection, then the *(geometric) number of branches* at x is $m_x = \sum_{\pi(x')=x} m_{x'}$.

(7.2) Genus at a Point on a Curve

Let x be a closed point on a curve X/k.

(7.2.1) Assume that the normalization $\tilde{X}$ of X is smooth. Let $\mathcal{O}_{x,red}$ be the reduced local ring at x on X and $\tilde{\mathcal{O}}_x$ the normalization of $\mathcal{O}_x$. Then, we set

$$\delta_x := \dim_k(\tilde{\mathcal{O}}_x/\mathcal{O}_{x,red}).$$

(7.2.2) With the assumptions of (7.2.1), if k' an extension of k and $\pi : X_{k'} \longrightarrow X$ the projection, then $\tilde{\mathcal{O}}_x \otimes_k k'$ is normal and therefore isomorphic to $\prod_{\pi(x')=x} \tilde{\mathcal{O}}_{x'}$. It follows that

$$\delta_x = \sum_{\pi(x')=x} \delta_{x'}.$$

(7.2.3) We have seen in (7.1.5) that if k' is a sufficiently big finite extension of k, then $X_{k'}$ has a smooth normalization. If $\pi : X_{k'} \longrightarrow X$ is the projection, we set $\delta_x = \sum_{\pi(x')=x} \delta_{x'}$.

(7.2.4) Definition The *genus* of X at x is

$$g_x := \delta_x - m_x + \deg x$$

and the *singular genus* of X is $s := \sum_x g_x$.

(7.2.5) Definition A rational point x on a curve X/k is an *ordinary m-tuple point (with normal tangents)* if the completion of the reduced local ring at x on X is isomorphic to $k[[t_1, \ldots, t_m]]/(t_i t_j,\ i \neq j)$. A closed point x on a curve X/k is an *ordinary multiple point (with normal tangents)* if after a sufficiently big finite extension k' of k, all the points x' over x on $X_{k'}$ are ordinary multiple points.

(7.3) Proposition *The genus (7.2.4) of a closed point* x *on a curve* X/k *is a non-negative integer. It is zero only for ordinary multiple points with normal tangents (7.2.5).*

Proof. We may clearly extend the base field as we wish and replace X by X_{red}. We may therefore assume that X is reduced with smooth normalization $\tilde{X}$, and that all the points $x_1, \ldots, x_m$ on $\tilde{X}$ over x, are rational. We have to show that $\delta_x \geq m_x - 1$ with equality if and only if x is an ordinary multiple point.

Let $\mathcal{O}_x$ be the local ring at x on X, $\tilde{\mathcal{O}}_x$ its normalization and for i = 1, . . , m let $\mathcal{O}_{x_i}$ the local ring at x_i on $\tilde{X}$. Let $\hat{\mathcal{O}}_x$ (resp. $\hat{\tilde{\mathcal{O}}}_x$, resp. , for all i, $\hat{\mathcal{O}}_{x_i}$) be the completion of $\mathcal{O}_x$ (resp. $\tilde{\mathcal{O}}_x$, resp.$\mathcal{O}_{x_i}$) so that

$$\hat{\tilde{\mathcal{O}}}_x \cong \prod_{i=1}^{m} \hat{\mathcal{O}}_{x_i}.$$

Since $\tilde{\mathcal{O}}_x/\mathcal{O}_x$ has finite length, we also have $\delta_x = \dim_k \hat{\tilde{\mathcal{O}}}_x/\hat{\mathcal{O}}_x$. There is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\mathcal{O}}_x & \longrightarrow & \prod_{i=1}^{m} \hat{\mathcal{O}}_{x_i} & \longrightarrow & \hat{\mathcal{O}}_x/\hat{\mathcal{O}}_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & k & \longrightarrow & \prod_{i=1}^{m} \hat{\mathcal{O}}_{x_i} \otimes_{\hat{\mathcal{O}}_x} k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & 0 & & & &
\end{array}$$

from which it results that

$$\delta_x \geq \dim_k \left[\prod_{i=1}^{m} \hat{\mathcal{O}}_{x_i} \otimes_{\hat{\mathcal{O}}_x} k\right] - 1 \geq m_x - 1.$$

One easily checks that for an ordinary multiple point, we have equality $\delta_x = m_x - 1$. Conversely, if $\delta_x = m_x - 1$ the above sequence of inequalities becomes a sequence of equalities

$$\delta_x = \dim_k \left[\prod_{i=1}^{m} \hat{\mathcal{O}}_{x_i} \otimes_{\hat{\mathcal{O}}_x} k\right] - 1 = m_x - 1.$$

This implies that for all i, $\hat{\mathcal{O}}_{x_i} \otimes_{\hat{\mathcal{O}}_x} k = k$ and, in the above diagram, we get an isomorphism of $\hat{\mathcal{O}}_x/\hat{\mathcal{O}}_x$ with the cokernel of the diagonal embedding of k into k^m.

Now, note that for all i, $\mathcal{O}_{x_i}$ is a discrete valuation ring, isomorphic to some $k[[t_i]]$ (or k if x is isolated). We can therefore identify $\hat{\mathcal{O}}_x$ with the subring of $\prod_{i=1}^{m} k[[t_i]]$ consisting of all m-tuples $(f_1, \ldots, f_m)$ of power series with identical constant terms. The canonical surjection of $k[[t_1, \ldots, t_m]]$ onto $k[[t_1]] \times \ldots \times k[[t_m]]$ gives an isomorphism of $\hat{\mathcal{O}}_x$ with $k[[t_1, \ldots, t_m]]/(t_i t_j)$.□

(7.4) Cyclomatic Number and Genus of a Curve

(7.4.1) Definition The *geometric number of connected* (resp. *of irreducible*) *components* of a k-scheme X is the number of connected (resp. irreducible) components of X after a sufficiently big finite extension of k.

(7.4.2) Definition Let c (resp. n) be the geometric number of connected (resp. irreducible) components of a curve X. The *cyclomatic number* (or *topological genus)* of X is

$$t := \sum_x (m_x - \deg x) - n + c,$$

where, for all closed point x on X, m_x denotes the (geometric) number of branches (7.1.6) at x.

(7.4.3) Definition Let X be a curve over k. It follows from (7.1.5) that there exists after a finite extension of K a proper smooth curve Y' which contains the normalization $\tilde{X}$ of X as a dense open subset. Let S be the singular locus of X_{red} and $\tilde{S}$ its inverse image in $\tilde{X}$. If we paste X_{red} with $Y' \setminus \tilde{S}$ along the canonical isomorphism $\tilde{X} \setminus \tilde{S} \xrightarrow{\sim} X_{red} \setminus S$, we get a geometrically reduced proper curve Y whose singularities are all in X_{red}. The *arithmetic (resp. geometric) genus* of X is $\dim_k H^1(Y, \mathcal{O}_Y)$ (resp. $\dim_k H^1(Y', \mathcal{O}_{Y'})$).

(7.4.4) Note that, if X is a curve over k, then the arithmetic genus, the geometric genus, the cyclomatic number (or topological genus) and the singular genus of X only depend on X_{red} and are invariant under finite extensions of k.

(7.5) Proposition *Let* X *be a proper curve over* k, g *its arithmetic genus (7.4.3),* d *its geometric genus (7.4.3),* t *its cyclomatic (7.4.2) number and* s *its singular genus (7.2.4). We have* g = d + t + s.

Proof. We may assume that X is reduced with smooth normalization $\tilde{X}$. Let $\pi : \tilde{X} \longrightarrow X$ be the projection. Since $\pi_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ is a torsion sheaf on X, it has 0-dimensional support. Thus, this sheaf has no higher cohomology and we have

$$H^0(X, \pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = \bigoplus_x (\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_x = \bigoplus_x \pi_*\mathcal{O}_{\tilde{X},x}/\mathcal{O}_{X,x} = \bigoplus_x \tilde{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}.$$

Since π is affine, there is an exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to \bigoplus_x \tilde{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x} \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}\ \mathcal{O}_{\tilde{X}}) \to 0\ .$$

Writing that the alternating sum of the dimensions is zero gives:

$$g = \dim_k H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) + \sum_x \dim_k \tilde{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x} - \dim_k H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) + \dim_k H^0(X, \mathcal{O}_X) =$$

$$d + \Sigma\, \delta_x - n + c = d + \Sigma\, g_x + \Sigma\, (m_x - \deg x) - n + c = d + s + t. \square$$

(8) RIGID COHOMOLOGY OF A PROPER CURVE

In this section, we assume that K has characteristic zero. We compute the dimension of the rigid cohomology spaces of a proper curve. Apart from some basic facts from rigid cohomology (e.g. Section 1 and 2), we will make use only of the results of the previous section. We write $h^i(X) := \dim_K H^i_{rig}(X)$ and $h^i_c(X) := \dim_K H^i_{rig,c}(X)$.

(8.1) Lemma *For a scheme* X/k, $h^0(X)$ *(resp.* $h^0_c(X)$*) is the geometric number of (resp. of proper) connected components of* X.

Proof. i) Let us first prove the assertion concerning rigid cohomology without support. As mentioned in (2.4.1) and (2.4.3), rigid cohomology is additive and commutes with finite extensions of K. We may therefore assume that X is connected with a k-rational point x and we have to show that $H^0_{rig}(X) = K$. Since we are only interested in schemes of dimension less than 1, we will assume for simplicity, that X is admissible (1.2.4).

Let $X \hookrightarrow P$ be an admissible embedding and $\bar{X}$ the closure of X in P. Let $\mathcal{O}^\dagger$ (resp. $\Omega^{1,\dagger}$) be the sheaf of overconvergent functions (1.3.1) (resp. overconvergent differentials $j^\dagger\Omega^1$) on X in P. By definition, the space $H^0_{rig}(X)$ is the kernel of the differentiation map

$$d\colon H^0(]\bar{X}[, \mathcal{O}^\dagger) \longrightarrow H^0(]\bar{X}[, \Omega^{1,\dagger}).$$

which contains at least the constants.

Since P is smooth in a neighborhood of X, the rational point x lifts to a $\mathcal{V}$-point of P. In particular,]X[has a K-rational point ξ. Since]X[is smooth, $\hat{\mathcal{O}}_{]X[,\xi}$ is isomorphic to a power series ring $K[[t_1, \ldots, t_r]]$ on which differentiation kills only the constants. There is therefore a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0_{rig}(X) & \xrightarrow{d} & H^0(]\bar{X}[, \mathcal{O}^\dagger) & \longrightarrow & H^0(]\bar{X}[, \Omega^{1,\dagger}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & K & \xrightarrow{d} & \hat{\mathcal{O}}_{]X[,\xi} & \longrightarrow & \hat{\Omega}^1_{]X[,\xi}. \end{array}$$

Since X is connected, so is]X[, as mentioned in (1.1.6), and it follows that there exists a fundamental system of connected strict neighborhoods of X in P. Therefore, the last two vertical arrows are necessarily injective. We thus see that $H^0_{rig}(X)$ injects into K.

ii) Let us now prove the assertion concerning rigid cohomology with support. Here again, we may assume that X is connected with a k-rational point x. If X is proper, the assertion is a consequence of part i) and property (2.4.4). We may therefore also assume that X is not proper and we have to show that $H^0_{rig,c}(X) = 0$.

Let $\bar{X}$ be a compactification of X. The beginning of the Gysin sequence reads since $\bar{X}$ and $\bar{X} \setminus X$ are proper

$$0 \longrightarrow H^0_{rig,c}(X) \longrightarrow H^0_{rig}(\bar{X}) \longrightarrow H^0_{rig}(\bar{X} \setminus X) \longrightarrow \cdots.$$

We therefore have to show that the restriction map $H^0_{rig}(\bar{X}) \longrightarrow H^0_{rig}(\bar{X} \setminus X)$ is injective. But $\bar{X}$ being connected with a rational point, it results from part i) that $H^0_{rig}(\bar{X}) = K$. On the other hand, since X is not proper, $\bar{X} \setminus X$ is not empty and we know from part i) that $H^0_{rig}(\bar{X} \setminus X)$ contains at least the constants.□

(8.2) Lemma *Let* k' *be a subfield of* k *which is absolutely of finite type. Then, after a finite extension of* K, *there exists a closed subfield* K' *of* K *with residue field* k' *on which the induced absolute value is discrete.*

Proof. Let p be the characteristic of k (read $\mathbf{Q}$ for $\mathbf{F}_0$). We first lift a transcendence basis $t_1, \ldots, t_r$ of k' over $\mathbf{F}_p$ to $T_1, \ldots, T_r$ in K. The elements $T_1, \ldots, T_r$ are algebraically independent over $\mathbf{Q}$ and it follows that the absolute value induced on $\mathbf{Q}(T_1, \ldots, T_r)$ is then necessarily given by

$$|\Sigma a_\mu T^\mu / \Sigma b_\mu T^\mu| = \max |a_\mu| / \max |b_\mu|.$$

In particular, it is discrete. Let K" be the topological closure of $\mathbf{Q}(T_1, \ldots, T_r)$ in K. Since $(t_1, \ldots, t_r)$ is a transcendence basis for k' over $\mathbf{F}_p$ we can write $k' = \mathbf{F}_p(t_1, \ldots, t_r, t_{r+1}, \ldots, t_n)$ with $t_{r+1}, \ldots, t_n$ algebraic over $\mathbf{F}_p(t_1, \ldots, t_n)$. Lift $t_{r+1}, \ldots, t_n$ to $T_{r+1}, \ldots, T_n$ in an algebraic closure of K" (inside an algebraic closure of K) and set $K' = K''(T_{r+1}, \ldots, T_n)$. This is a finite extension of K" and therefore a complete discretely valued field. By construction, K' has k' as residue field and $K(T_{r+1}, \ldots, T_n)$ is a finite extension of K containing K'.□

(8.3) Lemma *Let Y be a flat proper scheme over $\mathcal{V}$ whose fibres are geometrically reduced curves. Then the fibres Y_K and Y_k of Y have the same arithmetic genus (7.4.3).*

Proof. In this section, we will only need the case K discretely valued and will therefore make this assumption in the proof. The general case will be proven in Theorem (14.6.1).

We can make an extension of K and replace Y by one of its connected components in order to have Y_k geometrically connected. Since Y is flat, multiplication by a uniformizer π of $\mathcal{V}$ on $\mathcal{O}_Y$ provides us with a long exact sequence

$$\cdots \to H^0(Y, \mathcal{O}) \to H^0(Y_k, \mathcal{O}) \to H^1(Y, \mathcal{O}) \xrightarrow{\pi} H^1(Y, \mathcal{O}) \to H^1(Y_k, \mathcal{O}) \to H^2(Y, \mathcal{O}) \to \cdots.$$

Since Y_k is proper, geometrically connected and geometrically reduced, we know that $H^0(Y_k, \mathcal{O}) = k$ and the first map is therefore necessarily surjective. On the other hand, Y being Noetherian with one-dimensional fibres, we have $H^2(Y, \mathcal{O}) = 0$. Thus, we see that there is a short exact sequence

$$0 \longrightarrow H^1(Y, \mathcal{O}) \xrightarrow{\ \pi\ } H^1(Y, \mathcal{O}) \longrightarrow H^1(Y_k, \mathcal{O}) \longrightarrow 0.$$

from which it results that

$$H^1(Y_k, \mathcal{O}) = H^1(Y, \mathcal{O}) \otimes_{\mathcal{V}} k$$

and that $H^1(Y, \mathcal{O})$ has no torsion. Since $\mathcal{V}$ is a discrete valuation ring, this implies that $H^1(Y, \mathcal{O})$ is free of finite rank. Since

$$H^1(Y_K, \mathcal{O}) = H^1(Y, \mathcal{O}) \otimes_{\mathcal{V}} K,$$

we see that $\dim_k H^1(Y_k, \mathcal{O}) = \mathrm{rank}_{\mathcal{V}}\ H^1(Y, \mathcal{O}) = \dim_K H^1(Y_K, \mathcal{O})$□

(8.4) THEOREM *If* X *is a proper curve over* k, *then*

$$h^0(X) = c, \qquad h^1(X) = 2d + t,$$

$$h^2(X) = n, \qquad h^i(X) = 0 \text{ for } i > 2,$$

where c *(resp.* n*) is the geometric number of connected (resp. 1-dimensional irreducible) components of* X, t *its cyclomatic (7.4.2) number and* d *its geometric genus. Moreover,* $H^i_{rig}(X/K)$ *commutes with arbitrary isometric extensions of* K.

i) Let us first assume that X is smooth. Since X is of finite type over k, it is defined over a subfield k' of k which is absolutely of finite type. We have seen in Lemma (8.2) that, after a finite extension of K, there exists a closed subfield K' of K with residue field k' on which the induced absolute value is discrete. It therefore results from Corollary 7.4 in [Grothendieck 73] that X lifts to a smooth projective scheme Y over $\mathcal{V}$ defined over the valuation ring of K'. It follows from Lemma (8.3) that X and Y_K have the same genus. On the other hand, as mentioned in (1.1.6), the tube (1.1.5) of a smooth connected scheme is connected and the specialization map $Y_K \longrightarrow X$ is surjective on closed points. It results that X and Y_K have the same geometric number of connected components. As mentioned in (2.4.6), there are natural isomorphisms

$$H^i_{rig}(X/K) \cong H^i_{DR}(Y_K)$$

and the proposition in this case results from the analogous statement in de Rham cohomology.

ii) In general, we can make a finite extension of K and replace X by its maximal reduced subscheme in order to assume that X reduced with smooth normalization $\tilde{X}$. Choose a smooth dense open subset U of X and identify U and its inverse image under the projection $\pi : \tilde{X} \longrightarrow X$. The complement Z of U in X and its inverse image $\tilde{Z}$ in $\tilde{X}$ are 0-dimensional varieties and therefore do not have higher rigid cohomology spaces. It follows that the morphism of Gysin sequences induced by π (which is finite) reduces to

$$\begin{array}{ccccccccccccc} 0 \to & H^0_{rig,c}(U) & \to & H^0_{rig}(X) & \to & H^0_{rig}(Z) & \to & H^1_{rig,c}(U) & \to & H^1_{rig}(X) & \to 0 \\ & \| & & \downarrow & & \downarrow & & \| & & \downarrow & \\ 0 \to & H^0_{rig,c}(U) & \to & H^0_{rig}(\tilde{X}) & \to & H^0_{rig}(\tilde{Z}) & \to & H^1_{rig,c}(U) & \to & H^1_{rig}(\tilde{X}) & \to 0 \end{array}$$

and for $i \geq 2$,

$$\begin{array}{ccc} H^i_{rig,c}(U) & \xrightarrow{\sim} & H^i_{rig}(X) \\ \| & & \downarrow \\ H^i_{rig,c}(U) & \xrightarrow{\sim} & H^i_{rig}(\tilde{X}). \end{array}$$

Everything else is zero.

The dimension of $H^0_{rig}(X)$ has been computed in Lemma (8.1). Since the 1-dimensional irreducible components of X are in bijection with the 1-dimensional irreducible components of its normalization $\tilde{X}$ and that $H^i_{rig}(X) \xrightarrow{\sim} H^i_{rig}(\tilde{X})$ for $i \geq 2$, it result from part i) that $H^i_{rig}(X)$ has the right dimension in this case. It only remains to compute $h^1(X)$. We use the fact that for both Gysin sequences above, the alternating sum of the dimensions is zero. In particular, these alternating sums are equal. After obvious cancellations, we see that

$$- h^0(X) + h^0(Z) + h^1(X) = - h^0(\tilde{X}) + h^0(\tilde{Z}) + h^1(\tilde{X}).$$

Note that by lemma (8.1), we have $h^0(Z) = \sum_{x\in Z} \deg x$ and

$$h^0(\tilde{Z}) = \sum_{x\in \tilde{Z}} \deg y = \sum_{x\in Z} \sum_{\pi(y)=x} \deg y$$

so that

$$h^0(\tilde{Z}) - h^0(Z) = \sum_{x\in Z} \Big(\sum_{\pi(y)=x} \deg y - \deg x \Big) = \sum_{x\in Z} m_x' - \deg x.$$

Now, by Lemma (8.1) and part i) of the proof we have $h^0(X) = c$, $h^0(\tilde{X}) = n$ and $h^1(\tilde{X}) = 2d$. Thus,

$$h^0(X) = 2d + \Sigma\, (m_x - \deg x) - n + c = 2d + t.$$

Finally, using the Gysin sequence

$$0 \to H^0_{rig,c}(U) \to H^0_{rig}(\tilde{X}) \to H^0_{rig}(\tilde{Z}) \to H^1_{rig,c}(U) \to H^1_{rig}(\tilde{X}) \to 0$$

$$\text{and} \quad H^i_{rig,c}(U) \xrightarrow{\sim} H^i_{rig}(\tilde{X}) \quad \text{for } i \geq 2,$$

we easily deduce from part i) and Lemma (8.1) that $H^i_{rig,c}(U)$ commutes with arbitrary isometric extensions of K. Finally, using the Gysin sequence

$$0 \to H^0_{rig,c}(U) \to H^0_{rig}(X) \to H^0_{rig}(Z) \to H^1_{rig,c}(U) \to H^1_{rig}(X) \to 0$$

$$\text{and} \quad H^i_{rig,c}(U) \xrightarrow{\sim} H^i_{rig}(X) \quad \text{for } i \geq 2,$$

we find that that $H^i_{rig}(X/K)$ commutes to arbitrary isometric extensions of K.□

(8.5) Corollaries

(8.5.1) *If* X/k *is a proper curve over a perfect field, the natural homomorphism*

$$H^1_{rig}(X) \longrightarrow H^1_{rig}(\bar{X})$$

is surjective.

Proof. This is a corollary of the proof of Theorem (8.4) since when k is perfect, the normalization of X is smooth: We may assume that X is reduced and we have seen that the horizontal homomorphisms in the commutative diagram

$$\begin{array}{ccc} H^1_{rig,c}(U) & \longrightarrow & H^1_{rig}(X) \\ \| & & \downarrow \\ H^1_{rig,c}(U) & \longrightarrow & H^1_{rig,c}(\bar{X}), \end{array}$$

are surjective. So is therefore the right vertical homomorphism.□

(8.5.2) *If* k *is perfect and* X *is a smooth curve over* k, *then*

$$h^0_c(X) = c', \qquad h^1_c(X) = 2d + v - n,$$

$$h^2_c(X) = c \quad \textit{and} \quad h^i_c(X) = 0 \text{ for } i > 2,$$

where d *is the genus of* X, c (*resp.* c') *is the geometric number of 1-dimensional components (resp. of proper components) of* X *and* v *is the geometric number of missing points on* X (i.e. $v = \sum_{x \in \bar{X} \setminus X} \deg x$, *where* $\bar{X}$ *is the smooth compactification of* X). Moreover, $H^i_{rig,c}(X)$ *commutes with arbitrary isometric extensions of* K.

Proof. This follows at once from Theorem (8.4) and the Gysin sequence

$$0 \to H^0_{rig,c}(X) \to H^0_{rig}(\bar{X}) \to H^0_{rig}(\bar{X} \setminus X) \to H^1_{rig,c}(X) \to H^1_{rig}(\bar{X} \setminus X) \to 0$$

$$\text{and} \quad H^i_{rig,c}(X) \cong H^i_{rig}(\bar{X}) \quad \text{for } i \geq 2. \square$$

(8.5.3) *If* X *is a curve over* k, *then* $H^i_{rig,c}(X)$ *commutes with arbitrary isometric extensions of* K.

Proof. We can make a finite extension of k and replace X by X_{red} in order to assume that X contains a smooth dense open subset U. Then, we consider the Gysin sequence

$$0 \to H^0_{rig,c}(U) \to H^0_{rig}(X) \to H^0_{rig}(X \setminus U) \to H^1_{rig,c}(U) \to H^1_{rig}(X) \to 0$$

$$\text{and} \quad H^i_{rig,c}(U) \xrightarrow{\sim} H^i_{rig}(X) \quad \text{for } i \geq 2.$$

Our assertion immediately follows from Corollary (8.5.2).□

(9) RIGID COHOMOLOGY AND DE RHAM COHOMOLOGY WITH COMPACT SUPPORT

In this section which only requires some basic knowledge of rigid cohomology (e.g. Section 1 and 2), we study the notion of support in rigid analytic geometry and (when K has characteristic zero) its relation to rigid cohomology. We prove a technical result which will be used in the following section.

(9.1) Section with Support in Rigid Analytic Geometry

(9.1.1) Definition Let V/K be a rigid analytic space. A subset W of V is a *support* in V if its set-theoretic complement $V \setminus W$ is open in V.

(9.1.2) In terms of sites, if W is a support in V, then $V \setminus W$ has a closed complement in V (a site, explicitly given by: The open sets are the open subsets of V whose complement is contained in W and a sheaf is just a sheaf on V which is trivial outside W).

(9.1.3) Definition If W is a support in V, the *sheaf* (resp. *group*) *of sections* of an abelian sheaf E on V *with support* in W is the sheaf (resp. group) of sections of E with support in the closed complement of $V \setminus W$.

(9.1.4) Concretely, if W is a support in V, the sheaf (resp. group) of sections of E with support in W is the kernel $\underline{\Gamma}_W(E)$ (resp. $\Gamma_W(V, E)$) of the homomorphism $E \longrightarrow j_*j^*E$ where $j : V \setminus W \hookrightarrow V$ is the inclusion map (resp. the homomorphism $\Gamma(V, E) \longrightarrow \Gamma(V \setminus W, E)$). Moreover, we have $\Gamma_W(V, E) = \Gamma(V, \underline{\Gamma}_W(E))$.

(9.1.5) Notation Generalizing (2.2.1), we set $H^i_W(V, -) := \mathbf{R}^i\Gamma_W(V, -)$. Also, if V is smooth and K has characteristic zero, $H^i_{DR,W}(V) := \mathbf{R}^i\Gamma_W(V, \Omega^\bullet)$.

(9.1.6) It follows from Propositions 6.4 and 6.5. i) in [Verdier] that if W is a support in V and $E^{\bullet}$ a complex of abelian groups on V, then there are two distinguished triangles

$$R\underline{\Gamma}_W E^{\bullet} \longrightarrow E^{\bullet} \longrightarrow Rj_*j^*E^{\bullet},$$

$$R\Gamma_W(V, E^{\bullet}) \longrightarrow R\Gamma(V, E^{\bullet}) \longrightarrow R\Gamma(V \setminus W, E^{\bullet}).$$

(9.2) Sections with Support in a Family, with Compact Support

(9.2.1) Definition A *family of supports* in a rigid analytic space V over K is a directed set Φ of supports (9.1.1) in V.

(9.2.2) Definition If E is a sheaf of abelian groups on an analytic variety V/K and Φ a family of supports in V, the sheaf $\underline{\Gamma}_\Phi E = \varinjlim \underline{\Gamma}_W E$ (resp. the group $\Gamma_\Phi(V, E) := \varinjlim \Gamma_W(V, E)$) is the *sheaf* (resp. the *group*) *of sections* of E *with support in* Φ.

(9.2.3) Caution In definition (9.2.2), we might have $\Gamma_\Phi(V, E) \neq \Gamma(V, \underline{\Gamma}_\Phi E)$.

(9.2.4) Notation If Φ is a family of supports in an analytic variety V/K, then $\mathbf{H}^i_\Phi(V, -) := \mathbf{R}^i\Gamma_\Phi(V, -)$ and if V is smooth and K has characteristic zero, $H^i_{DR,\Phi}(V) := \mathbf{R}^i\Gamma_\Phi(V, \Omega^{\bullet})$.

(9.2.5) Definition An analytic variety V is *ind-compact* if the set c of (quasi-) compact open supports in V is a directed set.

(9.2.6) Definition Assuming K of characteristic zero, let V/K be a smooth ind-compact analytic variety and c the family of compact open supports in V. The K-vector space $H^i_{DR,c}(V)$ is the i-th *space of de Rham cohomology of* V *with compact support*.

(9.3) Rigid Cohomology and Support

Let $X \hookrightarrow P$ be an embedding of a k-scheme into a formal $\mathcal{V}$-scheme P and V a strict neighborhood (1.2.1) of X in P.

(9.3.1) Examples The tube (1.1.5) $]X[_\eta$ of radius η of X in P, with $\eta < 1$, is a support (9.1.1) in V. The set $\dagger$ of all $V \setminus V'$ where V' is a strict neighborhood of X in P contained in V is a family of supports in V. As in (2.1), we set $j^\dagger = \varinjlim j'_* j'^*$, where for $V' \in \dagger$, j' is the immersion of V' in V.

(9.3.2) It is shown in Chapter II of [Berthelot 89?] that the functors $j^\dagger$ and $\Gamma_\dagger$ (Caution: Berthelot writes $\dagger$ as a superscript) are both exact and that for any abelian sheaf E on V, the sequence

$$0 \longrightarrow \Gamma_\dagger E \longrightarrow E \longrightarrow j^\dagger E \longrightarrow 0$$

is exact.

(9.3.4) Remark If K has characteristic zero, if $X \hookrightarrow P$ is admissible (1.2.4) and if V is smooth, then $H^i_{\mathrm{rig},c}(X) = H^i_{\mathrm{DR},]X[}(V)$ and $H^i_{\mathrm{rig}}(X) = \mathbf{H}^i(V, j^\dagger\Omega^\bullet)$.

(9.3.5) With the assumptions of (9.3.4), the second triangle of (9.1.6) gives a long exact sequence

$$\cdots \longrightarrow H^i_{\mathrm{rig},c}(X) \longrightarrow H^i_{\mathrm{DR}}(V) \longrightarrow H^i_{\mathrm{DR}}(V \setminus]X[) \longrightarrow H^{i+1}_{\mathrm{rig},c}(X) \longrightarrow \cdots.$$

(9.4) Lemmas

Let $X \hookrightarrow P$ be an immersion of a k-scheme into a formal $\mathcal{V}$-scheme, V a strict neighborhood of X in P, S the complement of $]X[_P$ in V and i the inclusion map of S in V.

(9.4.1) *If* V' *is a strict neighborhood of* X *in* P *contained in* V *and* S' *the complement of* $]X[_P$ *in* V', *there is a* *natural isomorphism of functors*

$$\Gamma_{V\backslash V'} \cong i_*\Gamma_{S\backslash S'}i^* .$$

Proof. Let E be a sheaf of abelian groups on V and s a section of $i_*\Gamma_{S\backslash S'}i^*E$ over an open subset W of V. We can view s as a section of E over $W \cap S$ whose restriction to $W \cap S'$ is trivial. Since $V = V' \cup S$ is an admissible covering, so is

$$W = (W \cap V') \cup (W \cap S).$$

Since, on the other hand we have

$$(W \cap V') \cap (W \cap S) = W \cap S',$$

we can extend s uniquely to a section of E over $W \cap V$ whose restriction to $W \cap V'$ is trivial. In other words, s extends uniquely to a section of $\Gamma_{V\backslash V'}E$ over W.□

(9.4.2) *The set* c *of compact open supports of* S *is contained in the set* Φ *of of all* $S \setminus S'$ *with* V' *and* S' *as in (9.4.1).*

Proof. We have to show that if W is a compact open support of S, then $V' := V \setminus W$ is a strict neighborhood of X in P. We have mentioned in (1.1.6) that the tubes $]Z[_\eta$ of radius η of Z in P for $\eta < 1$ form an increasing admissible covering of $]Z[_P$. The open subset W of S being compact is necessarily contained in one of the $]Z[_\eta$ and this just means that V' is a strict neighborhood of $]X[_P$ in P.□

(9.5) Proposition *Let* X $\hookrightarrow$ P *be an embedding,* V *be a compact strict neighborhood of* X *in* P, S = V \ ]X[$_P$ and $E^\bullet$ *a complex of abelian groups on* V. *Then,* S *is ind-compact and there is a natural isomorphism*

$$\mathbf{R}\Gamma_c(S, i^* E^\bullet) \cong \mathbf{R}\Gamma(V, \underline{\Gamma}_\dagger E^\bullet).$$

Proof. It follows from Lemma (9.4.1) that if Φ is the set defined in (9.4.2), there is a functorial isomorphism

$$\underline{\Gamma}_\dagger \cong \varinjlim i_* \underline{\Gamma}_{S\backslash S'} i^*$$

Since V is compact, Corollaire 5.2 in [Grothendieck-Verdier] tells us that $H^i(V, -)$ commutes with direct limits. It follows that for any abelian sheaf E on V, we have

$$\Gamma_\Phi(S, i^* E) = \varinjlim \Gamma(S, \underline{\Gamma}_{S\backslash S'} i^* E) =$$

$$\varinjlim \Gamma(V, i_* \underline{\Gamma}_{S\backslash S'} i^* E) = \Gamma(V, \varinjlim i_* \underline{\Gamma}_{S\backslash S'} i^* E) \cong \Gamma(V, \underline{\Gamma}_\dagger E),$$

Also, if I is an injective sheaf of abelian groups on V, we have for all $i > 0$,

$$H^i(V, \underline{\Gamma}_\dagger I) \cong H^i(V, \varinjlim i_* \underline{\Gamma}_{S\backslash S'} i^* I) \cong \varinjlim H^i(S, \underline{\Gamma}_{S\backslash S'} i^* I) = 0,$$

since we know from [Verdier] that $\underline{\Gamma}_{S\backslash S'}$ preserves injectives.

Since both $\underline{\Gamma}_\dagger$ and i_* are exact and preserves injectives, the natural isomorphism $\Gamma_\Phi(S, i^* E) \cong \Gamma(V, \underline{\Gamma}_\dagger E)$ induces an isomorphism in the derived category

$$\mathbf{R}\Gamma_\Phi(S, i^* E^\bullet) \cong \mathbf{R}\Gamma(V, \underline{\Gamma}_\dagger E^\bullet).$$

To conclude, it is sufficient to check that the set c of compact open supports of S is cofinal in Φ: Since V is compact, any strict neighborhood V' of $]X[_p$ contained in V, contains some $V \setminus]Z[_\eta$. Hence $W := V \setminus V'$ is contained in $]Z[_\eta \cap S$ and a fortiori in $[Z]_\eta \cap S$ which is a compact open support in S.□

(10) SPECIALIZATION HOMOMORPHISMS BETWEEN DE RHAM AND RIGID COHOMOLOGY

In this section, we assume that K has characteristic zero. We study the relations between the de Rham cohomology of the generic fibre of a generically smooth proper formal scheme $\mathfrak{X}$ and various cohomology spaces associated to its special fibre X. In particular, when $\mathfrak{X}$ is projective, we describe a pairing between two Gysin sequences relating the de Rham cohomology of the generic fibre of $\mathfrak{X}$, the rigid cohomology of a dense open subset U of the special fibre of $\mathfrak{X}$ and the de Rham cohomology of the tube of the complement of U.

(10.1) Specialization Homomorphism

Let $\mathfrak{X}$ be a generically smooth (i.e. its generic fibre is smooth) proper formal $\mathcal{V}$-scheme, U a smooth open subset of the special fibre X of $\mathfrak{X}$ and Z the closed complement of U in X.

(10.1.1) As mentioned in (9.3.5), there is a long exact sequence

$$\cdots \longrightarrow H^i_{rig,c}(U) \xrightarrow{sp^i_c} H^i_{DR}(\mathfrak{X}_K) \longrightarrow H^i_{DR}(]Z[) \longrightarrow H^{i+1}_{rig,c}(U) \longrightarrow \cdots.$$

(10.1.2) Definition The homomorphism sp^i_c which appears in (10.1.1) is the i-th *specialization homomorphism* of U in $\mathfrak{X}$.

(10.1.3) Remark Let $\mathfrak{X} \hookrightarrow P$ be a closed immersion of $\mathfrak{X}$ into a formal $\mathcal{V}$-scheme which is smooth in a neighborhood of X. It induces an immersion $\mathfrak{X}_K \hookrightarrow]X[_P$ from which one obtains an homomorphism

$$sp^i : H^i_{rig}(X) \longrightarrow H^i_{DR}(\mathfrak{X}_K)$$

which is easily seen not to depend on P by the method of the diagonal embedding.

(10.1.4) Definition The homomorphism sp^i of (10.1.3) is the i-th *specialization homomorphism* of X in $\mathfrak{X}$.

(10.1.5) *There is a natural morphism*

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i_{rig,c}(U) & \longrightarrow & H^i_{rig}(X) & \longrightarrow & H^i_{rig}(Z) & \longrightarrow & H^{i+1}_{rig,c}(U) & \longrightarrow \cdots \\
& \| & & \downarrow sp^i & & \downarrow & & \| & \\
\cdots \longrightarrow & H^i_{rig,c}(U) & \xrightarrow{sp^i_c} & H^i_{DR}(\mathfrak{X}_K) & \longrightarrow & H^i_{DR}(]Z[_{\mathfrak{X}}) & \longrightarrow & H^{i+1}_{rig,c}(U) & \longrightarrow \cdots .
\end{array}$$

between the Gysin sequence and the long exact sequence of specialization.

Proof. With the notations of (10.1.3), we have a commutative diagram

$$\begin{array}{ccc}
]U[_{\mathfrak{X}} & \hookrightarrow &]U[_P \\
\uparrow & & \uparrow \\
\mathfrak{X}_K & \hookrightarrow &]X[_P
\end{array}$$

from which we deduce, using (9.1.6), a morphism of triangles

$$\begin{array}{ccccc}
R\Gamma_{]U[}(]X[_P, \Omega^\bullet) & \longrightarrow & R\Gamma(]X[_P, \Omega^\bullet) & \longrightarrow & R\Gamma(]Z[_P, \Omega^\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_{]U[}(\mathfrak{X}_K, \Omega^\bullet) & \longrightarrow & R\Gamma(\mathfrak{X}_K, \Omega^\bullet) & \longrightarrow & R\Gamma(]Z[_{\mathfrak{X}}, \Omega^\bullet).\square
\end{array}$$

(10.2) Cospecialization Homomorphism

We keep the hypothesis and notations of (10.1).

(10.2.1) From the functorial exact sequence

$$0 \longrightarrow \underline{\Gamma}_{\dagger} E \longrightarrow E \longrightarrow j^{\dagger} E \longrightarrow 0$$

of (9.3.2) on $\mathfrak{X}_K$ and Proposition (9.5), it follows that there is a long exact sequence

$$\cdots \longrightarrow H^i_{DR,c}(]Z[) \longrightarrow H^i_{DR}(\mathfrak{X}_K) \xrightarrow{\cos^i} H^i_{rig}(U) \longrightarrow H^{i+1}_{DR,c}(]Z[) \longrightarrow \cdots.$$

(10.2.2) Definition The homomorphism $\cos^i$ which appears in (10.2.1) is the *i-th cospecialization homomorphism of* U *in* $\mathfrak{X}$.

(10.2.3) Remark If $\mathfrak{X} \hookrightarrow P$ is a closed immersion in a formal $\mathcal{V}$-scheme which is smooth in a neighborhood of X, we have, as in the proof of (10.1.5), a commutative square

$$\begin{array}{ccc}]U[_{\mathfrak{X}} & \hookrightarrow &]U[_P \\ \uparrow & & \uparrow \\ \mathfrak{X}_K & \hookrightarrow &]X[_P \end{array}$$

From this, one deduces a commutative diagram

$$\begin{array}{ccc} H^i_{rig}(X) & \longrightarrow & H^i_{rig}(U) \\ \downarrow sp^i & & \| \\ H^i_{DR}(\mathfrak{X}_K) & \xrightarrow{\cos^i} & H^i_{rig}(U). \end{array}$$

(10.3) The Trace map in Rigid Cohomology

(10.3.1) Let V be a proper algebraic variety over K and V^{an} its analytification. It is shown in [Köpf] that if $\mathcal{F}$ is a coherent sheaf on V, then the natural maps

$$H^i(V, \mathcal{F}) \longrightarrow H^i(V^{an}, \mathcal{F}^{an})$$

are isomorphisms. It follows that when V is smooth, there are canonical isomorphism $H^i_{DR}(V) \xrightarrow{\sim} H^i_{DR}(V^{an})$.

(10.3.2) Construction (Berthelot) Let P be a closed subscheme of $\mathbf{P}^N_{\mathcal{V}}$ such that $Y := P \cap \mathbf{A}^N_{\mathcal{V}}$ is smooth of dimension d. The restriction to Y_K, which is smooth, of the dualizing complex ω on P_K is just $\Omega^d[d]$. If U is the special fibre of Y, we can therefore deduce from the trace map on P_K an homomorphism

$$H^d_{]U[}(Y_K^{an}, \Omega^d) \xrightarrow{\sim} H^0_{]U[}(P_K^{an}, \omega^{an}) \longrightarrow H^0(P_K^{an}, \omega^{an}) \xleftarrow{\sim} H^0(P_K, \omega) \longrightarrow K.$$

One can check that this homomorphism factors through the canonical map

$$H^{2d}_{]U[}(Y_K^{an}, \Omega^{2d}) \longrightarrow H^{2d}_{]U[}(Y_K^{an}, \Omega^{\bullet}) =: H^{2d}_{rig,c}(U)$$

to give an homomorphism

$$\mathrm{tr}: H^{2d}_{rig,c}(U) \longrightarrow K.$$

It can be shown that this homomorphism only depends on U.

(10.3.3) If X is a smooth variety of dimension d over k, it follows from Théorème 6 and Remarque 2 c) p 588 of [Elkik] that there exists a dense open subset U of X as above. Moreover, property (2.4.7) and the Gysin sequence (2.4.5) tell us that the restriction map $H^{2d}_{rig,c}(U) \longrightarrow H^{2d}_{rig,c}(X)$ is an isomorphism. We can therefore consider the composite map

$$\mathrm{tr}: H^{2d}_{rig,c}(X) \xleftarrow{\sim} H^{2d}_{rig,c}(U) \longrightarrow K.$$

(10.3.4) Definition With the notations of (10.3.3), if the homomorphism

$$\mathrm{tr}\colon H^{2d}_{rig,c}(X) \longrightarrow K$$

is well defined (i.e. independent of U), it is called the *trace map in rigid cohomology.*

(10.3.5) Berthelot shows in [Berthelot 89?] that when $h^{2d}_{rig,c}(X) = 1$, the trace map is well defined. It is actually sufficient to require that $h^{2d}_{rig,c}$ is the geometric number of connected components of X.

(10.4) Lemma *Let* $X \hookrightarrow P$ *be an immersion of a* k*-scheme into a formal* $\mathcal{V}$*-scheme,* V *a strict neighborhood (1.2.1) of* X *in* P *and* $E^\bullet$, $F^\bullet$ *two complexes of sheaves of* K*-vector spaces on* V*. There is a natural isomorphism*

$$R\underline{\Gamma}_{]X[}E^\bullet \otimes_K F^\bullet \cong R\underline{\Gamma}_{]X[}E^\bullet \otimes_K j^\dagger F^\bullet$$

$$(\text{resp. } E^\bullet \otimes_K \underline{\Gamma}_\dagger F^\bullet \cong Ri_*i^* E^\bullet \otimes_K F^\bullet).$$

Proof. Since, by (9.3.2), we have a functorial short exact sequence

$$0 \longrightarrow \underline{\Gamma}_\dagger F \longrightarrow F \longrightarrow j^\dagger F \longrightarrow 0$$

and by (9.1.6), a distinguished triangle

$$R\underline{\Gamma}_{]X[}E \longrightarrow E \longrightarrow Ri_*i^*E,$$

it is sufficient to show that, if E and F are sheaves of K-vector spaces on V, then

$$\underline{\Gamma}_{]X[}E \otimes_K \underline{\Gamma}_\dagger F = 0.$$

In other words, we have to show that the sheaf associated with

$$W \longmapsto \Gamma(W, \underline{\Gamma}_{]X[}E) \otimes_K \Gamma(W, \underline{\Gamma}_{\dagger}F)$$

is zero. If W is affinoid, and therefore compact, the value of this presheaf is

$$\varinjlim \Gamma_{]X[}(W, E) \otimes_K \Gamma_{V \setminus V'}(W, F)$$

Hence, any section s of this presheaf over W can be written $s = \sum_i s_i \otimes_K t_i$ where s_i (resp. t_i) is a section of E (resp. F) over W whose restriction to $S = V \setminus]X[$ (resp. V') is trivial. Since $V = S \cup V'$ is an admissible covering, this implies that s is necessarily zero.□

(10.5) The Poincaré Pairing

We assume that K has characteristic zero. Let X be a d-dimensional k-scheme, $X \hookrightarrow P$ an admissible (1.2.4) embedding and V a smooth strict neighborhood of X in P.

(10.5.1) It follows from Lemma (10.4) that the wedge product

$$\Omega^{\bullet}_V \otimes_K \Omega^{\bullet}_V \longrightarrow \Omega^{\bullet}_V$$

induces a commutative diagram

$$
\begin{array}{ccc}
\mathbf{R}\Gamma_{]X[}\Omega^{\bullet}_{V} \otimes_{K} j^{\dagger}\,\Omega^{\bullet}_{V} & \longrightarrow & \mathbf{R}\Gamma_{]X[}\Omega^{\bullet}_{V} \\
\downarrow\wr & & \| \\
\mathbf{R}\Gamma_{]X[}\Omega^{\bullet}_{V} \otimes_{K} \Omega^{\bullet}_{V} & \longrightarrow & \mathbf{R}\Gamma_{]X[}\Omega^{\bullet}_{V} \\
\downarrow & & \downarrow \\
\Omega^{\bullet}_{V} \otimes_{K} \Omega^{\bullet}_{V} & \longrightarrow & \Omega^{\bullet}_{V} \\
\uparrow & & \uparrow \\
\Omega^{\bullet}_{V} \otimes_{K} \Gamma_{\dagger}\Omega^{\bullet}_{V} & \longrightarrow & \Gamma_{\dagger}\Omega^{\bullet}_{V} \\
\downarrow\wr & & \downarrow\wr \\
\mathbf{R}\, i_{*}i^{*}\,\Omega^{\bullet}_{V} \otimes_{K} \Gamma_{\dagger}\Omega^{\bullet}_{V} & \longrightarrow & \Gamma_{\dagger}\Omega^{\bullet}_{V}.
\end{array}
$$

(10.5.2) Taking $H^{2d}(V, -)$ of the top arrow of the diagram (10.5.1), we obtain bilinear maps

$$H^{i}_{rig,c}(X) \times H^{2d-i}_{rig}(X) \longrightarrow H^{2d}_{rig,c}(X)$$

which can be seen to only depend on X by the method of diagonal embedding. These maps are also functorial with respect to open immersions.

(10.5.3) Definition Composing the bilinear maps of (10.5.2) with the trace map (10.3.4), if this is well defined, gives the *Poincaré pairings*

$$H^{i}_{rig,c}(X) \times H^{2d-i}_{rig}(X) \longrightarrow K.$$

(10.6) Proposition *Let $\mathfrak{X}/\mathcal{V}$ be a flat, projective, generically smooth formal scheme,* U *a smooth open subset of the special fibre* X *of* $\mathfrak{X}$ *and* Z *the closed complement of* U *in* X. *There is a natural morphism of long exact sequences*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i_{rig,c}(U) & \xrightarrow{sp^i_c} & H^i_{DR}(\mathfrak{X}_K) & \longrightarrow & H^i_{DR}(]Z[) & \longrightarrow & H^{i+1}_{rig,c}(U) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^{2d-i}_{rig}(U)^\vee & \xrightarrow{\cos^{2d-1}} & H^{2d-i}_{DR}(\mathfrak{X}_K)^\vee & \longrightarrow & H^{2d-i}_{DR,c}(]Z[)^\vee & \longrightarrow & H^{2d-i-1}_{rig}(U)^\vee & \longrightarrow \cdots
\end{array}$$

(in which the superscript $^\vee$ *denotes the dual vector space) in which the vertical maps in* U *and* $\mathfrak{X}_K$ *are induced by the Poincaré pairings (10.5.3) (provided the trace map (10.3.4) is well defined for* U)

Proof. Applying $\mathbf{R}\Gamma(\mathfrak{X}_K, -)$ to the big commutative diagram of (10.5.1) with $V = \mathfrak{X}_K$, and composing on the right with the trace map

$$\mathrm{tr} : \mathbf{R}\Gamma(\mathfrak{X}_K, \Omega^\bullet_V) \longrightarrow K[-2d],$$

which exists since $\mathfrak{X}_K$ is projective, yields a commutative diagram

$$\begin{array}{ccc}
\mathbf{R}\Gamma_{]U[}(\mathfrak{X}_K, \Omega^\bullet) \otimes_K \mathbf{R}\Gamma(\mathfrak{X}_K, j^\dagger\Omega^\bullet) & \longrightarrow & K[-2d] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(\mathfrak{X}_K, \Omega^\bullet) \otimes_K \mathbf{R}\Gamma(\mathfrak{X}_K, \Omega^\bullet) & \longrightarrow & K[-2d] \\
\uparrow & & \uparrow \\
\mathbf{R}\Gamma(]Z[, \Omega^\bullet) \otimes_K \mathbf{R}\Gamma(\mathfrak{X}_K, \underline{\Gamma}_! \Omega^\bullet) & \longrightarrow & K[-2d].
\end{array}$$

which induces a morphism of long exact sequences as asserted.

The vertical maps in $\mathfrak{X}_K$ are clearly induced by the Poincaré pairings in de Rham cohomology. Moreover, it is immediate from Construction (10.3.2) that the composite homomorphism

$$H^{2d}_{rig,c}(U) \longrightarrow H^{2d}_{DR}(\mathfrak{X}_K) \longrightarrow K$$

is the trace map in rigid cohomology if this is well defined. In particular the vertical homomorphisms corresponding to U are induced by the Poincaré pairings in rigid cohomology.□

(11) POINCARE DUALITY FOR CURVES

In this section, we assume that K has characteristic zero. We prove Poincaré duality for curves in rigid cohomology. Poincaré duality in rigid cohomology is known for (smooth) proper schemes when K is discrete and k perfect since it can then be deduced from Poincaré duality in crystalline cohomology. Berthelot can also prove it for (smooth) affine schemes of dimension at most three. Since a smooth curve is a disjoint union of affine and proper schemes, the main theorem of this section is not new. The idea in the present proof is to deduce the theorem from an analogous result in analytic de Rham cohomology.

(11.1) Class of a Point on a Smooth 1-Dimensional Analytic Space

Let S/K be a smooth analytic space of dimension one and ξ a point on S.

(11.1.1) Lemma *If* $f \in \Gamma(S', \mathcal{O})$ *is a defining equation for* ξ *in some neighborhood* S' *of* ξ *in* S, *then the image* $c_{\xi/S}$ *of* dlogf *under the canonical map*

$$\Gamma(S' \setminus \xi, \Omega^1) \longrightarrow H^1_\xi(S', \Omega^1) \xleftarrow{\sim} H^1_\xi(S, \Omega^1).$$

does not depend on f.

Proof. Let g be another defining equation for ξ in some neighborhood S'' of ξ. Since S is smooth of dimension one, the local ring at ξ is a discrete valuation ring. It follows that there exists a neighborhood S''' of ξ and $u \in \Gamma(S''', \mathcal{O}^\times)$ such that $g = uf$ on S'''. Then, on $S''' \setminus \xi$, we have $\operatorname{dlog} g = \operatorname{dlog} f + \operatorname{dlog} u$ with $\operatorname{dlog} u \in \Gamma(S''', \Omega^1)$ and the assertion follows from the exactness of the sequence

$$\Gamma(S''', \Omega^1) \longrightarrow \Gamma(S''' \setminus \xi, \Omega^1) \longrightarrow H^1_\xi(S''', \Omega^1). \square$$

(11.1.2) Definition The image $c_{\xi/S}$ of dlog f under the canonical map

$$\Gamma(S' \setminus \xi, \Omega^1) \longrightarrow H^1_\xi(S', \Omega^1) \xleftarrow{\sim} H^1_\xi(S, \Omega^1).$$

is the *class of the point* ξ in $H^1_\xi(S, \Omega^1)$.

(11.1.3) Definition If S is ind-compact, the image $s_{\xi/S}$ of $c_{\xi/S}$ under the composite map

$$H^1_\xi(S, \Omega^1) \longrightarrow H^1_c(S, \Omega^1) \longrightarrow H^2_c(S, \Omega^\bullet) = H^2_{DR,c}(S)$$

is the *de Rham-class of the point* ξ .

(11.2) Trace Map and Poincaré Pairing for a 1-Dimensional Ind-Compact Analytic Space

Let S be an analytic variety over K, C a non singular projective curve over K and $S \hookrightarrow C^{an}$ an open immersion. It is shown in [Liu] that S is necessarily ind-compact. We consider the composite map

$$tr : H^2_{DR,c}(S) \longrightarrow H^2_{DR}(C) \xrightarrow{\sim} K.$$

(11.2.1) Definition If tr is well defined (i.e. independent of the embedding), it is the *trace map in analytic de Rham cohomology.*

(11.2.2) Lemma *If* ξ *is a point on* S, *the de Rham class* $s_{\xi/S}$ *is sent to* 1 *under* tr.

Proof. Let f be a (algebraic) defining equation for ξ in some (algebraic) open subset V of C such that f is invertible outside ξ. Then, one knows that dlog f is sent to 1 under the composite map

$$\Gamma(V\setminus\xi,\Omega^1)\longrightarrow H^1_\xi(C,\Omega^1)\longrightarrow H^2_{DR}(C)\xrightarrow{\sim} K.$$

The assertion therefore follows from the commutativity of the diagram

$$\begin{array}{ccccc}
\Gamma(V\setminus\xi,\Omega^1) & \longrightarrow & H^1_\xi(C,\Omega^1) & \longrightarrow & H^2_{DR}(C) \\
\downarrow & & \downarrow & & \downarrow\wr \\
\Gamma(V^{an}\setminus\xi,\Omega^1) & \longrightarrow & H^1_\xi(C^{an},\Omega^1) & \longrightarrow & H^2_{DR}(C^{an}) \\
\downarrow & & \downarrow\wr & & \uparrow \\
\Gamma(S\cap V^{an}\setminus\xi,\Omega^1) & \longrightarrow & H^1_\xi(S,\Omega^1) & \longrightarrow & H^2_{DR,c}(S).\square
\end{array}$$

(11.2.3) It follows from (11.2.2) that if $h^2_{DR,c}(S) = 1$, then tr is well defined. Actually, it is sufficient to require that after a sufficiently big finite extension of K, then $h^2_{DR,c}$ is the number of connected components of S.

(11.2.4) Definition If the trace map is well defined, the pairings

$$H^i_{DR}(S)\times H^{2-i}_{DR,c}(S)\longrightarrow K.$$

obtained by composing the bilinear maps

$$H^i_{DR}(S)\times H^{2-i}_{DR,c}(S)\longrightarrow H^2_{DR,c}(S)$$

induced from the wedge product on S with the trace map are the *Poincaré pairings.*

(11.3) Proposition *For the open disc* **D** *of radius one, the trace (11.2.1) map is well defined and the Poincaré pairings (11.2.4) are perfect.*

Proof. Coherent sheaves have no higher cohomology on **D** since it is a quasi-Stein (as in [Kiehl 67, Inv. Math]) space. This implies that $H^*_{DR}(\mathbf{D})$ is the cohomology of the complex

$$\Gamma(\mathbf{D}, \mathcal{O}_{\mathbf{D}}) \longrightarrow \Gamma(\mathbf{D}, \Omega^1_{\mathbf{D}}).$$

Let us choose a parameter t for **D** so that

$$A = \{\sum_{i\geq 0} a_i t^i \,/\, a_i \in K \text{ and } \forall\, \varepsilon < 1 \,/\, \varepsilon^i |a_i| \to 0\}$$

is the algebra of functions on **D** and $H^*_{DR}(\mathbf{D})$ is the cohomology of the complex

$$d : A \longrightarrow A\, dt.$$

One immediately checks that the sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{\;d\;} A\, dt \longrightarrow 0$$

is exact, so that **D** has no higher de Rham cohomology and $H^0_{DR}(\mathbf{D}) = K$.

For $\varepsilon < 1$, let $\mathbf{D}_\varepsilon$ be the closed disc of radius ε and $\mathbf{C}_\varepsilon$ its complement in **D**. Since **D** and $\mathbf{C}_\varepsilon$ are Stein, we have an exact sequence

$$0 \longrightarrow H^0_{\mathbf{D}_\varepsilon}(\mathbf{D}, \mathcal{O}) \longrightarrow \Gamma(\mathbf{D}, \mathcal{O}) \longrightarrow \Gamma(\mathbf{C}_\varepsilon, \mathcal{O}) \longrightarrow H^1_{\mathbf{D}_\varepsilon}(\mathbf{D}, \mathcal{O}) \longrightarrow 0$$

and we also see that $H^i_{\mathbf{D}_\varepsilon}(\mathbf{D}, \mathcal{O})$ for $i > 1$. From the obvious injectivity of the middle homomorphism in this sequence, it results that $H^0_{\mathbf{D}_\varepsilon}(\mathbf{D}, \mathcal{O})$ as well is zero.

Note that the $\mathbf{D}_\varepsilon$'s form a fundamental system of compact open supports in **D**. Taking direct limits, we therefore obtain the nullity of $H^i_c(\mathbf{D}, \mathcal{O})$ for $i \neq 1$, and an exact sequence

$$0 \longrightarrow A \longrightarrow A' \longrightarrow H^1_c(\mathbf{D}, \mathcal{O}) \longrightarrow 0.$$

where $A' := \varinjlim \Gamma(C_\varepsilon, \mathcal{O})$. With respect to our parameter t, we have

$$A' = \{\sum_i a_i t^i \,/\, a_i \in K \,/\, \exists\, \varepsilon < 1 \,/\, \varepsilon^i |a_i| << \infty \text{ for } i << 0 \text{ and } \forall\, \varepsilon < 1 \,/\, \varepsilon^i |a_i| \to 0 \text{ for } i >> 0\}$$

and we can identify $H^1_c(\mathbf{D}, \mathcal{O})$ with the orthogonal complement

$$A_c := \{\sum_{i<0} a_i t^i \,/\, a_i \in K \text{ and } \exists\, \varepsilon < 1 \,/\, \varepsilon^i |a_i| << \infty\}.$$

of A in A' with respect to the topological basis $\{t^i\}_{i\in \mathbf{Z}}$.

By definition, $H^*_{DR,c}(\mathbf{D})$ is the cohomology with compact support of the complex

$$d : \mathcal{O}_{\mathbf{D}} \longrightarrow \Omega^1_{\mathbf{D}} = \mathcal{O}_{\mathbf{D}}\, dt.$$

We have just seen that $R\Gamma_c(\mathbf{D}, \mathcal{O}_{\mathbf{D}}) \cong A_c[-1]$ and it follows that $H^*_{DR,c}(\mathbf{D})$ is the cohomology of the complex

$$[\, A_c \xrightarrow{\ d\ } A_c\, dt\,]\,[-1].$$

It is easily seen that the sequence

$$\begin{array}{c} 0 \longrightarrow A_c \xrightarrow{\ d\ } A_c\, dt \xrightarrow{\ res\ } K \longrightarrow 0 \\ \sum a_i t^i\, dt \longmapsto a_{-1} \end{array}$$

is exact. This gives that $H^i_{DR,c}(\mathbf{D})$ is zero for $i \neq 2$ and an isomorphism

$$H^2_{DR,c}(\mathbf{D}) \xrightarrow{\ \sim\ } K$$

which is just the trace map since it clearly sends dlog t to 1.

The Poincaré pairing

$$H^0_{DR}(\mathbf{D}) \times H^2_{DR,c}(\mathbf{D}) \longrightarrow H^2_{DR,c}(\mathbf{D}) \xrightarrow{\sim} K$$

is induced by the bilinear map

$$< , > : A \times A_c\, dt \longrightarrow A_c\, dt$$

$$(\sum_{i\geq 0} a_i t^i, \sum_{i<0} b_i t^i\, dt) \longmapsto \sum_{i<0} c_i t^i dt$$

with $c_k = \sum_{i+j=k} a_i b_j$. This pairing is clearly non zero and therefore perfect since it involves 1-dimensional spaces.□

(11.4) THEOREM *If U is a smooth curve over k, then the trace map is well defined and the Poincaré pairings (10.5.3)*

$$H^i_{rig,c}(U) \times H^{2-i}_{rig}(U) \longrightarrow K$$

are perfect.

Proof. It follows from (10.3.5) and Corollary (8.5.2) that the trace map is well defined. Making if necessary a finite extension of K, we can use Lemma (8.2) to deduce from Proposition 7.4 of [Grothendieck 73] that there exists a smooth projective formal scheme $\mathfrak{X}$ over $\mathcal{V}$ whose special fibre X is a compactification of U. Let Z be the complement of U in X. We have seen in Proposition (10.5) that, if C is the generic fibre of $\mathfrak{X}$, then there is a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i_{rig,c}(U) & \xrightarrow{sp^i_c} & H^i_{DR}(C) & \longrightarrow & H^i_{DR}(]Z[) & \longrightarrow & H^{i+1}_{rig,c}(U) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^{2-i}_{rig}(U)^{\vee} & \xrightarrow{\cos^{1,\vee}} & H^{2-i}_{DR}(C)^{\vee} & \longrightarrow & H^{2-i}_{DR,c}(]Z[)^{\vee} & \longrightarrow & H^{1-i}_{rig}(U)^{\vee} & \longrightarrow \cdots
\end{array}$$

where the vertical homomorphisms corresponding to C and U are induced by the Poincaré pairings.

Poincaré duality in algebraic de Rham cohomology states that the vertical maps corresponding to C are bijective. In order to prove the theorem, it is therefore sufficient, by the five lemma, to show that the maps

$$H^i_{DR}(]Z[) \longrightarrow H^{2-i}_{DR,c}(]Z[)^{\vee}$$

are bijective. By construction, these maps are induced by the Poincaré pairings (provided the trace map is well defined) in analytic de Rham cohomology.

After a finite extension of K, the finite variety Z becomes a finite union of rational points, and by additivity, we are reduced to show the bijectivity of

$$H^i_{DR}(]x[) \longrightarrow H^{2-i}_{DR,c}(]x[)^{\vee}$$

when x is some rational point on X. In other words, we have to show that the trace map is well defined on $H^2_{DR,c}(]x[)$ and that the Poincaré pairings

$$H^i_{DR}(]x[) \times H^{2-i}_{DR,c}(]x[) \longrightarrow K$$

are perfect.

In general, if x is a smooth rational point on a formal $\mathcal{V}$-scheme, there is an étale map π from a neighborhood of x to the projective space and it follows from Proposition 1.3.1 in [Berthelot 89?] that the induced map between]x[to]π(x)[is an isomorphism. This implies that]x[is isomorphic to the open ball of radius one. In our one dimensional case, we see that]x[is isomorphic to the open disc of radius one, and the theorem is therefore a consequence of Lemma (11.3).□

(11.5) Corollaries

(11.5.1) It follows from Theorem (11.4) and Remark (8.5.2) that, when k is perfect, if X is a smooth curve over k, then

$$h^0(X) = c, \qquad h^1(X) = 2d + s,$$

$$h^2(X) = c' \quad \text{and} \quad h^i(X) = 0 \text{ for } i > 2,$$

where d is the arithmetic genus (7.4.3) of X, c (resp. c') is the geometric number of 1-dimensional connected (resp. proper connected) components of X and s is the geometric number of points needed to smoothly compactify X. This agrees with the results of [Monsky-Washnitzer]. We also see that $H^i_{rig,c}(X)$ commutes with isometric extensions of K.

(11.5.2) *If* U/k *is a smooth curve and* U' *a dense open subset, the restriction homomorphism*

$$H^1_{rig}(U) \longrightarrow H^1_{rig}(U')$$

is injective.

Proof. The complement of U' in U being zero-dimensional has no higher rigid cohomology. It therefore results from the Gysin sequence that the restriction map

$$H^1_{rig,c}(U') \longrightarrow H^1_{rig,c}(U)$$

is surjective. The assertion is therefore a consequence of Poincaré duality.□

(11.5.3) *Let* C/K *be a non singular projective curve,* $\mathfrak{X}$ *a flat, projective formal* $\mathcal{V}$*-scheme such that* $\mathfrak{X}_K \cong C^{an}$, U *a smooth open subset of the special fibre* X *of* $\mathfrak{X}$ *and* Z *the closed complement of* U *in* X. *Then, the trace map is well defined on* $H^2_{DR}(]Z[)$ *and the Poincaré pairings*

$$H^i_{DR}(]Z[) \times H^{2-i}_{DR,c}(]Z[) \longrightarrow K$$

are perfect. Moreover, the morphism of long exact sequences

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^i_{rig,c}(U) & \longrightarrow & H^i_{DR}(C) & \longrightarrow & H^i_{DR}(]Z[) & \longrightarrow & H^{i+1}_{rig,c}(U) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^{2d-i}_{rig}(U)^\vee & \xrightarrow{\cos^{1,\vee}} & H^{2-i}_{DR}(C)^\vee & \longrightarrow & H^{2-i}_{DR,c}(]Z[)^\vee & \longrightarrow & H^{1-i}_{rig}(U)^\vee & \longrightarrow \cdots \end{array}$$

of (10.5) *is an isomorphism where all the vertical homomorphisms are induced by the Poincaré pairings.*

Proof. It is clear by construction that the vertical maps corresponding to Z are induced by the Poincaré pairings provided the trace map is well defined. From Poincaré duality in de Rham and rigid cohomology, we know that vertical maps corresponding to C and U are bijective. The assertion follows immediately.□

(12) THE ORTHOGONALITY THEOREM FOR THE WEIGHT FILTRATION

In this section, starting at (12.2), the field K is assumed to have characteristic zero. We define the weight filtration on the first de Rham cohomology space of a smooth projective curve over K and prove an orthogonality theorem similar to Grothendieck's theorem in ℓ-adic cohomology.

(12.1) Formal Model for a Proper Scheme

We recall the definition (14.3.2) of a *(geometrically) distinguished formal $\mathcal{V}$-scheme*: It is a flat formal $\mathcal{V}$-scheme with (geometrically) reduced fibres.

(12.1.1) Definition A *formal model* for a proper scheme V over K is a formal scheme $\mathfrak{X}$ over $\mathcal{V}$ whose generic fibre $\mathfrak{X}_K$ is isomorphic to the analytification V^{an} of V.

(12.1.2) It is a consequence of the unpublished work of Raynaud (see [Raynaud 74], [Mehlman] and also [Nagata]) that any proper scheme over K has a formal model $\mathfrak{X}/\mathcal{V}$ and that then, $\mathfrak{X}$ is necessarily proper. Also, replacing it by its maximal flat closed subscheme, we may always assume that a formal model is flat.

(12.1.3) It immediately results from Proposition 7.2 in [Grothendieck 71] that if K is discretely valued, then any proper flat formal $\mathcal{V}$-scheme $\mathfrak{X}$ with one dimensional special fibre X is projective: There exists a unique projective scheme Y over $\mathcal{V}$ such that $\mathfrak{X}$ is the completion of Y along its special fibre X. And then, $\mathfrak{X}$ is a formal model for Y_K.

(12.1.4) I cannot show that (12.1.3) is still valid over an arbitrary complete ultrametric field K without assuming X geometrically distinguished (see Theorem (14.6.2)).

(12.2) Lemma *Let* C *be a projective non singular curve over* K *and* $\mathfrak{X}$ *a geometrically distinguished*

(14.3.2) formal model (12.1.1) for C *with special fibre* X. *Then, the specialization (10.1.4) homomorphism*

$$sp^1 : H^1_{rig}(X) \longrightarrow H^1_{DR}(C)$$

is injective. More precisely, if U *is a smooth dense open subset of* X, sp^1 *identifies* $H^1_{rig}(X)$ *with the image of the specialization (10.1.2) homomorphism*

$$sp^1_c : H^1_{rig,c}(U) \longrightarrow H^1_{DR}(C).$$

Proof. Let U be a smooth dense open subset of X and Z its closed complement. Since Z is 0-dimensional, we have $H^1_{rig}(Z) = 0$. Also, it follows from Theorem (14.6.4) that the natural map

$$H^0_{rig}(Z) \longrightarrow H^0_{DR}(]Z[_{\mathfrak{X}})$$

is bijective. From the morphism of long exact sequences of (10.1.5), we therefore obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} H^0_{rig}(Z) & \longrightarrow & H^1_{rig,c}(U) & \longrightarrow & H^1_{rig}(X) & \longrightarrow & 0 \\ \downarrow\wr & & \| & & \downarrow sp^1 & & \\ H^0_{DR}(]Z[_X) & \longrightarrow & H^1_{rig,c}(U) & \xrightarrow{sp^1_c} & H^1_{DR}(C) & & \end{array}$$

from which our assertion follows.□

(12.3) The Weight Filtration

We assume that k is perfect and let C/K be a projective non singular curve. Let $\mathfrak{X}$ be a distinguished formal model (12.1.1) for C with special fibre X.

(12.3.1) There is a unique filtration Fil_W^* on $H^1_{DR}(C)$ such that

i) $\mathrm{Fil}_W^0 H^1_{DR}(C)) := H^1_{DR}(C)$ and $\mathrm{Fil}_W^3 H^1_{DR}(C)) := 0$,

ii) using Lemma (12.2) to identify $H^1_{rig}(X)$ with its image in $H^1_{DR}(C)$, $\mathrm{Fil}_W^1 H^1_{DR}(C) := H^1_{rig}(X)$ and

iii) using the natural surjection $H^1_{rig}(X) \twoheadrightarrow H^1_{rig}(\tilde{X})$ which is surjective as mentioned in (8.5.1) to identify $H^1_{rig}(\tilde{X})$ with a quotient of $\mathrm{Fil}_W^1 H^1_{DR}(C)$,

$$\mathrm{Gr}_W^1 H^1_{DR}(C) = H^1_{rig}(\tilde{X}).$$

(12.3.2) Definition The filtration defined in (12.3.1) is the *weight filtration* associated to $\mathfrak{X}$ on $H^1_{DR}(C)$.

(12.3.3) Remark Let g be the arithmetic genus (7.4.3) of C, d the geometric genus and t the cyclomatic number (7.4.2) of X. It follows from lemma (8.3) and Theorem (8.4) that

$$\dim_K \mathrm{Fil}_W^0 H^1_{DR}(C) = 2g,$$

$$\dim_K \mathrm{Fil}_W^1 H^1_{DR}(C) = 2d + t,$$

$$\dim_K \mathrm{Fil}_W^2 H^1_{DR}(C) = t.$$

(12.3.4) Notation We will be interested in the orthogonal to the weight filtration for the Poincaré pairing

$$H^1_{DR}(C) \times H^1_{DR}(C) \longrightarrow K.$$

We will follow the usual conventions and write $\mathrm{Fil}^i_\perp := [\mathrm{Fil}^{1-i}]^\perp$.

(12.4) THEOREM (K of characteristic zero with perfect residue field) *Let* C/K *be a non singular projective curve,* $\mathfrak{X}$ *a distinguished model for* C *and* Fil^* *the weight filtration (12.3.2) associated to* $\mathfrak{X}$ *on* $H^1_{DR}(C)$.

Then, we have

$$\mathrm{Fil}^2 = \mathrm{Fil}^1 \cap \mathrm{Fil}^0_{\perp} \quad \text{and} \quad \mathrm{Fil}^{-1}_{\perp} = \mathrm{Fil}^0_{\perp} + \mathrm{Fil}^1.$$

Moreover, the filtration is auto dual (i.e. for all i, $\mathrm{Fil}^i_{\perp} = \mathrm{Fil}^{i+2}$*) if and only if the special fibre* X *of* $\mathfrak{X}$ *has only ordinary multiple points with normal tangents (7.2.5) as singularities.*

Proof. By definition, the specialization homomorphism

$$H^1_{\mathrm{rig}}(X) \longrightarrow H^1_{\mathrm{DR}}(C)$$

identifies $H^1_{\mathrm{rig}}(X)$ with Fil^1 and, if $\tilde{X}$ is the normalization of X, the kernel of the natural map

$$H^1_{\mathrm{rig}}(X) \longrightarrow H^1_{\mathrm{rig}}(\tilde{X})$$

is Fil^2.

Let U be a smooth dense open subset of X and $\tilde{U}$ its inverse image in $\tilde{X}$. Consider the commutative square

$$\begin{array}{ccc} H^1_{\mathrm{rig}}(X) & \longrightarrow & H^1_{\mathrm{rig}}(U) \\ \downarrow & & \| \\ H^1_{\mathrm{rig}}(\tilde{X}) & \longrightarrow & H^1_{\mathrm{rig}}(\tilde{U}). \end{array}$$

We know from Corollary (11.5.2) that the bottom homomorphism is injective. This implies that Fil^2 is also the kernel of the restriction homomorphism

$$H^1_{\mathrm{rig}}(X) \longrightarrow H^1_{\mathrm{rig}}(U).$$

We have seen in (10.2.3) that this restriction map factors as

$$H^1_{rig}(X) \xrightarrow{sp^1} H^1_{DR}(C) \xrightarrow{\cos^1} H^1_{rig}(U).$$

Thus, we see that Fil^2 is the intersection of Fil^1 with the kernel of $\cos^1$. In order to prove that $Fil^2 = Fil^1 \cap Fil^0_\perp$ it is therefore sufficient to show that the kernel of $\cos^1$ is $Fil^0_\perp$. From Corollary (11.5.3), it results that the Poincaré dual to the kernel of $\cos^1$ is the cokernel of sp^1_c. It is therefore sufficient to show that the cokernel of sp^1_c is the Poincaré Dual Gr^0 to $Fil^0_\perp$. This is an immediate consequence of Lemma (12.2) where we showed that the image of

$$sp^1_c : H^1_{rig,c}(U) \longrightarrow H^1_{DR}(C)$$

is exactly Fil^1.

Thus, we have $Fil^2 = Fil^1 \cap Fil^0_\perp$ and we immediately get that

$$Fil^{-1}_\perp = (Fil^2)^\perp = (Fil^1 \cap Fil^0_\perp)^\perp = (Fil^1)^\perp + (Fil^0_\perp)^\perp = Fil^0_\perp + Fil^1.$$

It remains to prove the last assertion. Note that for $i \neq 0$ and -1, we always have $Fil^i_\perp = Fil^{i+2}$. Moreover, if $Fil^0_\perp = Fil^2$ then necessarily $Fil^{-1}_\perp = (Fil^2)^\perp = Fil^1$. Therefore, the filtration will be auto dual if and only if $Fil^0_\perp = Fil^2$. From the first part of the proposition, we know that $Fil^0_\perp \supset Fil^2$. The filtration will therefore be auto dual if and only if $\dim_K Fil^2 = \dim_K Fil^0_\perp$.

Let g (resp. d, resp. t, resp. s) be the arithmetic genus of C (resp. the geometric genus of X, resp. the cyclomatic number of X, resp. the singular genus (7.2.4) of X). We have seen in (12.3.3) that $\dim_K Fil^2 = t$ and that $\dim_K Fil^1 = 2d + t$ and it follows that $\dim_K Fil^0_\perp = 2g - 2d - t$. We thus see that $\dim_K Fil^2 = \dim_K Fil^0_\perp$ if and only if $g = d + t$ or, using Proposition (7.5) and Theorem (14.6.1), if and only if $s = 0$. We showed in (7.3) that this happens exactly when X has only ordinary multiple points (with normal tangents) as singularities.□

(12.5) Remarks

(12.5.1) Using Theorem (12.4), one can give another definition of the weight filtration (12.3.2): Fil^1 is the image of

$$\mathrm{sp}_c^1 : H^1_{rig,c}(U) \longrightarrow H^1_{DR}(C)$$

and Fil^2 is the intersection of Fil^1 with the kernel of

$$\cos^1 : H^1_{DR}(C) \longrightarrow H^1_{rig}(U).$$

This definition makes sense even when k is not perfect and permits us to remove this assumption in the definition of the weight filtration.

(12.5.3) Using Corollary (11.5.3), we can give a third definition of the weight filtration: Fil^1 is the kernel of the restriction map

$$H^1_{DR}(C) \longrightarrow H^1_{DR}(]Z[)$$

and Fil^2 is the intersection of Fil^1 with the image of the restriction map

$$H^1_{DR,c}(]Z[) \longrightarrow H^1_{DR}(C).$$

This definition makes sense even when X is not reduced and permits us to remove this assumption in the definition of the weight filtration.

(12.5.4) Let us finally describe a very elementary way of defining the weight filtration: We have

$$\mathrm{Fil}^1 = \bigcap_{x \in X} \mathrm{Ker}\,[H^1_{DR}(C) \longrightarrow H^1_{DR}(]x[)]$$

and Fil^2 is the intersection of Fil^1 with its own orthogonal complement with respect to the Poincaré pairing.

APPENDICES

NUCLEAR SHEAVES
&
DISTINGUISHED FORMAL SCHEMES

(13) NUCLEAR SHEAVES

In this section, we introduce the notion of a nuclear sheaf on affine K-space which allows us to use cohomological arguments in order to (re-) prove some basic results about nuclear operator on a K-vector space. In this section, $\mathbf{A}^1_K$ denotes the affine line with a fixed origin 0.

(13.1) Trace of a Nuclear Sheaf

(13.1.1) Definitions Let $\mathcal{M}$ be a quasi-coherent sheaf on $\mathbf{A}^1_K$. A closed point on $\mathbf{A}^1_K \setminus 0$ is an *eigenvalue for* $\mathcal{M}$ if $\dim_K H^0_x(\mathcal{M}) > 0$. The sheaf $\mathcal{M}$ is *nuclear* if

i) For all $x \in \mathbf{A}^1_K \setminus 0$, $\dim_K H^0_x(\mathcal{M}) < \infty$ and $H^1_x(\mathcal{M}) = 0$, and

ii) The eigenvalues for $\mathcal{M}$ are discrete in $\mathbf{P}^1_K \setminus 0$ for the underlying analytic structure (i.e. any closed disk of $\mathbf{P}^1_K \setminus 0$ contains only a finite number of eigenvalues).

(13.1.2) Notations Since the origin of $\mathbf{A}^1_K$ is fixed, there is a canonical isomorphism $\mathbf{A}^1_K \cong \operatorname{spec} K[z]$. If x is a closed point on $\mathbf{A}^1_K$, then tr x is the trace of $z(x) \in K(x)$ over K and $\det(1\text{-}tx)) := t^{\deg x} P(1/t)$ where P is the characteristic polynomial of z(x) over K.

(13.1.3) Definitions Let $\mathcal{M}$ be a nuclear sheaf on $\mathbf{A}^1_K$. For a closed point $x \in \mathbf{A}^1_K \setminus 0$, let $\delta_x := \dim_K H^0_x(\mathcal{M})/\deg x$. Since $|\mathrm{tr}\ x| \le |x|$ and δ_x is an integer, the series

$$\mathrm{tr}\,\mathcal{M} := \sum_{x \in \mathbf{A}^1_K \setminus 0} \delta_x .\ \mathrm{tr}\ x$$

converges towards an element of K called the *trace of* $\mathcal{M}$. Now, if we write $\det(1\text{-}tx) = 1 + a_1 t + \ldots + a_{\deg x} t^{\deg x}$ then we have $|a_i| \le |x|^i$ for all $i = 1, \ldots, \deg x$ and it follows that the infinite product

$$D_{\mathcal{M}}(t) := \prod_{x \in A^1_K \setminus 0} \det(1-tx)^{\delta_x}$$

is an entire series called the *Fredholm Determinant of* $\mathcal{M}$.

(13.1.4) Remark If E is a K-vector space and u an endomorphism of E, there exists a unique quasi-coherent sheaf $\mathcal{M}$ on A^1_K = Spec K[z] such that $\Gamma(A^1_K, \mathcal{M}) = E$ and that for $m \in E$, $zm = u(m)$. If $\mathcal{M}$ is nuclear, we will say that u is a nuclear operator and write tr u := tr $\mathcal{M}$ and $\det(1-tu) := D_u(t) := D_{\mathcal{M}}(t)$. The translation of the next results in this more conventional language is left to the reader.

(13.2) Direct Image of a Nuclear Sheaf

(13.2.1) Let $\mathcal{M}$ be a nuclear sheaf on A^1_K and f a non zero regular function on A^1_K such that f(0) = 0. If we think of f as a finite map f: $A^1_K \longrightarrow A^1_K$, we have, for all closed point $y \in A^1_K$, $H^i_y(f_*\mathcal{M}) = \bigoplus_{f(x)=y} H^i_x(\mathcal{M})$ and it follows that $f_*\mathcal{M}$ is nuclear. For a closed point x on $A^1_K \setminus 0$, we write as in (13.1.3), $\delta_x := \dim_K H^0_x(\mathcal{M})/\deg x$ and we set $tr_x := tr_{K(x)/K}$. Then, we have

$$\mathrm{tr}\, f_*\mathcal{M} = \sum_{y \in A^1_K \setminus 0} (\dim_K H^0_y(f_*\mathcal{M})/\deg y)\ \mathrm{tr}\, y$$

$$= \sum_{y \in A^1_K \setminus 0} \sum_{f(x)=y} (\dim_K H^0_x(\mathcal{M})/\deg x)\ (\deg x/\deg y)\ \mathrm{tr}\, y$$

Therefore, if we think of f(x) as an element of K(x) via the natural embedding $K(f(x)) \hookrightarrow K(x)$, we have

$$\mathrm{tr}\, f_*\mathcal{M} = \sum_{x \in A^1_K \setminus 0} \delta_x .\ tr_x\, f(x).$$

(13.2.2) (We assume that K has characteristic zero) *Let* $\mathcal{M}$ *be a nuclear sheaf on* A^1_K. *For* $f = z^r$, *with* $r > 0$, *let* $\mathcal{M}_r := f_*\mathcal{M}$. *Then, we have the identity of formal power series*

$$D_{\mathcal{M}}(t) = \exp[-\sum_{r=1}^{\infty} (\mathrm{tr}\, \mathcal{M}_r)\, t^r/r].$$

Proof. Since both power series have constant term 1, it is sufficient to show they have the same logarithmic derivative. On one hand, we have

$$\mathrm{dlog}\,(D_{\mathcal{M}}(t))\,/\,dt = \sum_{x \in \mathbf{A}^1_K \setminus 0} \delta_x .\ \mathrm{dlog}\,(\det(1-tx)).$$

On the other hand, it follows from (13.2.1) that

$$\mathrm{dlog}\{\exp[-\sum_{r=1}^{\infty} (\mathrm{tr}\,\mathcal{M}_r)\, t^r/r]\}\,/\,dt = -\sum_{r=1}^{\infty} (\mathrm{tr}\,\mathcal{M}_r)\, t^{r-1} = \sum_{x \in \mathbf{A}^1_K \setminus 0} \delta_x \,(-\sum_{r=1}^{\infty} . (\mathrm{tr}_x x^r)\, t^{r-1})$$

and we are reduced to the well known (and easy to check) statement that $\det(1\text{-}tx) = \exp[-\sum_{r=1}^{\infty} . (\mathrm{tr}_x x^r)\, t^r]$.□

(13.3) Lemmas

(13.3.1) *Let* $\mathcal{M}$ *be a quasi-coherent sheaf on* $\mathbf{A}^1_K$ *and* f *a defining equation for a closed point* x *on* $\mathbf{A}^1_K$. *Then, we have*

i) *Any element of* $H^0_x(\mathcal{M})$ *is killed by some power of* f *and any power of* f *is surjective on* $H^1_x(\mathcal{M})$.

ii) *If there is a short exact sequence*

$$0 \longrightarrow N \longrightarrow \Gamma(\mathbf{A}^1_K, \mathcal{M}) \longrightarrow W \longrightarrow 0.$$

with f *bijective on* W *and such that a power of* f *kills* N, *then* $H^0_x(\mathcal{M}) = N$ *and* $H^1_x(\mathcal{M}) = 0$.

Proof. Since $\mathcal{M}$ is quasi-coherent, assertion i) is an immediate consequence of the existence of an exact sequence

$$0 \longrightarrow H^0_x(\mathcal{M}) \longrightarrow \Gamma(\mathbf{A}^1_K, \mathcal{M}) \longrightarrow \Gamma(\mathbf{A}^1_K \setminus x, \mathcal{M}) \longrightarrow H^1_x(\mathcal{M}) \longrightarrow 0.$$

The hypotheses in assertion ii) imply that the localizations at f of N and W are respectively 0 and W. Since localization at f is an exact functor, we see that $\Gamma(A^1_K \setminus x, \mathcal{M}) = W$.□

(13.3.2) *If $\{\mathcal{M}_i\}_{i\in N}$ is a decreasing sequence of nuclear sheaves, then the quasi-coherent sheaf $\mathcal{M}$ associated with $\bigcap_{i\in N}\Gamma(A^1_K, \mathcal{M}_i)$ is nuclear and we have, for all $x \neq 0$, $H^0_x(\mathcal{M}) = \bigcap_{i\in N} H^0_x(\mathcal{M}_i)$.*

Proof. Let $x \neq 0$ be a closed point of A^1_K. We have an inverse system of short exact sequences

$$0 \longrightarrow H^0_x(\mathcal{M}_i) \longrightarrow \Gamma(A^1_K, \mathcal{M}_i) \longrightarrow \Gamma(A^1_K \setminus x, \mathcal{M}_i) \longrightarrow 0.$$

Since, for all $i \in N$, $\mathcal{M}_i$ is nuclear, we know that $H^0_x(\mathcal{M}_i)$ is finite dimensional and it follows that the inverse system $\{H^0_x(\mathcal{M}_i)\}_{i\in N}$ satisfies the Mittag-Leffler condition. There is therefore a short exact sequence

$$0 \longrightarrow \bigcap_{i\in N} H^0_x(\mathcal{M}_i) \longrightarrow \Gamma(A^1_K, \mathcal{M}) \longrightarrow \bigcap_{i\in N} \Gamma(A^1_K \setminus x, \mathcal{M}_i) \longrightarrow 0.$$

It follows from part i) of Lemma (13.3.1) that a defining equation f for x will be bijective on $\bigcap_{i\in N} \Gamma(A^1_K \setminus x, \mathcal{M}_i)$ and that $\bigcap_{i\in N} H^0_x(\mathcal{M}_i)$ is killed by a power of f. Part ii) of Lemma (13.3.1) tells us that $H^0_x(\mathcal{M}) = \bigcap_{i\in N} H^0_x(\mathcal{M}_i)$ and $H^1_x(\mathcal{M}) = 0$. In particular, $H^0_x(\mathcal{M})$ is finite dimensional and zero when one of the $H^0_x(\mathcal{M}_i)$ is. Thus, we see that the sheaf $\mathcal{M}$ is nuclear.□

(13.4) Proposition *The category of nuclear sheaves on A^1_K is stable under kernel, cokernel and extension. Moreover, if $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ is a short exact sequence of nuclear sheaves, then $\operatorname{tr}\mathcal{M} = \operatorname{tr}\mathcal{M}' + \operatorname{tr}\mathcal{M}''$ and $D_{\mathcal{M}}(t) = D_{\mathcal{M}'}(t).\, D_{\mathcal{M}''}(t)$.*

Proof. Let $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ be a short exact sequence of sheaves on A^1_K. Then, for all closed point x on A^1_K, we have an exact sequence

$$0 \to H^0_x(\mathcal{M}') \to H^0_x(\mathcal{M}) \to H^0_x(\mathcal{M}'') \to H^1_x(\mathcal{M}') \to H^1_x(\mathcal{M}) \to H^1_x(\mathcal{M}'') \to 0.$$

It immediately follows that when $\mathcal{M}'$ is nuclear, the nuclearity of $\mathcal{M}$ is equivalent to the nuclearity of $\mathcal{M}''$.

Let us now show that if $\mathcal{M}$ and $\mathcal{M}''$ are nuclear, so is $\mathcal{M}'$. We use part i) of Lemma (13.3.1). Let f be a defining equation for a point x on $A^1_K \setminus 0$. Since $H^0_x(\mathcal{M})$ and $H^0_x(\mathcal{M}'')$ have finite dimension, there exists a power f^r of f which annihilates both. It follows from the right exactness of

$$H^0_x(\mathcal{M}) \longrightarrow H^0_x(\mathcal{M}'') \longrightarrow H^1_x(\mathcal{M}') \longrightarrow 0,$$

that f^r also annihilates $H^1_x(\mathcal{M}')$. Since f^r is always surjective on $H^1_x(\mathcal{M}')$, this implies that $H^1_x(\mathcal{M}') = 0$. Finally, we see that $\dim_K H^0_x(\mathcal{M}') < \infty$ and that any eigenvalue (13.1.1) for $\mathcal{M}'$ is an eigenvalue for $\mathcal{M}$. This shows that $\mathcal{M}'$ is nuclear.

To conclude, it is sufficient to show that if $\mathcal{M} \longrightarrow \mathcal{N}$ is a morphism of nuclear sheaves, then the image $\mathcal{I}$ of $\mathcal{M}$ in $\mathcal{N}$ is nuclear. Let x be a closed point on $A^1_K \setminus 0$. Since $\mathcal{M} \longrightarrow \mathcal{I}$ is surjective and $\mathcal{M}$ nuclear, then necessarily $H^1_x(\mathcal{I}) = 0$. On the other hand, since $\mathcal{I} \longrightarrow \mathcal{N}$ is injective, we see that $\dim_K H^0_x(\mathcal{I}) < \infty$ and that any eigenvalue (13.1.1) for $\mathcal{I}$ is also an eigenvalue for $\mathcal{N}$.

The second assertion follows from the fact that, for all closed point x on A^1_K, there is an exact sequence

$$0 \longrightarrow H^0_x(\mathcal{M}') \longrightarrow H^0_x(\mathcal{M}) \longrightarrow H^0_x(\mathcal{M}'') \longrightarrow 0. \square$$

(14) DISTINGUISHED FORMAL SCHEMES

In this section, we introduce the notion of distinguished formal $\mathcal{V}$-scheme. This will allow us to use rigid analytic geometry to remove Noetherian hypothesis from several theorems about formal schemes. We will write [BGR] for the reference [Bosch-Güntzer-Remmert].

(14.1) Algebraic and Analytic Reduction of a Formal Scheme

Let $\mathcal{A}$ be a complete $\mathcal{V}$-algebra (always assumed topologically finitely presented). We set $\mathcal{A}_K := \mathcal{A} \otimes_{\mathcal{V}} K$ and $A := \mathcal{A} \otimes_{\mathcal{V}} k$ and let $\| - \|$ be the spectral semi-norm on $\mathcal{A}_K$. It is a power multiplicative semi norm (i.e. it commutes with powers). We set

$$\mathcal{A}^o := \{f \in \mathcal{A}_K \,/\, \|f\| \leq 1\}, \qquad \mathcal{A}^{oo} := \{f \in \mathcal{A}_K \,/\, \|f\| < 1\}$$

and $\tilde{\mathcal{A}} := \mathcal{A}^o/\mathcal{A}^{oo}$.

(14.1.1) *$\mathcal{A}^o$ is the integral closure of (the image of) $\mathcal{A}$ in $\mathcal{A}_K$.*

Proof. Choose an epimorphism $\mathcal{V}\{\underline{t}\} \twoheadrightarrow \mathcal{A}$ with $\underline{t} = (t_1, t_2, \ldots, t_n)$ and consider the induced epimorphism $K\{\underline{t}\} \twoheadrightarrow \mathcal{A}_K$. Given any $f \in \mathcal{A}_K$, it results from Proposition 4, § 6.2.2. of [BGR], that there is an integral relation

$$f^r + g_1 f^{r-1} + \ldots + g_r = 0$$

for f over $K\{\underline{t}\}$ such that $\|f\| = \max \|g_i\|^{1/i}$. In particular, if $\|f\| \leq 1$ then for all i, $\|g_i\| \leq 1$ and $g_i \in \mathcal{V}\{\underline{t}\}$. Hence f is integral over $\mathcal{V}\{\underline{t}\}$ and a fortiori over $\mathcal{A}$.

Conversely, any f integral over $\mathcal{A}$ satisfies an integral relation $f^r + g_1 f^{r-1} + \ldots + g_r = 0$ over $\mathcal{V}\{\underline{t}\}$ and it results from the ultrametric inequality that

$$\|f\| \leq \max \|g_i\|^{1/i} \leq 1. \square$$

(14.1.2) Since $\mathcal{A}^{oo}$ is an ideal in $\mathcal{A}^o$ containing $\mathfrak{m}\mathcal{A}^o$, we see that $\bar{\mathcal{A}}$ is a k-algebra and that there is a natural map

$$A \longrightarrow \bar{\mathcal{A}}.$$

(14.1.3) Definition The k-algebras A and $\bar{\mathcal{A}}$ are called respectively the *algebraic* and *analytic reductions* of $\mathcal{A}$.

(14.1.4) Note that $\bar{\mathcal{A}}$ is reduced. (Since $\| - \|$ is power- multiplicative , we have for $f \in \mathcal{A}^o$, $\|f^n\| < 1 \Rightarrow \|f\|^n < 1 \Rightarrow \|f\| < 1$)

(14.2) Distinguished Complete $\mathcal{V}$-Algebras

We keep the notations of (14.1).

(14.2.1) *If* $\| - \|_{res}$ *is the quotient norm for an epimorphism* $K\{\underline{t}\} \twoheadrightarrow \mathcal{A}_K$ *induced by an epimorphism* $\mathcal{V}\{\underline{t}\} \twoheadrightarrow \mathcal{A}$, *then*

i) $\|\mathcal{A}_K\|_{res} = |K|$,

ii) $\mathcal{A} = \{f \in \mathcal{A}_K / \|f\|_{res} \leq 1\}$ and

iii) $\mathfrak{m}\mathcal{A} = \{f \in \mathcal{A}_K / \|f\|_{res} < 1\}$.

Proof. Given any $f \in \mathcal{A}_K$, it results from Corollary 8, § 5.2.7 of [BGR] that there exists $g \in K\{\underline{t}\}$ whose image in $\mathcal{A}_K$ is f and such that $\|f\|_{res} = \|g\|$. Since $\|K\{t\}\| = |K|$, this implies that $\|\mathcal{A}_K\|_{res} = |K|$. Moreover,

$$\|f\|_{res} \le 1 \Leftrightarrow \|g\| \le 1 \Leftrightarrow g \in \mathcal{V}\{t\} \Leftrightarrow f \in \mathcal{A}$$

$$(\text{resp. } \|f\|_{res} < 1 \Leftrightarrow \|g\| < 1 \Leftrightarrow g \in \mathfrak{m}\mathcal{V}\{t\} \Leftrightarrow f \in \mathfrak{m}\mathcal{A}).\square$$

(14.2.2) *If* A *is reduced, the quotient norm* $\|\cdot\|_{res}$ *for an epimorphism* $K\{t\} \twoheadrightarrow \mathcal{A}_K$ *induced by an epimorphism* $\mathcal{V}\{t\} \twoheadrightarrow \mathcal{A}$ *is power-multiplicative.*

Proof. Using the Remark following Proposition 1, § 1.5.3. of [BGR], we have to show that

(i) Given $f \in \mathcal{A}_K$, there exists $\alpha \in K$ and $n \in \mathbf{N}$ such that $\|\alpha f^n\|_{res} = |\alpha|\,\|f^n\|_{res} = 1$,

(ii) The algebra $\{f \in \mathcal{A}_K / \|f\|_{res} \le 1\} / \{f \in \mathcal{A}_K / \|f\|_{res} < 1\}$ is reduced.

Both conditions are consequences of (14.2.1): Condition (i) follows from the fact that $\|\mathcal{A}_K\|_{res} = |K|$ and condition (ii) from the fact that $\{f \in \mathcal{A}_K / \|f\|_{res} \le 1\} / \{f \in \mathcal{A}_K / \|f\|_{res} < 1\} = A.\square$

(14.2.3) Definition An epimorphism $\mathcal{V}\{t\} \twoheadrightarrow \mathcal{A}$ is *distinguished* if the induced epimorphism $K\{t\} \twoheadrightarrow \mathcal{A}_K$ is distinguished. This means (e.g. Definition 6.4.1 in [BGR]) that the quotient norm coincides with the spectral semi-norm. The algebra $\mathcal{A}$ is *distinguished* if if there exists such an epimorphism.

(14.2.4) It follows from (14.2.1) that if $\mathcal{A}$ is distinguished, then $\mathcal{A} = \mathcal{A}^o$, $\mathfrak{m}\mathcal{A} = \mathcal{A}^{oo}$ and $A = \tilde{\mathcal{A}}$.

(14.2.5) *If $\mathcal{A}$ is distinguished, then $\mathcal{A}_K$ and* A *are reduced.*

Proof. In this case, $\|\cdot\|$ is power-multiplicative and it is a norm. Therefore, for $f \in \mathcal{A}_K$, we have

$$f^n = 0 \Rightarrow \|f^n\| = 0 \Rightarrow \|f\|^n = 0 \Rightarrow \|f\| = 0 \Rightarrow f = 0.$$

On the other hand, we have seen in (14.2.4) that $A = \bar{\mathcal{A}}$, which is reduced by (14.1.4).□

(14.3) Distinguished Formal Schemes

(14.3.1) Proposition *For a flat formal $\mathcal{V}$-scheme X, the following are equivalent:*

(i) X has reduced fibres.

(ii) Any epimorphism $\pi: \mathcal{V}\{\underline{t}\} \twoheadrightarrow \mathcal{A}$ with Spf $\mathcal{A}$ an affine open subscheme of X, is distinguished (14.2.3).

(iii) There exists an affine open covering $X = \cup$ Spf $\mathcal{A}_i$ with all the $\mathcal{A}_i$'s distinguished.

Proof. Of course, (ii) implies (iii) and it results from (14.2.5) that (iii) implies (i). Let us show that (i) implies (ii):

The question is local, so we are given a complete flat $\mathcal{V}$-algebra $\mathcal{A}$ with $\mathcal{A}_K$ and A reduced, and an epimorphism $\mathcal{V}\{\underline{t}\} \twoheadrightarrow \mathcal{A}$. We have to show that the induced epimorphism $K\{\underline{t}\} \twoheadrightarrow \mathcal{A}_K$ is distinguished.

We have seen in (14.2.2) that, when A is reduced, $\| - \|_{res}$ is a power-multiplicative norm on $\mathcal{A}_K$. On the other hand, it is shown in Proposition 4 (iii) , § 6.2.1. of [BGR] that, when $\mathcal{A}_K$ is reduced, $\| - \|$ is a power-multiplicative norm on $\mathcal{A}_K$. But, we know from Lemma 3, § 3.8.3 of [BGR] that there can only be one power-multiplicative norm on $\mathcal{A}_K$.□

(14.3.2) Definition A flat formal $\mathcal{V}$-scheme X is *distinguished* if it satisfies the equivalent conditions of Proposition (14.3.1). It is *geometrically distinguished* if given any finite extension K' of K with valuation ring $\mathcal{V}'$, the formal scheme $X \otimes_{\mathcal{V}} \mathcal{V}'$ is distinguished.

(14.4) Formal Analytic Spaces

The reference for the theory of formal analytic spaces is [Bosch 77, Manuscripta Math.].

(14.4.1) Construction and Definitions Let R be an affinoid K-algebra and $\|\cdot\|$ the spectral semi-norm on R. A *strict Laurent domain* in Spm R is a subset of the form $D_f = \{x \in \text{Spm } R \,/\, |f(x)| = 1\}$ with $f \in R$ and $\|f\| = 1$. These subsets form a basis for a topology called the *formal topology* on SpmR. The *formal space* of R is the locally ringed space S = Spf R whose set of points is Spm R, whose topology is the formal topology and whose structural sheaf satisfies $\Gamma(D_f, \mathcal{O}) = R\{f^{-1}\}$ (with the obvious compatibilities for different f's). The locally ring space Spf R is called a *formal affinoid space*. A *formal analytic space* is a locally ringed space which is locally isomorphic to a formal affinoid space.

(14.4.2) Definitions The *reduction* of an affinoid algebra R is the k-algebra $\bar{R} = R^o/R^{oo}$, where

$$R^o := \{f \in R \,/\, \|f\| \le 1\} \quad \text{and} \quad R^{oo} := \{f \in R \,/\, \|f\| < 1\}.$$

The *reduction* $\bar{S}$ of S := Spf R is Spec $\bar{R}$. The *reduction* $\bar{S}$ of a formal analytic space S/K is the k-scheme obtained by pasting the reductions of the formal affinoid subspaces of S.

(14.4.3) Let S/K be a formal analytic space. By functoriality, any point x on S (thought of as a morphism Spf K' $\longrightarrow$ S with K' a finite extension of K) gives rise to a closed point $\bar{x}$ on $\bar{S}$. Thus, we obtain the *reduction map*

$$r: S \longrightarrow \bar{S}.$$

It is shown in [Bosch 77, Manuscripta Math.] to be continuous and surjective on closed points.

(14.5) Formal Analytic Spaces and Formal Schemes

(14.5.1) Let S/K be a formal analytic space and $\mathcal{O}_S^o$ the subsheaf of $\mathcal{O}_S$ satisfying $\Gamma(U, \mathcal{O}_S^o) = R^o$ for any formal affinoid subspace U := Spf R of S. Then the locally ringed space $S_{\mathcal{V}} := (\bar{S}, r_* \mathcal{O}_S^o)$ is a flat formal scheme (not necessarily topologically of finite type) and we have $\bar{S} = (S_{\mathcal{V}} \otimes_{\mathcal{V}} k)_{red}$.

(14.5.2) If $S = \bigcup_i \text{Spf } R_i$ is a covering of a formal analytic space S by formal affinoid subspaces, then one can glue the $S_i^{an} = \text{Spm } R_i$ to get a rigid analytic space S^{an} (independent of the covering). By construction, if $S_{\mathcal{V}}$ is finitely presented, we have $S^{an} = (S_{\mathcal{V}})_K$.

(14.5.3) **Definition** A formal analytic space S is *distinguished* if, whenever Spf R is a formal affinoid subspace of S, then R is a distinguished affinoid algebra. In this case, $S_{\mathcal{V}}$ is a distinguished (14.3.2) formal scheme.

(14.5.4) Let $\mathfrak{X}/\mathcal{V}$ be a distinguished formal scheme and $\mathfrak{X}^o$ the locally ringed space whose set of points is the set of points of $\mathfrak{X}_K$, whose topology is the inverse image of the topology of $\mathfrak{X}$ under the specialization map sp: $\mathfrak{X}_K \longrightarrow \mathfrak{X}$ and whose structural sheaf is the restriction of the structural sheaf of $\mathfrak{X}_K$. Then $\mathfrak{X}^o$ is a distinguished formal analytic space.

Proof. Since the question is local, we may assume that $\mathfrak{X}$ is affine, say $\mathfrak{X} = \text{Spf } \mathcal{A}$. The point is to check that $\mathfrak{X}^o$ has the right topology. A basis for this topology is given by the $sp^{-1}(D(f)) = \{x \in \text{Spm } \mathcal{A}_K \,/\, |f(x)| = 1\}$ with $f \in \mathcal{A}$ and we have seen in (14.2.4) that $\mathcal{A} = \mathcal{A}^o$.□

(14.5.5) The functor $S \longmapsto S_{\mathcal{V}}$ and $\mathfrak{X} \longmapsto \mathfrak{X}^o$ establish an equivalence between the category of distinguished formal analytic spaces and distinguished formal schemes. Moreover, under this equivalence, we have $S^{an} = \mathfrak{X}_K$. It also follows from (14.2.4) that if X is the special fibre of $\mathfrak{X}$, then $\bar{S} = X$.

(14.6) Applications of Rigid Analytic Geometry to Formal Schemes

Using the equivalence of (14.5.5), one deduces the following theorems from the corresponding results in rigid analytic geometry.

(14.6.1) THEOREM *If a proper flat formal scheme has geometrically reduced curves as fibres, its fibres have the same (arithmetic) genus.*

Proof. Follows from Theorem 2.8 c) of [Bosch M.M. 77].□

(14.6.2) THEOREM *A proper flat formal scheme whose fibres are geometrically reduced curves is projective.*

Proof. Follows from Théorème 2 in Chapitre 7 of [Fresnel].□

(14.6.3) THEOREM *A geometrically reduced projective scheme over K has, after a finite extension of* K, *a flat formal model with geometrically reduced special fibre.*

Proof. Follows from Proposition 1.3 in [Bosch-Lütkebohmert I] (Using the main theorem of [Raynaud] or [Mehlman]).□

(14.6.4) THEOREM *In a flat formal scheme with geometrically reduced fibres of pure dimension, the tube of a point is connected.*

Proof. Follows from Satz 6.1 in [Bosch 77 Math. Ann. 77].□

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