The analytic site of a scheme

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1 Rigid cohomology

Rigid cohomology associates to a variety X over a field k of characteristic $p > 0$ some vector spaces $H^q_{\mathrm{rig}}(X/K)$ over a p-adic field K of characteristic 0. More generally, one defines *overconvergent isocrystals* E over X and their *rigid cohomology* $H^*_{\mathrm{rig}}(X/K, E)$.

The method is as follows : embed X in some proper smooth formal scheme P over the ring of integers $\mathcal{V}$ of K and choose a sufficiently small strict neighborhood V of X in the generic fiber of P. An overconvergent isocrystal on X is a coherent module with an integrable "overconvergent" connection E on V and

$$H^*_{\mathrm{rig}}(X/K, E) := H^*_{\mathrm{dR}}(V, E).$$

2 New interpretation

A *(pre-) topology* on a category $\mathcal{C}$ is a collection of *coverings* $\{X_k \to X\}_k$ which is stable by pull back and composition and contains the identites. A *site* is a category $\mathcal{C}$ endowed with a topology. The *sheaves* on $\mathcal{C}$ are the contravariant functors from C to Sets that satisfy gluing with respects to coverings. They form a *topos* $\widetilde{\mathcal{C}}$. A *ringed site* is a site $\mathcal{C}$ endowed with a sheaf of rings.

We will show how one can associate in a *natural* way a ringed site $(\mathrm{AN}(X), \mathcal{O}_X^\dagger)$ to an algebraic variety X so that the category of overconvergent isocrystals on X can be identified with the category of finitely presented $\mathcal{O}_X^\dagger$-modules.

3 The conventions

K	:	a complete ultrametric field
$\mathcal{V}$	:	the valuation ring of K
k	:	the residual field of $\mathcal{V}$

Algebraic variety	:	k-scheme locally of finite type
Formal scheme	:	$\mathcal{V}$-formal scheme locally topologically finitely presented
Analytic space	:	Berkovich K-analytic space

If S is a formal scheme, its special fiber S_k is an algebraic variety, its generic fiber S_K is an analytic space and there is a specialization map $sp : S_K \to S$.

4 Example

$$K := \mathbf{C}((t)) \quad \mathcal{V} = \mathbf{C}[[t]] \quad k = \mathbf{C} \quad \pi = t \quad |F| = e^{-ord_t(F)},$$
$$K := \mathbf{Q}_p \quad \mathcal{V} = \mathbf{Z}_p \quad k = \mathbf{F}_p \quad \pi = p \quad |n| = p^{-ord_p(n)}.$$

$$\begin{array}{lcl|l|l} S & : & \widehat{\mathbf{A}_{\mathcal{V}}^n} & V(f_1, \ldots, f_r) \subset \widehat{\mathbf{A}_{\mathcal{V}}^n} & \widehat{\mathbf{P}_{\mathcal{V}}^N} \\ S_k & : & \mathbf{A}_k^n & V(\tilde{f}_1, \ldots, \tilde{f}_r) \subset \mathbf{A}_k^n & \mathbf{P}_k^n \\ S_K & : & \mathbf{B}^n(0, 1^+) & V(f_1, \ldots, f_n) \subset \mathbf{B}^n(0, 1^+) & \mathbf{P}_K^{n\ \mathrm{an}} \end{array}$$

$$\begin{array}{ccc|ccc} sp : \mathbf{B}^n(0,1^+) & \longrightarrow & \mathbf{A}_k^n & sp : \mathbf{P}_K^{n\ \mathrm{an}} & \longrightarrow & \mathbf{P}_k^n \\ (x_1, \ldots, x_n) & \mapsto & (\tilde{x}_1, \ldots, \tilde{x}_n) & (x_0, \ldots, x_n) & \mapsto & (\tilde{x}_0, \ldots, \tilde{x}_n) \\ & & & \multicolumn{3}{c}{\text{if } \max(|x_1|, \ldots, |x_n|) = 1} \end{array}$$

5 The objects

Let S be a formal scheme and X an algebraic variety over S_k.

An *analytic variety over* X/S is a couple $(U \hookrightarrow P, \lambda : V \to P_K)$ where $U \hookrightarrow P$ is a *good* embedding of an X-scheme into a formal scheme over S and λ is a morphism of analytic spaces over S_K. When there is no risk of confusion, we will simply write (U, V).

Remark : The formal embedding is said *good* if any point in $]U[_P := sp^{-1}(U)$ has an affinoid neighborhood (this is the case for example, if P is affine or proper over $\mathcal{V}$).

6 Example

We let

$$S := \mathrm{Spf}(\mathcal{V}) \text{ and } A := \mathcal{V}[T_1, \ldots, T_N]/I.$$

We want

$$U = X = \mathrm{Spec}(A \otimes_{\mathcal{V}} k) \text{ and } V := [\mathrm{Spec}(A \otimes_{\mathcal{V}} K)]^{\mathrm{an}}.$$

$$\begin{array}{lll} P = \mathrm{Spf}(\hat{A}) & ? & \text{No, because } [\mathrm{Spf}(\hat{A})]_K = V \cap \mathbf{B}^N(0, 1^+). \\ P = \widehat{\mathbf{A}_{\mathcal{V}}^N} & ? & \text{Not better, because } [\widehat{\mathbf{A}_{\mathcal{V}}^N}]_K = B^N(0, 1^+). \\ P = \widehat{\mathbf{P}_{\mathcal{V}}^N} & ? & \text{Yes since } [\widehat{\mathbf{P}_{\mathcal{V}}^N}]_K = \mathbf{P}_K^{N\mathrm{an}} \supset \mathbf{A}_K^{N\mathrm{an}} \supset V. \end{array}$$

7 The notion of tube

If $(U \hookrightarrow P, \lambda : V \to P_K)$ is an analytic variety over X/S, the *tube* of U in V is

$$]U[_V := \lambda^{-1}(sp^{-1}(U)).$$

We will denote by

$$i_U :]U[_V \hookrightarrow V$$

the canonical embedding and by

$$]U[_V \longrightarrow U, \quad x \mapsto \tilde{x} = sp(\lambda(x))$$

the *specialization map*.

8 The morphisms

A *hard morphism*

$$(f,u):(U',V')\to(U,V)$$

is a couple where $f:U'\to U$ is a morphism of algebraic varieties over X and $u:V'\to V$ is a morphism of analytic spaces over S_K such that

$$x'\in]U'[_{V'}\Rightarrow f(\widetilde{x'})=\widetilde{u(x')},$$

and the same is true after any isometric extension of K.

We obtain a category $\overline{\mathrm{AN}}(X/S)$ which has non-empty finite inverse limits.

9 The category

If we localize on the right the category $\overline{\mathrm{AN}}(X/S)$ with respect to hard morphisms

$$(Id_U, j) : (U, V') \hookrightarrow (U, V)$$

where j is an open immersion of analytic spaces that induces

$$]U[_{V'} \simeq]U[_V,$$

we obtain the category $\mathrm{AN}(X/S)$ of *analytic varieties over* X/S.

Remark : Right localization with respect to a subcategory made of monomorphisms and stable under pull-back is always possible.

10 The topology

A family

$$(Id_U, j_k) : \{(U, V_k) \to (U, V)\}_{k \in I}$$

is an *open covering* if there exists open neighborhoods W_k of $]U[_{V_k}$ in V_k such that j_k induces an open immersion $W_k \hookrightarrow V$ and $]U[_V \subset \cup W_k$.

This is a pretopology on $\mathrm{AN}(X/S)$. The corresponding site has fibered products and the topology is coarser than the canonical topology. We will denote by $(X/S)_{\mathrm{AN}}$ the corresponding topos.

This construction is functorial in $X/S/\mathcal{V}$.

11 The sheaves

If E is a sheaf on $\mathrm{AN}(X/S)$ and (U, V) an analytic variety over X/S, the *realization* of E on (U, V) is the sheaf on $]U[_V$ defined by

$$E_{U,V} :]U[_{V'} \mapsto E(U, V').$$

Giving E is equivalent to the following : the sheaves $E_{U,V}$ and the transitions morphisms

$$\phi_{f,u} : u^{-1}E_{U,V} \to E_{U',V'}$$

coming from morphisms $(f, u) : (U', V') \to (U, V)$.

12 The crystals

The presheaf

$$\mathcal{O}^{\dagger}_{X/S} : (U, V) \mapsto \Gamma(]U[_V, i_U^{-1}\mathcal{O}_V)$$

of *dagger functions* on X/S is a sheaf and a module on this ring is called a *dagger module.*

Note that any morphism $(f, u) : (U', V') \to (U, V)$ induces a morphism of ringed spaces

$$(u^{\dagger}, u_*) : (]U'[_{V'}, i_{U'}^{-1}\mathcal{O}_{V'}) \to (]U[_V, i_U^{-1}\mathcal{O}_V).$$

$\cdots/\cdots$

Giving a dagger module E is equivalent to the following : the $i_U^{-1}\mathcal{O}_V$-modules $E_{U,V}$ and the transition morphisms

$$\phi_{f,u} : u^\dagger E_{U,V} \to E_{U',V'}.$$

A dagger module E is a *crystal* if the transition morphisms are bijective. Actually, a dagger module E is finitely presented if and only if it is a crystal and the $E_{U,V}$ are coherent $i_U^{-1}\mathcal{O}_V$-modules.

Remark : we could also consider the convergent situation with

$$\hat{\mathcal{O}}_{X/S} : (U,V) \mapsto \Gamma(]U[_V, \mathcal{O}_{]U[_V}).$$

13 Varieties over a formal scheme

If $X \hookrightarrow P$ is a good embedding, we denote by

$$\mathrm{AN}(X_P/S)$$

the full subcategory of objects (U, V) of $\mathrm{AN}(X/S)$ which admit some morphism $(U, V) \to (X, P_K)$.

We can do with this category exactly the same as what we did with $\mathrm{AN}(X/S)$. In particular, we have a ringed site

$$(\mathrm{AN}(X_P/S), \mathcal{O}^\dagger_{X_P/S}),$$

the notions of dagger module or dagger crystals on X_P/S, and their realizations.

14 The Taylor isomorphism

If E is a dagger crystal on X_P and $E_P := E_{X,P_K}$, the isomorphism

$$\epsilon_P : p_2^\dagger E_P \simeq E_{P^2} \simeq p_1^\dagger E_P$$

on $]X[_{P^2}$ is the *Taylor isomorphism* of E_P. If E_P is coherent, it extends to a module with an integrable connexion on some neighborhood V of $]X[_P$ in $]\bar{X}[_P$ (where $\bar{X}$ is the Zariski closure of X).

Proposition 1 *(Char$K = 0$) If P is smooth in the neighborhood of X, we obtain an equivalence of categories between finitely presented $\mathcal{O}^\dagger_{X_P/S}$-modules and isocrystals on $\bar{X}$ in P which are overconvergent along $\bar{X}\backslash X$.*

15 The main theorem

Theorem 1 *If $X \hookrightarrow P$ is a good admissible embedding with P/S proper and smooth on X, there is a canonical equivalence of toposes*

$$(X_P/S)_{AN} \simeq (X/S)_{AN}.$$

From this, we deduce, when $\mathrm{Char} K = 0$, an equivalence between the category of finitely presented $\mathcal{O}^{\dagger}_{X/S}$-modules and the category of overconvergent isocrystals on X/S.

Remarks : 1) $X \hookrightarrow P$ is *admissible* if P has no $\mathcal{V}$-torsion. 2) $u : P' \to P$ is *proper on* $X \subset P'$ if the Zariski closure $\overline{X}$ of X in P'_k has proper irreducible components over P_k.

16 The method

The theorem reduces successively to the next two theorems.

Theorem 2 *If* $X \hookrightarrow P$ *is a good admissible embedding with* P/S *proper and smooth on* X*, then* (X, P_K) *is a covering of (the final object of)* $(X/S)_{AN}$*.*

Theorem 3 *Let* $u : P' \to P$ *be a morphism of good admissible embeddings of* X*. If* u *is proper and smooth on* X*, the morphism*

$$(Id_X, u_K) : (X, P'_K) \to (X, P_K)$$

has locally a section in $AN(X/S)$*.*

17 The proof

Using results of Berkovich, Temkin and Ducros, one shows that the morphism of the last theorem induces a smooth morphism $u_K : V' \to V$ between good neighborhoods of the tubes.

The smoothness of u_K gives for each $x \in]X[_V$ an isomorphism

$$\phi_x : u_K^{-1}(x) \simeq \mathbf{B}^d_K(0, 1^-).$$

One then needs an approximation theorem in order to show that there exists neighborhoods W of x in V and W' of $\phi_x^{-1}(x)$ in V' such that ϕ_x extends to an open immersion

$$\phi : W' \hookrightarrow \mathbf{B}^d_W(0, 1^-).$$