Université de Rennes 2025-2026

Homological algebra
Examen 1 du 16 décembre 2025
Début 15h – Durée 1h

You can compose in english or in french. You can freely (no need to make a precise reference) use any result (exercises included) obtained prior to the appearance of the question in the (online) course. You can use any document you wish (electronic or not).

1. (5 points) Let $\mathcal{B}$ be an abelian category. Show that, if a fully faithful functor $F : \mathcal{A} \rightarrowtail \mathcal{B}$ has an exact adjoint or coadjoint G, then $\mathcal{A}$ also is abelian.

Solution: We already know that $\mathcal{A}$ is additive. Also, since F is fully faithful, we have $G \circ F \simeq \mathrm{Id}$. Let $f : M \to N$ be any morphism in $\mathcal{A}$. Since G is exact, we have $G(\ker F(f)) = \ker G(F(f)) \simeq \ker f$ and $G(\mathrm{coker} F(f)) = \mathrm{coker} G(F(f)) \simeq \mathrm{coker} f$ showing that $\mathcal{A}$ is preabelian. Moreover, since G is exact and $F(f)$ is strict, $G(F(f))$ is strict and the same holds for f.

We work in an abelian category $\mathcal{A}$.

2. (5 points) Show that $M \xrightarrow{f} N \xrightarrow{g} Q \xrightarrow{h} R$ is exact if and only if it splits into right and left exact sequences

$$M \xrightarrow{f} N \to P \to 0 \quad \text{and} \quad 0 \to P \to Q \xrightarrow{h} R.$$

Solution: By definition, the original sequence is exact if and only if $\mathrm{im}(f) = \ker(g)$ and $\mathrm{im}(g) = \ker(h)$. But, by duality, the first condition is equivalent to $\mathrm{coker}(f) = \mathrm{im}(g)$. If we set $P := \mathrm{im}(g)$, we can then rewrite both conditions as $\mathrm{coker}(f) = P = \ker(h)$. The implication follows. Conversely, if we have such a splitting, then there exists an epi-mono factorization $g : N \twoheadrightarrow P \rightarrowtail Q$ and therefore $\mathrm{im}(g) = P$.

3. Show that (and dual)
   (a) (5 points) if $g : N \to N'$, then $g_* : \mathrm{Ext}(M, N) \to \mathrm{Ext}(M, N')$ is a morphism of groups,

Solution: We shall write $\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Since $g \circ \nabla = \nabla \circ (g \oplus g)$, if $E \in \mathrm{Ext}(M, N)$ and $E' \in \mathrm{Ext}(M, N')$, we shall have

$$g_*(E + E') := g_* \Delta^* \nabla_*(E \oplus E') = \Delta^*(g \circ \nabla)_*(E \oplus E')$$

$$= \Delta^*(\nabla \circ (g \oplus g))_*(E \oplus E') = \Delta^* \nabla_* (g \oplus g)_*(E \oplus E')$$
$$= \Delta^* \nabla_* (g_* E \oplus g_* E') =: g_* E + g_* E'.$$

Moreover, we have $g_*(-E) = -g_* E$ since the opposite of $0 \to N \xrightarrow{i} E \xrightarrow{\pi} M \to 0$ is simply $0 \to N \xrightarrow{i} E \xrightarrow{-\pi} M \to 0$.

(b) (5 points) if E is an extension of M by N then the map

$$\mathrm{Hom}(M', M) \to \mathrm{Ext}(M', N), \quad f \mapsto f^* E$$

is a morphism of groups.

Solution: Now, if $f, g \in \mathrm{Hom}(M', M)$ and we write $F := \begin{bmatrix} f \\ -g \end{bmatrix}$, then we have

$$(f - g)^* E = (\nabla \circ F)^* E = F^* \nabla^* E$$

and, on the other hand,

$$f^* E - g^* E = \nabla_* \Delta^* (f^* E \oplus -g^* E) = \nabla_* \Delta^* (f \oplus -g)^* (E \oplus E)$$
$$= ((f \oplus -g) \circ \Delta)^* \nabla_* (E \oplus E) = F^* \nabla_* (E \oplus E).$$

It only remains to show that $\nabla^* E = \nabla_*(E \oplus E)$. But the commutative diagram

$$\begin{array}{ccc} E \oplus E & \longrightarrow & M \oplus M \\ \downarrow & & \downarrow \nabla \\ E & \longrightarrow & M \end{array}$$

provides a morphism $E \oplus E \to \nabla^* E$ that fits into a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \oplus N & \longrightarrow & E \oplus E & \longrightarrow & M \oplus M & \longrightarrow & 0 \\ & & \downarrow \nabla & \lrcorner & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & \nabla^* E & \longrightarrow & M \oplus M & \longrightarrow & 0. \end{array}$$

The dual statements mean respectively that if $f : M' \to M$ then $f^* : \mathrm{Ext}(M, N) \to \mathrm{Ext}(M', N)$ is a morphism of groups and that if E is an extension of M by N, then the map

$$\mathrm{Hom}(N, N') \to \mathrm{Ext}(M, N'), \quad g \mapsto g_* E$$

is a morphism of groups.