Workshop on Berkovich theory
Introduction and examples

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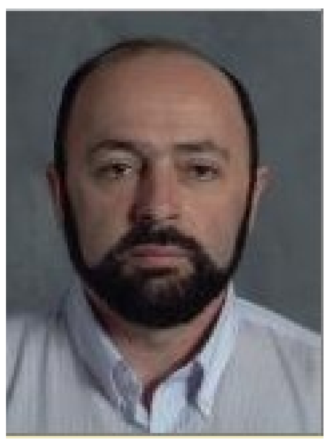

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1 Discovery of the p-adic numbers by K. Hensel (1905)

If p is a prime and we endow for each $n \in \mathbf{N}$, the ring $\mathbf{Z}/p^{n+1}$ with the discrete topology, we can form

$$\mathbf{Z}_p := \varprojlim \mathbf{Z}/p^{n+1}$$

in the category of topological rings. We get a compact integral domain : the ring of *p-adic integers*. Moreover, the canonical map $\mathbf{Z} \to \mathbf{Z}_p$ is injective and $\mathbf{Z}$ is dense in $\mathbf{Z}_p$. The fraction field $\mathbf{Q}_p$ of $\mathbf{Z}_p$ is the field of *p-adic numbers*.

Hensel's expansion theorem says that any element in $\mathbf{Z}_p$ has a unique expression of the form $\sum_0^\infty a_n p^n$ with $a_n \in \mathbf{N}$ and $a_n < p$.

2 Classification of valued fields by J. Kürschack (1913) and A. Ostrowski (1918)

An *absolute value* on a field K is a group homomorphism

$$|-|: K^* \to \mathbf{R}_{>0} \text{ such that } |x+y| \leq |x| + |y|.$$

It is extended to K by $|0| = 0$. It is said *ultrametric* if

$$|x+y| \leq \max(|x|, |y|).$$

This turns K into a metric space and a topological ring. The completion $\hat{K}$ of K (which is a topological ring) is a field and the absolute value extends uniquely to $\hat{K}$. Two absolute values are *equivalent* if the corresponding metrics are. There exists only one absolute value on $\mathbf{Q}$ such that $|p| = p^{-1}$. Its completion is $\mathbf{Q}_p$.

An ultrametric field is totally discontinuous : any point has a basis of open-closed neighborhood. More precisely, any closed ball is open. Moreover, any triangle is isosceles. Any point in a ball is “the” center.

The theorem of Ostrowski says that, up to equivalence, the non trivial absolute values of $\mathbf{Q}$ are the usual one and the p-adic ones. Their completions are respectively $\mathbf{R}$ and $\mathbf{Q}_p$.

Let K be a valued field. If K'/K is finite and K complete, the absolute value of K extends uniquely to K' and K' is complete. If K is algebraically closed, so is its completion. Thus, there exists a smaller complete algebraically closed extension of K. For $\mathbf{R}$ and $\mathbf{Q}_p$, we get $\mathbf{C}$ and $\mathbf{C}_p$, respectively.

3 Non archimedean function theory by R. Strassman ($\geq$ 1920) and W. Shöbe (1930)

Let K be a valued field. A *semi-norm* (resp. a *norm*) on a K-vector space E is a map $\| - \| : E \to \mathbf{R}_{\geq 0}$, such that

$$\begin{cases} \|ax\| = |a|\|x\| \\ \|x+y\| \leq \|x\| + \|y| \\ \text{(resp. } \|x\| = 0 \text{ iff } x = 0) \end{cases}$$

Then E is a (semi-)metric space. It is called a *Banach space* if it is complete.

A semi-norm on a K-algebra A is always assumed to satisfy $\|fg\| \leq \|f\|\|g\|$. It is called *multiplicative* if equality holds.

Let K be non trivial complete ultrametric field. If $\underline{r} \in \mathbf{R}^n_{>0}$, then

$$K\{\underline{T}/\underline{r}\} := K\{T_1/r_1, \ldots, T_n/r_n\}$$
$$= \{\sum_{\underline{n}\geq 0} a_{\underline{n}}\underline{T}^{\underline{n}}, |a_{\underline{n}}|\underline{r}^{\underline{n}} \to 0\} \subset K[[\underline{T}]].$$

This is a subalgebra with a multiplicative norm

$$\|\sum_{\underline{n}\geq 0} a_{\underline{n}}\underline{T}^{\underline{n}}\| = \max |a_{\underline{n}}|\underline{r}^{\underline{n}}.$$

When $r_1 = \cdots = r_n = 1$, we write $K\{\underline{T}\}$.

The *radius of convergence* of $f \in K[[T]]$ is

$$R := R(f) := \sup\{r \in \mathbf{R}, f \in K\{T/r\}\}.$$

Consider the function

$$f : x \mapsto \frac{1}{p - x^2}.$$

It is defined on all $\mathbf{Q}_p$, it has a power series expansion

$$f := \sum T^{2n}/p^{n+1} \text{ on } D(0, 1^-) \subset \mathbf{Q}_p,$$

and its radius of convergence is $\frac{1}{\sqrt{p}}$.

Assume for a while that $\mathrm{car}(K) = 0$. A function is *analytic* on a closed disk $D \subset K$ of radius r at $a \in D$ if it has a power series expansion f at a with $R := R(f) \geq r$ and even $R > r$ if $f \notin K\{T/R\}$. Then $R(f)$ does not depend on a. There is no hope for analytic continuation.

Note that if K is algebraically closed, then the algebra of analytic functions on D is $K\{T/r\}$.

4 Discovery of analytic elements by M. Krasner ($\geq$ 1954)

Assume K algebraically closed. If W is a bounded infinite subset of K, then the ring $\mathcal{R}(W) \subset K^W$ of rational functions with no pole in W is endowed with the uniform topology of uniform convergence on W. Its completion $\mathcal{H}(W) \subset K^W$ is the set of *analytic elements* on W. It is a Banach K-algebra if and only if W is closed.

If D a closed disk, then $\mathcal{H}(D)$ is the ring of analytic functions on D. In particular, if $D \subset W$ is any closed disc, then any analytic element on W is analytic on D.

Krasner's theory is a good theory of non archimedean functions. However, it does not give a reasonable theory for analytic continuation : a non-zero function can be zero on some open disk.

5 Discovery of rigid analytic spaces by J. Tate (1961)

Let K be a non trivial complete ultrametric field. A strictly affinoid algebra over K is a quotient of $K\{\underline{T}\}$. Any morphism $u : A \to B$ of strictly affinoid algebras induces a map

$$\mathrm{Spm} u : Y = \mathrm{Spm} B \mapsto X := \mathrm{Spm} A.$$

It is called an *admissible embedding* if it is injective and universal for maps $\mathrm{Spm} v : Z = \mathrm{Spm} C \mapsto X$ whose image is contained in Y.

For example, assume K algebraically closed, and call a disk *strict* if its radius belongs to $|K^*|$. An admissible embedding in $W \hookrightarrow D(0,1) = \mathrm{Spm} K\{T\}$ is an inclusion of a finite union of complements of finitely many strict open disks in stricts closed disks. Moreover, in this case, $\mathcal{H}(W)$ is an affinoid algebra and we have $W = \mathrm{Spm}\mathcal{H}(W)$.

Tate's theorem says that if

$$\{X_i = \mathrm{Spm} A_i \hookrightarrow X = \mathrm{Spm} A\}_{i \in I}$$

is a finite surjective family of admissible embeddings and M an A-module of finite presentation, then the sequence

$$0 \mapsto M \to \prod A_i \otimes_A M \to \prod A_i \hat{\otimes}_A A_j \otimes_A M \to \cdots$$

is exact.

This gives a good theory for coherent sheaves with theorem A and B. Pasting maximal spectrums gives rigid analytic spaces. However, the usual topology has to be replaced by a Grothendieck topology.

6 Interpretation of rigid spaces in terms of formal schemes by Raynaud ($\geq$ 1970)

Let $\mathcal{V}$ be the ring of integers of K and π a non zero, non invertible element of $\mathcal{V}$. There is a functor $\mathcal{X} \mapsto \mathcal{X}_K$ from the category of π-torsion free π-adic formal schemes of finite presentation to the category of rigid analytic spaces. It sends Spf$A$ to SpmA_K.

It induces an equivalence between the first category localized with respect to generic isomorphisms and the category of quasi-compact quasi-separated rigid analytic spaces.

7 Discovery of the ultrametric spectrum by A. Escassut, G. Garandel and B. Guenebaud ($\geq$ 1973)

Let K be a complete ultrametric field. If A be a K-algebra, then $\mathcal{M}^{alg}(A)$ is the set of multiplicative seminorms on A. It is given the uniform topology of simple convergence. If A is a topological algebra, then its *analytic spectrum* is the subset $\mathcal{M}(A)$ of continuous seminorms.

The *analytic affine space* (resp. *closed disk of radius* r) is

$$\mathbf{A}^{n,an} := \mathcal{M}^{alg}(K[\underline{T}])$$

$$(\text{resp. } D(0,r) := \mathcal{M}(K\{\underline{T}/r\})$$

and we have

$$\mathbf{A}^{n,an} = \cup_r D(0,r).$$

8 Discovery of the "new" points by M. van der Put (1982)

If X is a rigid analytic space, we can consider the set $\mathcal{F}(X)$ of filters (family of nonempty subsets stable by finite intersection and enlargement) of admissible subsets. The subset of *prime filters* $\mathcal{P}(X)$ is defined by the localization condition (if an element of the filter has an admissible covering, some element of the covering must belong to the filter).

The set $\mathcal{P}(X)$ is endowed with the weakest topology such that $\mathcal{P}(Y)$ is open whenever Y is an admissible affinoid subset of X. The inclusion map $X \hookrightarrow \mathcal{P}(X)$ that sends x to the set of its neighborhoods induces an equivalence of toposes. In particular, we get a sheaf of rings on $\mathcal{P}(X)$.

One can also consider the set $\mathcal{M}(X) \subset \mathcal{P}(X)$ of *maximal filters*. There is an obvious retraction $\mathcal{P}(X) \to \mathcal{M}(X)$ and $\mathcal{M}(X)$ is endowed with the quotient structure.

9 Discovery of analytic spaces by V. Berkovich (1990)

If $x \in \mathcal{M}(A)$, then

$$\mathfrak{p}_x := \{f \in A, x(f) = 0\}$$

is a prime ideal and x induces a multiplicative norm on $A/\mathfrak{p}_x$. It extends to an absolute value $| - |$ on the completion $\mathcal{H}(x)$ of the fraction field. Thus x factors as

$$f \mapsto f(x), A \to \mathcal{H}(x)$$

followed by $| - |$. In particular, we have $x(f) = |f(x)|$.

By definition, the topology of $\mathcal{M}(A)$ is the weakest topology making continuous all maps

$$\mathcal{M}(A) \to \mathbf{R}, x \mapsto |f(x)|.$$

If U is an open subset of $\mathbf{A}^{n,an}$ and $\mathcal{R}(U)$ denotes the set of rational functions with no pole on U, we get a natural map

$$\mathcal{R}(U) \hookrightarrow \prod_{x \in U} \mathcal{H}(x)$$

and we denote by $\mathcal{H}(U)$ the closure of $\mathcal{R}(U)$ for the topology of uniform convergence. The sheaf associated to $\mathcal{H}$ is a sheaf of local rings $\mathcal{O}$ on $\mathbf{A}^{n,an}$.

This is Berkovich affine space. The definition generalizes as follows.

An affinoid algebra over K is a quotient of $K\{\underline{T}/\underline{r}\}$. One defines a sheaf of rings on $\mathcal{M}(A)$ as in the rigid situation. The category of analytic spaces is built by glueing $\mathcal{M}(A)$ with A affinoid, exactly as in the rigid situation.

Actually, if A is a strictly affinoid algebra, there is a natural isomorphism

$$\mathcal{M}(A) \simeq \mathcal{M}(\mathrm{Spm} A).$$

More generally, we get is an equivalence $X \mapsto \mathcal{M}(X)$ between quasi-separated paracompact rigid analytic spaces and paracompact strictly analytic spaces.

10 Raynaud-type interpretation of Berkovich theory by Deligne (1992)

Assume (for simplification) that the valuation is discrete. If $\mathcal{X}$ is a π-torsion free π-adic formal scheme of finite presentation, then one can form the inverse limit $\mathcal{P}(\mathcal{X})$ in the category of locally ringed spaces on all generic isomorphisms $\mathcal{X}' \to X$. We can also form the Hausdorff quotient $\mathcal{M}(\mathcal{X})$ of $\mathcal{P}(\mathcal{X})$ with the direct image of $\mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{O}_{\mathcal{P}(X)}$.

We get a Raynaud-type descriptions $\mathcal{P}(\mathcal{X}) \simeq \mathcal{P}(\mathcal{X}_K)$ and $\mathcal{M}(\mathcal{X}) \simeq \mathcal{M}(\mathcal{X}_K)$ of the spaces of prime and maximal filters on $\mathcal{X}_K$.

In particular, $\mathcal{M}(\mathcal{X})$ is an analytic space. More precisely, we always have $\mathcal{M}(\mathrm{spf}(\mathcal{A})) = \mathcal{M}(\mathcal{A}_K)$.

11 Introduction of analytic spaces à la Huber by R. Huber (1994)

If A is a strictly affinoid K-algebra, then $\mathcal{P}(A)$ is the set of equivalence classes of continuous valuations $v : A \to \Gamma \cup 0$ such that $\|f\| \leq 1 \Rightarrow v(f) \leq 1$. Continuity means that

$$\forall \epsilon \in \Gamma, \exists \eta > 0, \|f\| \leq \eta \Rightarrow v(f) \leq \epsilon.$$

It is endowed with a topology, a sheaf of rings and a family of valuations on local rings. An analytic space in Huber sense is obtained by glueing such spaces.

We have a natural isomorphism $\mathcal{P}(A) \simeq \mathcal{P}(\mathrm{Spm}(A))$ that turns the functor $X \mapsto \mathcal{P}(X)$ into a fully faithful functor from rigid analytic spaces to Huber analytic spaces.

12 Example : the projective line

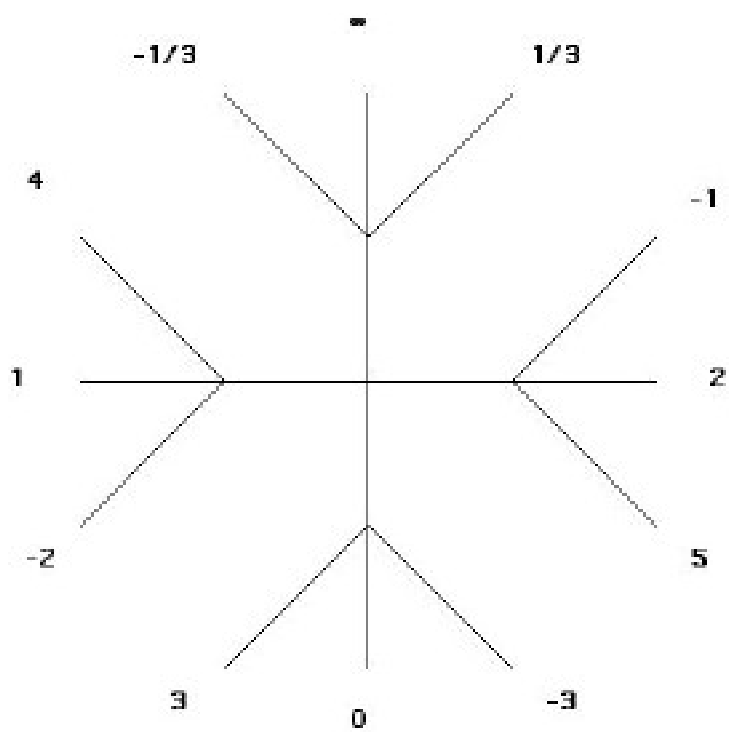
Connected admissible affinoid subsets of $\mathbf{P}^1(\mathbf{C}_p)$ are finite intersections of strict closed disks. Prime filters, and therefore points of $\mathbf{P}^{1,an}_{\mathbf{C}_p}$, correspond bijectively to decreasing sequences of (strict) closed disks up to equivalence.

There are four cases. The intersection of these disks is a strict closed disk, a non-strict closed disk (cutpoints), a (rigid) point or empty (endpoints). A point is a *cutpoint* (resp. an *endpoint*) if $\mathbf{P}^{1,an}_{\mathbf{C}_p} \backslash x$ has more than (resp. 1) connected components.

Given $x, y, z \in \mathbf{P}^{1,an}_{\mathbf{C}_p}$, we say that z is *between* x and y if $z = x$, $z = y$ or x and y belong to different components of $\mathbf{P}^{1,an}_{\mathbf{C}_p} \backslash z$. The subset $[x, y]$ of such z is the only arc (homeomorphic to $[0, 1] \subset \mathbf{R}$) joining x and y.

We will finish with a picture, but first we give an example of analytic function. The standard $\log : \mathbf{C}_p^* \to \mathbf{C}_p$ extends uniquely to a an analytic function on $\mathbf{P}^{1,an}_{\mathbf{C}_p} \backslash [0, \infty]$.

The Berkovich line on the 3-adic field



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