

Homological algebra
Homework (due November 3rd)

You can compose in english or in french. You can freely (no need to make a precise reference) use any result proved in the (online) course.

We recall that Δ denotes the category of ordered integers $[n] := \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$ with order preserving maps as morphisms.

1. Show that, for fixed $m, n \in \mathbb{N}$, a morphism $\alpha : [m] \rightarrow [n]$ in Δ is uniquely determined by the sets

$$I(\alpha) = \{i \in [n], \forall j \in [m], \alpha(j) \neq i\} \quad \text{and} \quad J(\alpha) = \{j \in [m-1], \alpha(j) = \alpha(j+1)\}.$$

We shall denote by

$$\delta_i^n : [n-1] \rightarrow [n] \quad (\text{resp. } \sigma_i^n : [n+1] \rightarrow [n])$$

*i*th *face* (resp. *degeneracy*) map that forgets (resp. repeats) *i*.

2. Show that any morphism $\alpha : [m] \rightarrow [n]$ in Δ can be uniquely written as

$$\alpha = \delta_{i_s}^n \delta_{i_{s-1}}^{n-1} \dots \delta_{i_1}^{n-s+1} \sigma_{j_t}^{m-t} \sigma_{j_{t-1}}^{m-t+1} \dots \sigma_{j_1}^{m-1}$$

with $m-1 \geq j_1 > j_2 > \dots > j_t \geq 0$ and $0 \leq i_1 < i_2 < \dots < i_s \leq n$ (and $n-s = m-t$).

We recall that a *simplicial set* is a family of sets S_n endowed with maps $d_n^i : S_n \rightarrow S_{n-1}$ for $n > 0$ and $s_n^i : S_n \rightarrow S_{n+1}$ for $n \geq 0$ with $0 \leq i \leq n$ satisfying

$$\begin{aligned} d_n^i \circ d_{n+1}^j &= d_n^{j-1} \circ d_{n+1}^i \quad \text{for } 0 \leq i < j \leq n+1, \\ s_{n+1}^i \circ s_n^j &= s_{n+1}^{j+1} \circ s_n^i \quad \text{for } 0 \leq i \leq j \leq n, \\ d_{n+1}^i \circ s_n^j &= \begin{cases} s_{n-1}^{j-1} \circ d_n^i & \text{for } 0 \leq i < j \leq n \\ \text{Id}_{S_n} & \text{for } 0 \leq j \leq i \leq j+1 \leq n+1 \\ s_{n-1}^j \circ d_n^{i-1} & \text{for } 0 \leq j < i-1 \leq n. \end{cases} \end{aligned}$$

Moreover, if $T_\bullet$ is another (semi) simplicial set, then a *map of simplicial sets* $f : S_\bullet \rightarrow T_\bullet$ is a family of maps $f_n : S_n \rightarrow T_n$ satisfying $f_{n-1} \circ d_n^i = d_n^i \circ f_n$ for $n > 0$ and $f_{n+1} \circ s_n^i = s_n^i \circ f_n$ for $n \geq 0$.

3. Show that simplicial sets form a category $\mathbf{sSet}$ isomorphic to the category $\widehat{\Delta} := \mathbf{Hom}(\Delta^{\text{op}}, \mathbf{Set})$ of presheaves of sets on Δ .

We shall make the *identification* $\mathbf{sSet} = \widehat{\Delta}$ and write $S_n := S([n])$. We shall also write $\Delta^n := h_{[n]} : [m] \mapsto \text{Hom}([m], [n])$ so that $\Delta^n \in \mathbf{sSet}$.

For $S \in \mathbf{sSet}$, we consider the following category $\Delta_{/S}$:

- an object is an $s \in S_n$ for *some* $n \in \mathbb{N}$,
- a morphism from $t \in S_m$ to $s \in S_n$ is a morphism $\alpha : [m] \rightarrow [n]$ in Δ such that $S(\alpha)(s) = t$.

We then consider the diagram in $\mathbf{sSet}$ indexed by $\Delta_{/S}$ sending $s \in S_n$ to Δ^n and $\alpha : [m] \rightarrow [n]$ to $h_\alpha : \Delta^m \rightarrow \Delta^n$.

4. Show that if $S \in \mathbf{sSet}$, then $S \simeq \varinjlim_{\Delta_{/S}} \Delta^n$.

We let $\mathcal{T}$ be a cocomplete category and $r : \Delta \rightarrow \mathcal{T}$ a functor.

5. Show that

- (a) the functor

$$G : \mathcal{T} \rightarrow \text{Hom}(\Delta^{\text{op}}, \mathbf{Set}) = \mathbf{sSet}$$

obtained by currying from

$$\Delta^{\text{op}} \times \mathcal{T} \xrightarrow{r^{\text{op}} \times \text{Id}} \mathcal{T}^{\text{op}} \times \mathcal{T} \xrightarrow{\text{Hom}} \mathbf{Set}$$

has an adjoint F ,

- (b) if $S \in \mathbf{sSet}$, then $F(S) \simeq \varinjlim_{\Delta_{/S}} r([n])$,
- (c) if $X \in \mathcal{T}$ and $n \in \mathbb{N}$, then $G(X)_n = \text{Hom}(r([n]), X)$.

We finish with some MCQs:

6. (Proof required) In case $r : \Delta \rightarrow \mathbf{sSet}, [n] \mapsto \Delta^n$ is the Yoneda embedding, then G is

☐ faithful ☐ fully faithful ☐ an equivalence ☐ none.

We recall that a topological space is said to be *compactly generated* if it is a colimit of compact Hausdorff spaces and that there exists topological spaces such as $\mathbb{R}^{\mathbb{R}}$ that are *not* compactly generated.

7. (Proof required) In case $r : \Delta \rightarrow \mathbf{Top}, [n] \mapsto |\Delta_n|$ sends $[n]$ to the convex envelope of the canonical basis $e_0, \dots, e_n$ of $\mathbb{R}^{n+1}$ and α to the linear map $e_i \mapsto e_{\alpha(i)}$, then G is

☐ faithful ☐ fully faithful ☐ an equivalence ☐ none.

8. (**No** proof required) In case $r : \Delta \rightarrow \mathbf{Cat}, [n] \mapsto [\mathbf{n}]$ is the canonical embedding, then G is

☐ faithful ☐ fully faithful ☐ an equivalence ☐ none.