Université de Rennes 2025-2026

Homological algebra
Examen 2 du 16 décembre 2025
Début 16h – Durée 1h

You can compose in english or in french. You can freely (no need to make a precise reference) use any result (exercises included) obtained prior to the appearance of the question in the (online) course. You can use any document you wish (electronic or not).

We let X be a topological space.

1. (2 points) Show that, if $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}(\emptyset) = 1$.

> **Solution:** In the case $U = \emptyset$ and $R = \emptyset$, then the condition reads $\mathcal{F}(\emptyset) = \varprojlim_{V \subset \emptyset} \mathcal{F}(V) = 1$.

2. (4 points) Show that, if E is a set, then $E_X \simeq \mathcal{C}_E^X$ when E is endowed with the discrete topology, and there exists an adjunction

$$\mathrm{Hom}(E_X, \mathcal{F}) \simeq \mathrm{Hom}(E, \Gamma(X, \mathcal{F}))$$

if $\mathcal{F}$ is a sheaf.

> **Solution:** If $\mathcal{F}$ is a sheaf, there exists a sequence of natural isomorphisms
>
> $$\mathrm{Hom}(E_X, \mathcal{F}) \simeq \mathrm{Hom}(\widehat{E}_X, \mathcal{F}) \simeq \mathrm{Hom}(E, \Gamma(X, \mathcal{F})).$$
>
> This proves the second assertion. But then, E_X and $\mathcal{C}_E^X$ have the same universal property and must be isomorphic.

We let $\mathcal{A}$ be a sheaf of rings on X and $\mathcal{M}^\bullet$ be a complex of $\mathcal{A}$-modules.

3. (4 points) Show that $\mathrm{H}^n(X, \mathcal{M}^\bullet) \simeq \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{M}^\bullet)$ for all $n \in \mathbb{Z}$.

> **Solution:** We know that there exists a natural isomorphism $\Gamma(X, \mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{M})$. This provides an isomorphism at the derived functor level and therefore also on cohomology.

We assume that X is a manifold of class $\mathcal{C}^k$.

4. (a) Let E be a vector bundle of rank n on X. Show that
   i. (2 points) if we denote by $E(U)$ the set of $\mathcal{C}^k$-differentiable sections on an open subset U, then we obtain an $\mathcal{O}_X$-module on X,

Solution: This is clearly a sheaf because a section on an open subset U is simply a map $s : U \to E$ and that a map is uniquely defined by its values. Given another section s', we define $(s + s')(x) := s(x) + s'(x)$ for $x \in U$. Also, if we are given a section $f : U \to \mathbb{R}$ of $\mathcal{O}_X$, then one sets $(fs)(x) := f(x)s(x)$. Since each E_x is an $\mathbb{R}$-vector space, this defines a structure of $\mathcal{O}_X(U)$-module on $E(U)$ and they are compatible by definition.

ii. (2 points) E is a locally free $\mathcal{O}_X$-module of rank n (any point has an open neighborhood U such that $E_{|U} \simeq \mathcal{O}_U^n$).

Solution: Any point has an open neighborhood U such that $E_{|U} \simeq U \times \mathbb{R}^n$. If V is an open subset of U, then $E(V) \simeq \mathcal{C}^k(V, \mathbb{R}^n) = \mathcal{C}^k(V, \mathbb{R})^n = \mathcal{O}_U(V)^n = \mathcal{O}_U^n(V)$.

(b) Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of rank n. Show that

i. (2 points) $E_x := \mathcal{F}(x)/\mathfrak{m}_X(x)\mathcal{F}(x)$ is a vector space of dimension n,

Solution: The question is local and we may therefore assume that $\mathcal{F} = \mathcal{O}_X^n$. The question is additive and we may therefore assume that $n = 1$. Finally, we have $\mathcal{O}_X(x)/\mathfrak{m}_X(x) \simeq \mathbb{R}$.

ii. (4 points) there exists a structure of vector bundle of rank n on $E := \coprod_{x \in X} E_x$.

Solution: Our space X is covered by open subsets U such that $\mathcal{F}_{|U} \simeq \mathcal{O}_U^n$. When $x \in U$, this isomorphism induces an isomorphism $\mathcal{F}(x) \simeq \mathcal{O}_X(x)^n$ and then $E_x \simeq \mathbb{R}^n$ on the quotients. All together, they provide a chart $E \simeq U \times \mathbb{R}^n$. If we are given another open subset V, then the glueing map

$$(U \cap V) \times \mathbb{R}^n \simeq (U \cap V) \times \mathbb{R}^n, \quad (x, v) \mapsto (x, f(x, v))$$

comes from an automorphism of $\mathcal{O}_{U \cap V}^n$ given by $\mathcal{C}^k$-differentiable maps $f_1, \ldots, f_n$ on $U \cap V$ via

$$\mathrm{Hom}(\mathcal{O}_{U\cap V}^n, \mathcal{O}_{U\cap V}^n) \simeq \mathrm{Hom}(\mathcal{O}_{U\cap V}, \mathcal{O}_{U\cap V}^n)^n \simeq \mathcal{O}_{U\cap V}^n(U \cap V)^n.$$

(c) (Bonus points) the category of vector bundles on X is equivalent to the category of locally free $\mathcal{O}_X$-modules of finite rank.