The local Simpson correspondence in positive characteristic

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Outline

Characteristic zero

Let X be a non-singular projective variety over **C**. Then, we may consider various equivalences:

$$\{\text{Coherent modules with integrable connexion on } X\}$$
$$\Big\updownarrow \text{Riemann}$$
$$\{\text{Finite dimensional linear representations of } \pi_1(X^{\text{an}})\}$$
$$\Big\updownarrow \text{Simpson}$$
$$\{\text{Higgs bundles on } X\}$$

(restricted to "semi-simple" and "stable with trivial Chern classes")

The composite correspondence is not algebraic although both the upper and lower categories are.

General situation

Let X be smooth scheme over some scheme S.

Giving an integrable connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

on an $\mathcal{O}_X$ module $\mathcal{E}$ is equivalent to a $\mathcal{D}^{(0)}_{X/S}$-module structure: locally,

$$\nabla(m) = \sum \partial_i m \otimes dt_i.$$

Recall that a Higgs field

$$\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

is simply an $\mathcal{O}_X$-linear map satisfying $\theta^2 = 0$. If we denote by $\mathcal{T}_{X/S}$ the tangent sheaf and build its symmetric algebra $S^\bullet\mathcal{T}_{X/S}$, the Higgs field corresponds to a $S^\bullet\mathcal{T}_{X/S}$-structure: locally,

$$\theta(m) = \sum \xi_i m \otimes dt_i.$$

Positive characteristic

Let X be a smooth scheme over a scheme S of characteristic $p > 0$. We consider the usual commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X' & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_S} & S. \end{array} \qquad (F_X : X \to X)$$

Our aim is to understand how $F_{X/S}^*$ can induce a correspondence

$$\left\{ \begin{array}{c} \text{Higgs bundles} \\ \text{on } X'/S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Differential modules} \\ \text{on } X/S \end{array} \right\}.$$

Or more precisely

$$\{S^\bullet \mathcal{T}_{X'/S}\text{–mod}\} \leftrightarrow \{\mathcal{D}^{(0)}_{X/S}\text{–mod}\}.$$

Azumaya Algebras

Definition

Let $\mathcal{R}$ be a commutative ring and $\mathcal{A}$ an $\mathcal{R}$-algebra. $\mathcal{A}$ is called a matrix ring if there exists a locally free $\mathcal{R}$ module $\mathcal{M}$ of finite rank such that $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{R}}(\mathcal{M})$. $\mathcal{A}$ is called an Azumaya algebra if it is locally free of finite rank and

$$\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^{\mathrm{op}} \simeq \mathcal{E}nd_{\mathcal{R}}(\mathcal{A}).$$

A matrix ring is an Azumaya algebra. Conversely, if X is a scheme, an Azumaya algebra over $\mathcal{O}_X$ is an algebra that is locally for the flat (or étale) topology, a matrix ring.

Lemma

If $\mathcal{A}$ is a matrix ring on $\mathcal{R}$ as above, the map $\mathcal{F} \mapsto \mathcal{M} \otimes_{\mathcal{R}} \mathcal{F}$ induces an equivalence between $\mathcal{R}$-modules and $\mathcal{A}$-modules.

Differentials of level 0

Definition

The sheaf of differentials of level 0 of X/S is the $\mathcal{O}_X$-algebra $\mathcal{D}^{(0)}_{X/S}$ locally generated by the ∂_i's with the usual commutation rules.

Of course, here the ∂_i's denote the derivatives corresponding to some local coordinates $t_1, \ldots, t_n$.

Definition

$\mathcal{E}$ has zero p-curvature if, locally, the ∂_i^p's act trivially.

Note that the p-curvature map $S^\bullet \mathcal{T}_{X'/S} \to \mathcal{D}^{(0)}_{X/S}$ locally defined by $\xi_i \mapsto \partial_i^p$ is an isomorphism onto the center $\mathcal{Z}^{(0)}_{X/S}$. Finally, we will denote by $\mathcal{K}^{(0)}_{X/S}$ the two-sided ideal of $\mathcal{D}^{(0)}_{X/S}$ locally generated by the ∂_i^p.

Cartier equivalence

Lemma

The natural action of $\mathcal{D}^{(0)}_{X/S}$ on $\mathcal{O}_X$ induces an isomorphism

$$\mathcal{D}^{(0)}_{X/S}/\mathcal{K}^{(0)}_{X/S} \xrightarrow{\simeq} \mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{O}_X).$$

Using the above theorem on matrix rings, we get:

Theorem (Cartier)

The functor

$$\mathcal{F} \longmapsto \mathcal{E} := F^*_{X/S}\mathcal{F}$$

is an equivalence between $\mathcal{O}_{X'}$-modules and $\mathcal{O}_X$-modules with and integrable connection with trivial p-curvature.

Lifting Cartier

In order to get a Simpson correspondence, we want a bijective $\mathcal{O}_X$-linear map making commutative the following diagram:

$$
\begin{array}{ccc}
\mathcal{Z}^{(0)}_{X/S} & \xleftarrow{\simeq} & S^\bullet \mathcal{T}_{X'/S} \\
\downarrow & & \downarrow \\
\mathcal{D}^{(0)}_{X/S} & \dashrightarrow & \mathcal{E}nd_{S^\bullet \mathcal{T}_{X'/S}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^\bullet \mathcal{T}_{X'/S}) \\
\downarrow & & \downarrow \\
\mathcal{D}^{(0)}_{X/S}/\mathcal{K}^{(0)}_{X/S} & \xrightarrow{\simeq} & \mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{O}_X)
\end{array}
$$

In other words, we want an integrable connection on $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^\bullet \mathcal{T}_{X'/S}$ such that, locally,

Local conditions

the following conditions are satisfied:

1. $\partial_i \bullet (f \otimes Q) = (\partial_i \bullet (f \otimes 1))(1 \otimes Q)$
2. $\partial_i \bullet (f \otimes 1) = \partial_i(f) \otimes 1 \mod \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^+\mathcal{T}_{X'/S}$
3. $\partial_i^p \bullet \mathbf{1} = \xi_i$.

Our map is determined by

$$\partial_i \bullet \mathbf{1} = H_i \in \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^+\mathcal{T}_{X'/S}$$

subject to the condition $\partial_i^p \bullet \mathbf{1} = \xi_i$. We may endow $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^\bullet\mathcal{T}_{X'/S}$ with the trivial structure $P(f \otimes Q) = P(f) \otimes Q$. Then, the condition may be rewritten

$$\partial_i^{p-1}(H_i) + H_i^p = \xi_i.$$

Of course, we also need the integrability condition $\partial_i H_j = \partial_j H_i$.

van der Put's method

If one reads through the lines, we have:

Theorem

Locally, the Cartier isomorphism

$$\mathcal{D}^{(0)}_{X/S}/\mathcal{K}^{(0)}_{X/S} \simeq \mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{O}_X)$$

lifts to an isomorphism

$$\widehat{\mathcal{D}^{(0)}_{X/S}} \simeq \mathcal{E}nd_{\widehat{S^\bullet\mathcal{T}_{X'/S}}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \widehat{S^\bullet\mathcal{T}_{X'/S}})$$

compatible with the inverse of the p-curvature isomorphism

$$\widehat{\mathcal{Z}^{(0)}_{X/S}} \simeq \widehat{S^\bullet\mathcal{T}_{X'/S}}.$$

An idea of the proof

Of course, $\widehat{S^{\bullet}\mathcal{T}_{X'/S}}$ denote the completion with respect to its augmentation ideal $S^{+}\mathcal{T}_{X'/S}$ and $\widehat{\mathcal{D}^{(0)}_{X/S}}$ denotes the completion with respect to $\mathcal{K}^{(0)}_{X/S}$.

Proof.

By successive approximation, one can choose

$$H_i := -\sum_{i=1}^{\infty} t_i^{(p^i-1)} \xi_i^{p^{i-1}}.$$

It only remains to check that we do obtain an isomorphism. □

As a corollary, we see that $\mathcal{D}^{(0)}_{X/S}$ is an Azumaya algebra (question is local and completion is faithfully flat).

Differentials of higher level

Definition

The sheaf of differentials of level m of X/S is the $\mathcal{O}_X$-algebra $\mathcal{D}^{(m)}_{X/S}$ locally generated by the

$$\partial_i^{[p^l]} := \frac{\partial_i^{p^l}}{p^l!} \quad \text{with} \quad l \leq m.$$

The ring $\mathcal{D}_{X/S} := \varinjlim \mathcal{D}^{(m)}_{X/S}$ is nothing but Grothendieck's ring of differential operators. We will denote by $\mathcal{Z}^{(m)}_{X/S}$ the center of $\mathcal{D}^{(m)}_{X/S}$ and by $\mathcal{O}\mathcal{Z}^{(m)}_{X/S}$ the centralizer of $\mathcal{O}_X$ in $\mathcal{D}^{(m)}_{X/S}$. We will also iterate the Frobenius diagram and denote by

$$F^{m+1}_{X/S} : X^{(m+1)} \to X$$

the corresponding map.

Azumaya structure

After pulling back by Frobenius, the ring $\mathcal{D}^{(m)}_{X/S}$ becomes a matrix algebra:

Theorem (Kaneda/Bezrukavnikov-Mircovic-Rumynin)

There is a $\mathcal{O}\mathcal{Z}^{(m)}_{X/S}$-isomorphism

$$\begin{array}{ccc} F^{m+1*}_{X/S}\mathcal{D}^{(m)}_{X/S} & \xrightarrow{\simeq} & \mathcal{E}nd_{\mathcal{O}\mathcal{Z}^{(m)}_{X/S}}(\mathcal{D}^{(m)}_{X/S}) \\ f \otimes Q & \longmapsto & (P \mapsto fPQ). \end{array}$$

In particular, we see that $\mathcal{D}^{(m)}_{X/S}$ is an Azumaya Algebra. Note that the Frobenius provides a global Azumaya splitting but only for the flat topology. It is way more difficult to find an étale splitting.

The étale splitting

Assume for a while that $m = 0$. We briefly present the local theory of Ogus and Vologodsky. Locally, one can choose a splitting of the Cartier operator, meaning a lifting $\Omega^1_{X'/S} \to \Omega^1_{X/S}$ of the inverse Cartier

$$\begin{aligned} \Omega^1_{X'/S} &\xrightarrow{\simeq} \mathcal{H}^1(\Omega^\bullet_{X/S}) \\ 1 \otimes f &\longmapsto f^{p-1}df. \end{aligned}$$

Geometrically, this splitting induces a (non linear) morphism on the cotangent space

$$h : \check{\mathbf{T}}_{X'/S} \to \check{\mathbf{T}}_{X'/S}.$$

Lemma (Ogus-Vologodsky)

The map $\alpha = h - \mathrm{id}$ is surjective étale.

Local isomorphism

Since $S^\bullet\mathcal{T}_{X'/S}$ is the ring of functions on $\check{\mathbf{T}}_{X'/S}$ with respect to X', we may see $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^\bullet\mathcal{T}_{X'/S}$ as a locally free sheaf of finite rank on $\check{\mathbf{T}}_{X'/S}$. Also, we can use the p-curvature map $S^\bullet\mathcal{T}_{X'/S} \to \mathcal{D}^{(0)}_{X/S}$ in order to see $\mathcal{D}^{(0)}_{X/S}$ as a sheaf on $\check{\mathbf{T}}_{X'/S}$.

Theorem (Ogus-Vologodsky)

There is an isomorphism

$$\alpha^*\mathcal{D}^{(0)}_{X/S} \simeq \mathcal{E}nd_{\mathcal{O}_{\check{\mathbf{T}}_{X'/S}}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} S^\bullet\mathcal{T}_{X'/S}).$$

This is a local étale splitting. Note that $\widehat{\alpha *}$ is a non trivial automorphism of $\widehat{S^\bullet\mathcal{T}_{X'/S}}$ and that one recovers van der Put formulas by composing with its inverse on the left.

Duality

If $\mathcal{M}$ is an $\mathcal{R}$-module, we already met the symmetric algebra $S^\bullet\mathcal{M}$. We will denote by $\Gamma_\bullet\mathcal{M}$ the divided power algebra on $\mathcal{M}$. When $\mathcal{M}$ is free on $s_1, \ldots, s_n$, we get the divided polynomial ring $\mathcal{R}\langle s_1, \ldots, s_n\rangle$, generated as a free module by the products of $s_i^{[k]}$ with the multiplication rule

$$s_i^{[k]} s_i^{[l]} = \binom{k+l}{k} s_i^{[k+l]}.$$

There is a canonical paring of bialgebras

$$\Gamma_\bullet \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{R}) \times S^\bullet\mathcal{M} \to \mathcal{R}.$$

In the free case, the products of $\check{s}_i^{[k]}$ is dual to the usual basis of the polynomial ring. It is formally perfect.

Dual approach (following Berthelot)

We will denote by $\mathcal{I}$ the ideal of X in $X \times_S X$ which is locally generated by the elements $\tau_i = 1 \otimes t_i - t_i \otimes 1$. The algebra $\mathcal{P}_{X/S,m}$ is the divided power envelope of the ideal $\mathcal{I}^{(p^m)}$ locally generated by the elements $\tau_i^{p^m}$. Locally, it is the free module generated by the products of

$$\tau_i^{\{qp^m+r\}} = (\tau_i^{p^m})^{[q]}\tau_i^r := \frac{\tau_i^{qp^m+r}}{q!}, \quad q \geq 0, \quad 0 \leq r < p^m.$$

The natural map

$$\mathcal{O}_{X \times_{X^{(m+1)}} X} \to \mathcal{P}_{X/S,m}$$

dualizes to a morphism

$$\mathcal{D}^{(m)}_{X/S} \to \mathcal{E}nd_{\mathcal{O}_{X^{(m+1)}}}(\mathcal{O}_X)$$

whose kernel is the two-sided ideal $\mathcal{K}^{(m)}_{X/S}$ locally generated by the $(\partial_i^{[p^m]})^p$.

p^m-CURVATURE

LEMMA

There is a canonical isomorphism of divided power algebras

$$(*) \qquad F_{X/S}^{m+1*}\Gamma_{\bullet}\Omega^1_{X^{(m+1)}/S} \xrightarrow{\simeq} \mathcal{P}_{X/S,m}/\mathcal{I}\mathcal{P}_{X/S,m}(\twoheadleftarrow \mathcal{P}_{X/S,m})$$

sending locally dt_i to the class of $\tau_i^{\{p^{m+1}\}}$.

DEFINITION

The (linearized) p^m-curvature map of X is the map

$$F_{X/S}^{m+1*}S^{\bullet}\mathcal{T}_{X^{(m+1)}/S} \xleftarrow{\simeq} \mathcal{O}\mathcal{Z}^{(m)}_{X/S} \hookrightarrow \mathcal{D}^{(m)}_{X/S}$$

obtained by duality from $(*)$. Locally, it sends ξ_i to $(\partial_i^{[p^m]})^p$.

Lifting of Frobenius

We assume from now on that there is a fixed (strong) lifting of Frobenius modulo p^2:

$$\widetilde{F}^{(m+1)}_{X/S} : \widetilde{X}^{(m+1)} \to \widetilde{X}.$$

Lemma

There is a morphism of divided power algebras

$$(**) \qquad F^{m+1*}_{X/S}\Gamma_\bullet\Omega^1_{X^{(m+1)}/S} \longrightarrow \mathcal{P}_{X/S,m}$$

sending dt_i to the reduction of $\frac{1}{p!}\widetilde{F}^{m+1*}_{X/S}(\widetilde{\tau}_i)$.

Note that this is not a lifting of $(*)$ because the quotient map

$$\mathcal{P}_{X/S,m} \twoheadrightarrow \mathcal{P}_{X/S,m}/\mathcal{I}\mathcal{P}_{X/S,m}$$

is not compatible with divided powers.

Frobenius on differentials

Definition

The Frobenius of $\mathcal{D}^{(m)}_{X/S}$ is the map

$$\Phi^{(m)}_{X/S} : \mathcal{D}^{(m)}_{X/S} \longrightarrow F^{m+1*}_{X/S} S^{\bullet}\mathcal{T}_{X^{(m+1)}/S} \hookrightarrow \mathcal{D}^{(m)}_{X/S}$$

deduced by duality from $(**)$ (and the p^m-curvature map).

Locally, we have

$$\partial_i^{[p^m]} \longmapsto \frac{1}{p!} \sum_{j=1}^{r} \partial_i^{[p^m]}(\widetilde{F}^*(\widetilde{t}_j))(\partial_j^{[p^m]})^p$$

and $\partial_i^{[p^l]} \mapsto 0$ for $l < m$. This Frobenius is not $\mathcal{Z}^{(m)}_{X/S}$-linear but does

$$(\partial_i^{[p^m]})^p \longmapsto (\partial_i^{[p^m]})^p + \Phi(\partial_i^{[p^m]})^p.$$

Main theorem

Lemma

The morphism $(**)$ *extends to an isomorphism*

$$(***) \quad \mathcal{O}_{X \times_{X^{(m+1)}} X} \otimes \Gamma_\bullet \Omega^1_{X^{(m+1)}/S} \xrightarrow{\simeq} \mathcal{P}_{X/S,m}.$$

As a formal consequence, we obtain:

Theorem

There is an isomorphism of $\mathcal{O}_X$*-algebras*

$$\widehat{\mathcal{D}}^{(m)}_{X/S} \xrightarrow{\simeq} \mathcal{E}nd_{S^\bullet \widehat{\mathcal{T}_{X^{(m+1)}/S}}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{m+1}}} S^\bullet \widehat{\mathcal{T}_{X^{(m+1)}/S}}).$$

Note that the map is defined before completion but this is not an isomorphism.

Comments

1. The Frobenius $\Phi^{(m)}_{X/S}$ of $\mathcal{D}^{(m)}_{X/S}$ restricts to an endomorphism $\alpha^{(m)}_{X/S}$ of $\mathcal{Z}^{(m)}_{X/S}$ and we have an isomorphism

$$\mathcal{Z}^{(m)}_{X/S\,\underset{\alpha}{\nwarrow}} \otimes_{\mathcal{Z}^{(m)}_{X/S}} \mathcal{D}^{(m)}_{X/S} \xrightarrow{\simeq} \mathcal{E}nd_{S^\bullet\mathcal{T}_{X^{(m+1)}/S}}(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{m+1}}} S^\bullet\mathcal{T}_{X^{(m+1)}/S}).$$

   It corresponds via the p^m-curvature to the trivialization of Ogus-Vologodsky when $m = 0$.

2. Completion provides an automorphism $\widehat{\alpha}^{(m)}_{X/S}$ of $\widehat{\mathcal{Z}}^{(m)}_{X/S}$. And we can twist the isomorphism of the theorem in order to make it $\widehat{\mathcal{Z}}^{(m)}_{X/S}$-linear. Then, we recover van der Put's situation when $m = 0$.

The correspondence

By definition, a Higgs bundle on $X^{(m+1)}$ is nothing but a $S^\bullet\mathcal{T}_{X^{(m+1)}/S}$-module. It follows that :

Theorem

There exists an equivalence between quasi-nilpotent Higgs bundles on $X^{(m+1)}$ and quasi-nilpotent $\mathcal{D}^{(m)}_{X/S}$-modules on X.

This equivalence is given explicitly by

$$\mathcal{E} = F^{(m+1)*}_{X/S}\mathcal{F} \quad \text{and} \quad \mathcal{F} = \mathcal{E}^{\Phi=\mathrm{Id}}.$$

with

$$\partial_i^{[p^l]}(1 \otimes s) = \begin{cases} 0 & \text{if} \quad l < m \\ \frac{1}{p!}\sum_{j=1}^r \partial_i^{[p^m]}(\widetilde{F}^*(\tilde{t}'_j)) \otimes \xi'_j(s) & \text{if} \quad l = m. \end{cases}$$

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