Homological Algebra

(Master course)

Bernard Le Stum (September 5, 2026)

– If you're capable of doing the math yourself with confidence and efficiency, you're often better off doing it yourself.

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General invariant theory associates a simple mathematical object to a more complex one. For instance, one can associate to a set E its number of elements $\sharp E$.

Invariant theory is frequently employed to establish impossibilities. For instance, Cantor-Bernstein theorem demonstrates that when the cardinality of a set E is less than that of another set F ($\sharp E < \sharp F$), there exists no injective map from F to E.

Another basic invariant is the dimension $\dim V$ of a finite-dimensional real vector space V. It can be shown that when the dimensions of V and another vector space W are different ($\dim V \neq \dim W$), there exists no homeomorphism between V and W (Brouwer's open mapping theorem).

However, to prove this, a more sophisticated invariant is required: to a topological manifold X, one can associate a family of abelian groups $\mathrm{H}_i(X)$, called homology groups, which are invariant up to isomorphism under homeomorphism.

From a finite-dimensional vector space V endowed with an Euclidean norm, one can construct a sphere $S := (V \smallsetminus \{0\})/\| - \|$. It can then be shown that $\mathrm{H}_n(S) = 0$ unless $n = 0$ or $n = \dim(V) - 1$ in which case it is free of rank one. This implies Brouwer's theorem.

Homology and cohomology theories provide invariants of abelian nature (for instance, abelian groups) for objects of geometrical nature (for example, manifolds). The purpose of homological algebra is to elucidate the theory behind these invariants. It appears that the most suitable setting for this study is category theory.

Introduction

It is common in mathematics to present an object by generators and relations.

For example, the field $\mathbb{C}$ of complex numbers may be seen as the $\mathbb{R}$-algebra generated by an element x satisfying $x^2+1=0$. We may think of $x=i$ but can as well choose $x=-i$ if we wish. In any case, we can write $\mathbb{C}=\mathbb{R}[X]/(X^2+1)$: it means that there exists a surjection from the polynomial ring $\mathbb{R}[X]$ whose kernel is the ideal generated by X^2+1. More generally, any commutative algebra over a commutative ring R may be written on the form $R[\{X_i\}_{i\in I}]/(f_j)_{j\in J}$.

Let us now consider the dihedral group D_n. This is the group generated by (a rotation of order n) r and (a reflection) s subject to the relations $r^n=1$, $s^2=1$ and $rsrs=1$. One usually writes

$$D_n=\langle r,s \mid r^n=1, s^2=1, rsrs=1\rangle=\langle R,S\rangle/\langle R^n,S^2,RSRS\rangle.$$

It means that there exists a surjection from the free group generated by two elements R,S whose kernel is the subgroup generated by R^n, S^2 and $RSRS$. This last group is in turn the target of a surjection from the free group on three elements X,Y,Z sending them respectively to R^n, S^2 and $RSRS$. More generally, any group has a free *presentation*

$$F'\stackrel{\psi}{\to}F\stackrel{\phi}{\to}G\to 0.$$

It means that both F and F' are free groups, im $\psi=\ker\phi$ and ϕ is surjective (im $\phi=\ker 0$).

Let us consider now the *commutative* group μ_n of nth roots unity. If we choose a primitive nth root of unity ζ, then there exists a surjection $\mathbb{Z}\to\mu_n$ sending k to ζ^k. Of course, $\mathbb{Z}$ is a free abelian group. But the kernel $n\mathbb{Z}$ of our surjective map also is a free abelian group. This is a general fact: any commutative group has a free *(left) resolution*

$$0\to P_1\stackrel{i}{\to}P_0\stackrel{p}{\to}M\to 0$$

(of length 2). It means that P_0 and P_1 are free abelian groups, i is injective (im $0=\ker i$), im $i=\ker p$ and p is surjective (im $p=\ker 0$). One also calls $P_1\to P_0$ a free *(left) replacement* for M.

Before going any further, let us consider the effect of a "transformation" on a free resolution. For example, for a fixed integer n, we may turn an abelian group M into its quotient M/n and consequently, any morphism of abelian groups $f:M\to N$ into the corresponding map $\overline{f}:M/n\to N/n$. If we apply this transformation to the resolution $0\to\mathbb{Z}\stackrel{n}{\to}\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\to 0$, then we get $0\to\mathbb{Z}/n\mathbb{Z}\stackrel{0}{\to}\mathbb{Z}/n\mathbb{Z}\stackrel{\sim}{\to}\mathbb{Z}/n\mathbb{Z}\to 0$ which is *not* a resolution anymore because the first map is not injective. The point is that, if we are given $M\stackrel{f}{\to}N\stackrel{g}{\to}P$, it may happen that im $f=\ker g$ but im $\overline{f}\neq\ker\overline{g}$. This is called a default of exactness. However, it should be noticed that our transformation is compatible with composition: we have $\overline{g\circ f}=\overline{g}\circ\overline{f}$. As a consequence, the fact that im $f\subset\ker g$ will always imply im $\overline{f}\subset\ker\overline{g}$.

Let us now consider the case of a (left) R-module M where R is a ring. To generalize the previous case (which is essentially the case $R = \mathbb{Z}$), we need to allow 1) direct summands of free modules, which are called *projective* modules and 2) longer, possibly infinite, *projective* (left) resolutions

$$\cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0,$$

meaning that $\operatorname{im} d_{n+1} = \ker d_n$ (so that, in particular, $\operatorname{im} d_0 = \ker 0 = M$ which means that d_0 is surjective). And we shall call $P_\bullet := \cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0$ a projective (left) replacement for M.

Again, a transformation will not preserve resolutions in general but it will preserve chain complexes (that we will shortly define) as long as the transformation is compatible with composition. A *chain complex* is a sequence of maps

$$K_\bullet := \cdots \to K_2 \overset{d_2}{\to} K_1 \overset{d_1}{\to} K_0 \overset{d_0}{\to} K_{-1} \to \cdots$$

satisfying $\operatorname{im} d_{n+1} \subset \ker d_n$, or equivalently $d_n \circ d_{n+1} = 0$. To measure the default of exactness, we introduce the *homology* groups

$$\mathrm{H}_n(K_\bullet) := \ker d_n / \operatorname{im} d_{n+1}.$$

Now, a transformation will preserve complexes but shake homology groups and this is what we want to control.

Let us briefly explain how these techniques can be used in topology. We shall not enter the details at this point, but one may consider on any topological space X, the notion of an abelian sheaf M: it is a compatible[1] local[2] family of abelian groups $\Gamma(U, M)$ for all open subsets U of X. As a baby example, one may consider the sheaf $\mathcal{C}_\mathbb{R}$ of continuous maps on X with values in $\mathbb{R}$ so that $\Gamma(U, \mathcal{C}_\mathbb{R}) = \mathcal{C}(U, \mathbb{R})$ or the *constant sheaf* $\mathbb{Z}_X$ given by

$$\Gamma(U, \mathbb{Z}_X) = \mathbb{Z}^{\pi_0(U)}.$$

It happens that there are *not* enough projective abelian sheaves in general. We need to rely on the dual construction and introduce the notions of an *injective* abelian sheaf as well as an injective (right) resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

(and injective (right) replacement). One also replaces chain complex with *cochain complexes*

$$K^\bullet := \cdots \to K^{-1} \overset{d_{-1}}{\to} K^0 \overset{d_0}{\to} K^1 \overset{d_1}{\to} K^2 \to \cdots$$

meaning $d_n \circ d_{n-1} = 0$ and homology with *cohomology*

$$\mathrm{H}^n(K^\bullet) = \ker d_n / \operatorname{im} d_{n-1}.$$

[1] There exists restrictions maps $s \mapsto s_{|V}$ for s on U and $V \subset U$ with $(s_{|V})_W = s_{|W}$ and $s_{|U} = s$.

[2] Given s_i on U_i such that s_i and s_j coincide on $U_i \cap U_j$, then there exists a unique s on $\bigcup U_i$ extending all s_i simultaneously.

One can show that if $I^\bullet$ is an injective replacement for an abelian sheaf M, then

$$\mathrm{H}^n(X, M) := \mathrm{H}^n(\Gamma(X, I^\bullet))$$

only depends on M and not on $I^\bullet$. This is the *nth cohomology group* of M. When X is locally contractible, we recover *singular cohomology*:

$$\mathrm{H}^n(X, \mathbb{Z}_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X).$$

Sheaf cohomology however is a lot more flexible since it allows coefficients in any abelian sheaf (and not merely abelian group) and this may even be extended to cochain complexes of abelian sheaves. There also exists a relative version: given any sheaf M on X and any continuous map $f : X \to Y$, there exists cohomology sheaves $R^n f_* M$ on Y which may be used to compute the cohomology of M. There is no such thing in singular cohomology (unless f is a fibration).

One can also apply these methods to differential geometry. On a smooth manifold X, there exists the de Rham complex:

$$\Omega^\bullet_X := \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_X.$$

Poincaré lemma states that this is actually a (right) replacement for the constant sheaf $\mathbb{R}_X$. The cohomology of the complex of global sections

$$\Gamma(X, \Omega^\bullet_X) := \Gamma(X, \mathcal{O}_X) \xrightarrow{\mathrm{d}} \Gamma(X, \Omega^1_X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Gamma(X, \Omega^n_X)$$

is called *de Rham cohomology.* It happens actually that

$$\mathrm{H}^n_{\mathrm{dR}}(X) := \mathrm{H}^n(\Gamma(X, \Omega^\bullet_X)) \simeq \mathrm{H}^n(X, \Omega^\bullet_X) \simeq \mathrm{H}^n(X, \mathbb{R}_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X, \mathbb{R}),$$

showing that de Rham and singular cohomology coincide. In particular, a differential invariant appears to actually be a topological invariant. This generalizes to a Riemann-Hilbert correspondence between differential systems and local systems but this is another story.

Throughout the course, there will be numerous exercises. Some of these exercises will illustrate previous properties or definitions, while others will present general results that are usually straightforward to prove. It's important to ensure that you can complete all these exercises and write down full solutions.

Some material is optional and can be skipped on a first lecture: this will be indicated by the use of 10 points font.

Many thanks to the students who helped make this course more readable by their comments and questions.

Background

We briefly review some fundamental concepts from general mathematics as well as specific developments that will be used throughout the course. Our main goal is to refresh the reader's memory with familiar definitions and standard properties, as well as introduce some notation. We assume that the reader is proficient in elementary logical operations, including quantifiers, and has mastered the induction process. Additionally, we expect the reader to be familiar with the concepts of sets, elements, inclusion, intersection, union, and (possibly infinite) product[3] of sets. Also, we will not revisit the definitions and basic properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ nor delve into issues related to logic or set theory. Finally, we shall use von Neumann notation $n := \{0, 1, \dots, n-1\}$.

0.1 Relation

A *(binary) relation* $R : X \to Y$ is the data of two sets X, Y called respectively *domain* and *codomain*, and another set $R \subset X \times Y$ called the *graph* of the relation. We will write xRy instead of $(x, y) \in R$ and call y an *image* of x (resp. x a *preimage* of y).

There exists the notion of a *(direct) image* by R of a subset A of X which is defined as[4]

$$R(A) := \{y \in Y, \exists x \in X, xRy\}.$$

We also set $\operatorname{im}(R) := R(X)$. We shall simply write $R(x) := R(\{x\})$ and more generally remove all embedded parenthesis when there is no risk of confusion.

[3]Let us however recall that the *product* $\prod_{i\in I} X_i$ is the set of all *families* $(x_i)_{i\in I}$ of *terms* $x_i \in X_i$ for all $i \in I$ and we shall write $X^I := \prod_I X$.

[4]To avoid confusion later, we should write $R_*(A)$ in which case, we would write $R^*(B) = R_*^{-1}(B)$ below.

There exists an *inverse* relation $R^{-1} : Y \to X$ for R obtained by switching factors so that

$$\forall x \in X, y \in Y, \quad yR^{-1}x \Leftrightarrow xRy.$$

If $B \subset Y$, then $R^{-1}(B)$ is called the preimage or *inverse image* of B by R.

One can make the *product* of relations[5] $\prod_{i\in I} R_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ so that

$$(x_i)_{i\in I} \left(\prod_{i\in I} R_i\right) (y_i)_{i\in I} \quad \Leftrightarrow \quad \forall i \in I, x_i R_i y_i.$$

We may consider the *conjunction* $R := \bigcap_{i\in I} R_i : X \to Y$ of relations so that

$$\forall x \in X, y \in Y, \quad xRy \quad \Leftrightarrow \quad \forall i \in I, xR_i y.$$

A relation $R' : X' \to Y'$ is *a restriction* (resp. *the restriction*) of R to $X' \subset X$ and $Y' \subset Y$ if

$$\forall x \in X', y \in Y', \quad xR'y \Rightarrow xRy \quad (\text{resp. } \Leftrightarrow xRy).$$

We may also say that R' is *induced* by R or that R is an *extension* of R'.

One can also *compose* a relation $R : X \to Y$ with a relation $S : Y \to Z$ by setting

$$x(S \circ R)z \Leftrightarrow \exists y \in Y, xRy \text{ and } ySz.$$

A *map* $f : X \to Y$ is a relation satisfying

$$\forall x \in X, \exists y \in Y, \quad f(\{x\}) = \{y\}.$$

We will then write $f(x) = y$ or $f : x \mapsto y$.

For a set X, we shall consider the set $\mathcal{P}(X)$ of subsets of X so that $A \subset X \Leftrightarrow A \in \mathcal{P}(X)$. If $R : X \to Y$ is any relation, then the direct and inverse images introduced above are maps

$$R : \mathcal{P}(X) \to \mathcal{P}(Y), \quad A \mapsto R(A) \quad \text{and} \quad R^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \quad B \mapsto R^{-1}(B)$$

and any of them uniquely determines the relation.

A map $f : X \to Y$ is said to be *injective* (resp. *surjective*, resp. *bijective*) if, for all $y \in Y$, $f^{-1}(y)$ has at most (resp. at least, resp. exactly) one element. The map f is bijective if and only if f^{-1} is a map.

We shall denote by $\mathcal{F}(X, Y)$ (resp. $\mathcal{S}(X, Y)$) the set of all maps (resp. bijective maps) from X to Y and $\mathcal{F}(X)$ (resp. $\mathcal{S}(X)$) when $X = Y$. There always exists a bijection

$$\mathcal{F}(X, Y) \simeq Y^X, \quad f \mapsto (f(x))_{x\in X}.$$

It means that a map from X to Y is essentially the same thing as a family of elements of Y indexed by X. By the way, to any set X, we can associate a family $(x)_{x\in X}$

[5] This is indeed the set-theoretic product up to switching factors.

corresponding to the map Id_X, and conversely, to any family $(x_i)_{i\in I}$, corresponding to the map $f : I \to X, i \mapsto x_i$, we can associate the set $\mathrm{im}(f) = \{x_i\}_{i\in I}$.

Also, there always exists a bijection (called *currying*)

$$\mathcal{F}(X \times Y, Z) \simeq \mathcal{F}(X, \mathcal{F}(Y, Z)), \quad f \mapsto (x \mapsto (f_x : y \mapsto f(x, y))).$$

The maps f_x are called *partial* maps. A map $X \times Y \to Z$ is also called an *operation* of X on Y with values in Z. This is the essentially the same thing as a map $X \mapsto \mathcal{F}(Y, Z)$.

0.2 Relation in a set

A *relation in* a set X is a relation $R : X \to X$. For example, the *identity* Id_X, also called *equality*, is the relation in X defined by $x\mathrm{Id}_X y \Leftrightarrow x = y$.

A relation in a set X is said to be

1. *reflexive* if $\forall x \in X, xRx$,
2. *transitive* if $\forall x, y, z \in X, (xRy \text{ and } yRz) \Rightarrow xRz$,
3. *symmetric* if $\forall x, y \in X, xRy \Rightarrow yRx$,
4. *antisymmetric* if $\forall x, y \in X, (xRy \text{ and } yRx) \Rightarrow x = y$,
5. *total* (or *linear*) if $\forall x, y \in X, xRy$ or yRx,

All five properties are inherited by (the restriction to) any subset and satisfied by the opposite relation. The first four properties are stable under conjunction. Therefore, there always exists a smallest relation in X extending R and having one or more of the first three properties.

A *preorder* is a relation in X which is both reflexive and transitive. This is called an *equivalence relation* (resp. a *(partial) order*) if, moreover, it is symmetric (resp. antisymmetric). A set endowed with a partial order is called a *poset* but we shall concentrate on preordered sets. A preorder $\leq$ on $X \neq \emptyset$ is *directed* if $\forall x, y \in X, \exists z \in X, x \leq z$ and $y \leq z$. A set endowed with a directed partial order is called a *directed* set. An element x of a preordered set $(X, \leq)$ is said to be *maximal* if $\forall y \in X, x \leq y \Rightarrow x = y$ (and *minimal* if it is maximal for the opposite preorder $\geq$).

An element x of a preordered set $(X, \leq)$ is an *upper bound* for a subset $A \subset X$ if $\forall y \in A, y \leq x$. An upper bound for X itself is called a *maximum* (or *greatest element*). There exists the opposite notions of *lower bound* and *minimum*. The *supremum* (or *join*) $\sup(A)$ (resp. *infimum* (or *meet*) $\inf(A)$) is the least upper bound (greatest lower bound). A poset is said to be *inductive* if any totally ordered subset has an upper bound. Zorn's lemma states that any inductive set has a maximal element.

If R is a relation on X and $\widetilde{R}$ denotes the equivalence relation generated by[6] R, then the *quotient* of X by R is $X/R := \{\widetilde{R}(x), x \in X\}$ and $\widetilde{R}(x)$ is called an *equivalence class*. There exists an obvious surjective map

$$\pi : X \twoheadrightarrow X/R, \quad x \mapsto \widetilde{R}(x)$$

[6]The smallest equivalence relation containing.

called the *quotient map*. A map $f : X \to Y$ is said to be *compatible* with relations R and S in X and Y respectively if

$$\forall x, x' \in X, \quad xRx' \Rightarrow f(x)Sf(x').$$

When this is the case, f provides a map $\overline{f} : X/R \to Y/S$. And conversely when S is an equivalence relation. When a map is compatible with partial orders, it is said to be *order preserving*.

The *direct image* $f(R)$ of a relation R on X by a relation $f : X \to Y$ is defined by

$$\forall y, y' \in Y, \quad yf(R)y' \Leftrightarrow \exists x, x' \in X, xRx', xfy, x'fy'.$$

The direct image by f^{-1} of a relation S on Y is called the *inverse image* by f. We shall only use these notions in the case f is a map in which case f will be compatible with R and $f(R)$ (and with $f^{-1}(S)$ and S if S is a relation on Y). The inverse image under a map of a preorder (resp. an equivalence relation) is still a preorder (resp. an equivalence relation). In particular, the inverse image of equality on Y is an equivalence relation on X. Conversely, if $\sim$ is an equivalence relation on X, then $\sim$ is the inverse image of equality under the quotient map $\pi : X \twoheadrightarrow X/\sim$. Finally, if $\leq$ is a preorder on a set X, then the image of $\leq$ is a partial order on $X/\sim$ where $x \sim y \Leftrightarrow x \leq y$ and $y \leq x$.

A *filter* on a poset P is a non empty proper subset $\mathcal{F} \subsetneq P$ such that
1. $\forall x, y \in \mathcal{F}, \exists z \in \mathcal{F}, z \leq x$ and $z \leq y$,
2. $\forall x \in \mathcal{F}, \forall y \in X, x \leq y \Rightarrow y \in \mathcal{F}$.

An *ultrafilter* is a maximal filter (for inclusion). If I is any set, then a filter on $\mathcal{P}(I)$ is also called a *filter* on I (this may be confusing when I itself is endowed with some preorder).

0.3 Group

A *magma* is a set G endowed with an internal operation

$$G \times G \to G, \quad (g, h) \mapsto gh$$

or, equivalently, a map

$$G \to \mathcal{F}(G), \quad g \mapsto (h \mapsto gh).$$

Note that $\mathcal{P}(G)$ then becomes also a magma for the rule $AB := \{gh, g \in A, h \in B\}$. A *homomorphism of magmas* is a map $\varphi : G \to G'$ such that

$$\forall g, h \in G, \varphi(gh) = \varphi(g)\varphi(h).$$

The homomorphism φ is called an *isomorphism* if it is bijective. If G is a magma, then the *opposite magma* is the magma G^{op} with the same elements as G but multiplication $(g, h) \mapsto hg$. A magma G is said to be *commutative* or *abelian* if $G^{\text{op}} = G$. A magma is called a *semigroup* if multiplication is *associative*:

$$\forall g, h, k \in G, \quad (gh)k = g(hk).$$

The notion of (non empty finite) *product* $g_1 \dots g_n$ is then defined by induction.

A *monoid* is a semigroup which is *unital*:

$$\exists 1 \in G, \forall g \in G, \quad 1g = g = g1.$$

By convention, the *empty product* is 1. A *homomorphism of monoids* is required to preserve *all* finite products (including the empty product). A *submonoid* of a monoid G is a subset which is stable under (all) finite products. Any intersection of submonoids is a submonoid and we may always consider the submonoid generated by a given subset. If X is any set, then the set $\mathcal{F}(X)$ of all maps from X to itself is a monoid for composition. If X is any set and G is a monoid, then $\mathcal{F}(X, G)$ (or equivalently G^X) is a monoid for the termwise multiplication rule: $\forall x \in X, (\varphi\psi)(x) = \varphi(x)\psi(x)$. The subset $G \cdot X \subset \mathcal{F}(X, G)$ of maps with finite support (the support of φ is $\{x \in X, \varphi(x) \neq 1\}$) is a submonoid (and similarly $G^{(X)} \subset G^X$). More generally, given a family of monoids $\{G_i\}_{i \in I}$, the subset $\oplus_{i\in I} G_i \subset \prod_{i \in I} G_i$ of families $(g_i)_{i \in I}$ that are almost always 1, is a submonoid.

An element g of a monoid G is said to be *regular* (resp. *invertible*) if multiplication by g is injective (resp. bijective) on both sides. A monoid is *integral* (resp. a *group*) if all elements are regular (resp. invertible). A *subgroup* of a group G is a submonoid which is a group. In general, we shall denote by $G^\times$ the group of invertible elements of a monoid G. Any group homomorphism $\varphi : G \to H$ induces a bijection between the set of subgroups of G containing the *kernel* $\ker \varphi := \varphi^{-1}(1)$ and the set of subgroups of H contained in the *image* $\operatorname{im} \varphi := \varphi(G)$. If M an N are two *abelian* groups, then the set $\mathrm{Hom}_{\mathrm{Ab}}(M, N)$ of all homomorphisms form M to N is a subgroup of $\mathcal{F}(M, N)$. We shall write $\mathrm{End}_{\mathrm{Ab}}(M)$ in the case $M = N$. A monoid M isomorphic to $\mathbb{Z} \cdot I$ is called a *free abelian group of rank* I. Alternatively, there exists a family $(e_i)_{i\in I}$ of elements of M called a *basis* such that each element of M car be uniquely written as a finite sum $\sum n_i e_i$ with $n_i \in \mathbb{Z}$. A subgroup of a free abelian group is also a free abelian group (this is not trivial).

If G is a monoid, then a *G-set* is a set X endowed with a homomorphism of monoids

$$G \to \mathcal{F}(X), \quad g \mapsto (x \mapsto gx).$$

It means that X is endowed with an *action* of G (an operation on X with values in X itself) which is associative ($(gh)x = g(hx)$) and unital ($1.x = x$). One may then define the notions of a *G-morphism* between G-sets as well as a *sub-G-set* in the usual way (exercise). If G is a *group*, then a G-set X comes with an equivalence relation

$$x \sim y \Leftrightarrow \exists g \in G, y = gx.$$

One can then consider the *quotient* X/G which is the set of *orbits* $Gx := \{gx, g \in G\}$ as well as the quotient map

$$X \to X/G, \quad x \mapsto Gx.$$

As a particular case, if H is a subgroup of a group G, then H acts by translation on G and we may consider the quotients G/H as well as $H\backslash G := G^{\mathrm{op}}/H^{\mathrm{op}}$. The subgroup H is said to be *normal* in G if $G/H = H\backslash G$.

An abelian group si said to be *divisible* if $\forall a \in Q, \forall n \in \mathbb{N} \setminus 0, \exists x \in Q, a = nx$. A $\mathbb{Q}$-vector space is divisible. A quotient of a divisible group is divisible.

0.4 Ring and module

A *(unital associative) ring* is an (additive) abelian group A endowed with a monoid structure that comes from a homomorphism of abelian groups

$$A \to \mathrm{End}_{\mathrm{Ab}}(A) \subset \mathcal{F}(A), \quad a \mapsto (b \mapsto ab).$$

Note that this extra condition simply means that *distributivity* holds on both sides:

$$\forall a, b, c \in A, \quad a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc.$$

A *ring homomorphism* is a map which is a homomorphism for both operations (*isomorphism* if it is bijective). A *subring* is a subset which is at the same time a subgroup of $(A, +)$ and a submonoid of $(A, \times)$. If M is an abelian group, then $\mathrm{End}_{\mathrm{Ab}}(M)$ is a ring for composition.

If A is a ring, then a *(left) A-module* is an abelian group M endowed with a ring homomorphism $A \to \mathrm{End}_{\mathrm{Ab}}(M), a \mapsto (x \mapsto ax)$. One then defines the notions of an *A-linear map* between A-modules as well as a *sub-A-module* and *quotient A-module* in the usual way (exercise). The kernel of a morphism of A-modules $\varphi : M \to N$ is a sub-A-module and its *cokernel* is defined as $\mathrm{coker}\, \varphi := N/\mathrm{im}\, \varphi$. If I is any set, then both A^I and $A \cdot I$ have a structure of A-module and a *free module of rank I* is a module isomorphic to $A \cdot I$ providing the notion of a basis. One also defines the notion of a submodule (S) generated by a subset S and call an A-module M *finite (type)* if it is generated by a finite subset. A *(left) ideal* $\mathfrak{a}$ of A is a submodule of the A-module A. Be careful that, if $\mathfrak{a}$ is an ideal of A and M an A-module, then $\mathfrak{a}M$ will denote the submodule *generated by* the naive product and not merely the product itself. A *right A-module* is a (left) A^{op}-module. The ring A is said to be *commutative* if the multiplicative monoid of A is commutative, or equivalently if $A^{\mathrm{op}} = A$. When $\mathfrak{a}$ is an ideal in a commutative ring A, there exists a unique structure of (commutative) ring on $A/\mathfrak{a}$ making the quotient map $A \to A/\mathfrak{a}$ commutative.

When k is a *field* (meaning that k is a commutative ring and $k \setminus \{0\}$ is a group), a k-module M is also called a *vector space*. If k is a commutative ring, then the set $\mathrm{L}(M, N)$ of all k-linear maps from M to N is automatically a k-module and $M^\vee := \mathrm{L}(M, k)$ is called the *dual* of M. If M, N, P are k-modules, then a map $M \times N \to P$ is said to be *k-bilinear* if all partial maps are k-linear. The set $\mathrm{B}(M, N; P)$ of all k-bilinear maps is naturally a k-module and currying induces an isomorphism

$$\mathrm{B}(M, N; P) \simeq \mathrm{L}(M, \mathrm{L}(N, P)), \quad \varphi \mapsto (y \mapsto \varphi(x, y))$$

As an example, the action $k \times M \to M$ is bilinear and corresponds to a linear map $k \to L(M) := \mathrm{L}(M, M)$. A *(unital associative) k-algebra* is a k-module A endowed with a k-bilinear map $A \times A \to A$ that turns A into a (unital associative) ring. For example, $\mathrm{L}(M)$ is a k-algebra (for composition) and we shall write $\mathrm{GL}(M) := \mathrm{L}(M)^\times$. One sets $\mathrm{M}_n(k) := \mathrm{L}(k^n)$ and $\mathrm{GL}_n(k) := \mathrm{GL}(k^n)$.

If A is a ring, then the *tensor product* of a right A-module M with a left A-module N is the quotient $M \otimes_A N$ of the free abelian group $\mathbb{Z} \cdot (M \times N)$ by the following relation:

1. $\forall m \in M, n, n' \in N, \quad (m, n + n') \sim (m, n) + (m, n')$,
2. $\forall m, m' \in M, n \in N, \quad (m + m', n) \sim (m, n) + (m', n)$,
3. $\forall a \in A, m \in M, n \in N, \quad (ma, n) \sim (m, an)$.

We shall denote by $m \otimes n$ the class of (m, n) in $M \otimes_A N$. It should be noticed that

$$(M/M') \otimes_A N \simeq (M \otimes_A N)/(M' \otimes_A N)$$

if $M' \subset M$ is a submodule and that

$$(\oplus_{i \in I} M_i) \otimes_A N \simeq \oplus_{i \in I}(M_i \otimes_A N)$$

if $(M_i)_{i \in I}$ is a family of right A-modules. Of course we have the same results on the other side.

If k is a *commutative* ring and M, N are two k-modules, then $M \otimes_k N$ is endowed with the structure of a k-module through $a(m \otimes n) = am \otimes n$. Composition with the map

$$M \times N \to M \otimes_k N, \quad (m, n) \mapsto m \otimes n$$

provides an isomorphism

$$\mathrm{L}(M \otimes_k N, P) \simeq \mathrm{B}(M, N; P)$$

for all k-module P. Note that we always have

$$M \otimes_k N \simeq N \otimes_k M, \quad (M \otimes_k N) \otimes_k P \simeq M \otimes_k (N \otimes_k P) \quad \text{and} \quad k \otimes_k M \simeq M.$$

Also,

$$(k \cdot E) \otimes_k (k \cdot F) \simeq k \cdot (E \times F)$$

when E, F are two sets (which gives tensor product of vector spaces a very simple description).

If $f : A \to B$ is a ring homomorphism, then B becomes a right A-module via $ba = bf(a)$ and, if M is a left A-module, then $B \otimes_A M$ becomes a B-module via $b(b' \otimes s) = bb' \otimes s$. If k is a commutative ring and A and B are two k-algebras, then $A \otimes_k B$ is endowed with a structure of a k-algebra through $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

If k is a commutative ring, then the nth *tensor power* of a k-module M is

$$\mathrm{T}^n(M) := \underbrace{M \otimes_k \cdots \otimes_k M}_{n \text{ times}}.$$

If M is a k-module, then there exists an obvious bilinear map

$$\mathrm{T}^i(M) \times \mathrm{T}^j(M) \to \mathrm{T}^{i+j}(M)$$

and the composite map

$$\left(\bigoplus_{i=0}^{\infty} \mathrm{T}^i(M)\right) \times \left(\bigoplus_{j=0}^{\infty} \mathrm{T}^j(M)\right) = \bigoplus_{i,j=0}^{\infty} \mathrm{T}^i(M) \times \mathrm{T}^j(M) \to \bigoplus_{n=0}^{\infty} \mathrm{T}^n(M)$$

endows the *tensor algebra* $\mathrm{T}(M) := \bigoplus_{n=0}^{\infty} \mathrm{T}^n(M)$ with the structure of a ring. The *symmetric* (resp. *exterior*) algebra of M is the quotient $\mathrm{S}(M)$ (resp. $\Lambda(M)$) $:= \mathrm{T}(M)/I$ where I is the ideal generated by $x \otimes y - y \otimes x$ for $x, y \in M$ (resp. $x \otimes x$ for $x \in M$). The module of nth *symmetric* (resp. *exterior*) powers of M is the image $\mathrm{S}^n(M)$ (resp. $\Lambda^n(M)$) of $\mathrm{T}^n(M)$ in $\mathrm{S}(M)$ (resp. $\Lambda(M)$). We have $\mathrm{S}(M) = \bigoplus_{n=0}^{\infty} \mathrm{S}^n(M)$ (resp. $\Lambda(M)) = \bigoplus_{n=0}^{\infty} \Lambda^n(M)$). Any k-linear map $u : M \to N$ produces in an obvious way a k-linear map $\mathrm{T}^n(u) : \mathrm{T}^n(M) \to \mathrm{T}^n(N)$ (resp. $\mathrm{S}^n(u) : \mathrm{S}^n(M) \to \mathrm{S}^n(N)$, resp. $\Lambda^n(u) : \Lambda^n(M) \to \Lambda^n(N)$).

If M is free of rank n, then $\Lambda^n(M)$ is free of rank 1 and the "canonical" map $k \to \mathrm{L}(\Lambda^n(M))$ is therefore an isomorphism. The *determinant* is the composite map

$$\det : \mathrm{L}(M) \xrightarrow{\Lambda^n} \mathrm{L}(\Lambda^n(M)) \simeq k.$$

On the other hand, there exists a general bilinear map

$$\mathrm{L}(M,N) \times P \to \mathrm{L}(M, N \otimes_k P), \quad (f,p) \mapsto (m \mapsto f(m) \otimes p)$$

providing a k-linear map $\mathrm{L}(M,N) \otimes_k P \to \mathrm{L}(M, N \otimes_k P)$. This is an isomorphism when P is free of finite rank. In particular, if M is a free module of finite rank, then there exists an isomorphism $M^\vee \otimes_k M \simeq \mathrm{L}(M)$. Now, there exists an obvious bilinear map

$$M^\vee \times M \to k, \quad (\varphi, m) \mapsto \langle \varphi, m \rangle := \varphi(m)$$

providing the *trace map*

$$\mathrm{tr} : \mathrm{L}(M) \simeq M^\vee \otimes_k M \to k$$

when M is free of finite rank.

If k is a commutative ring and G is a monoid, then the composite map

$$k \cdot G \times k \cdot G \to k \cdot G \otimes_k k \cdot G \simeq k \cdot (G \times G) \to k \cdot G$$

turns the free k-module $k \cdot G$ into a k-algebra. The *polynomial ring* over k on a set E is $k[E] := k \cdot G$ where $G = \mathbb{N} \cdot E$ (free abelian monoid). Alternatively, $k[E] \simeq \mathrm{S}(M)$ with $M = k \cdot E$.

A *lattice* is a set X endowed with two internal operations $\vee$ and $\wedge$ which are associative, commutative and *absorbant* with respect to each other:

$$\forall x, y \in X, \quad x \vee (x \wedge y) = x = x \wedge (x \vee y).$$

It is said to be *distributive* if

$$\forall x, y, z \in X, \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{and} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is said to be *bounded* if both $\vee$ and $\wedge$ are unital (with units denoted by 0 and 1 respectively). Then, a *complement* for an $x \in X$ is an $\neg x \in X$ such that $x \vee \neg x = 1$ and $x \wedge \neg x = 0$. A *morphism* *of (bounded) lattices* is a homomorphism for both operations. The relation $x \leq y \Leftrightarrow x = x \wedge y$ is a partial order on X. Conversely, a partial order such that any pair has a supremum and an infimum defines a lattice by $x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$. A morphism of lattices is an order preserving map that also preserves maxima and minima for pairs.

A *boolean ring* is a ring A such that all $a \in A$ are *idempotent*: $a^2 = a$. If we set $a \wedge b = ab$ and $a \vee b = a + b + ab$ and $\neg a = 1 + a$, then A becomes a bounded distributive lattice with complements. Conversely, if A is a bounded distributive lattice with complements, then it is a boolean ring.

0.5 Topology

A *topology* on a set X is a subset $\mathrm{Open}(X) \subset \mathcal{P}(X)$ of the set of all subsets of X which is stable under any union (and in particular the empty union) and any finite intersection (and in particular empty intersection). A set X endowed with a topology is called a *topological space* and a subset $U \in \mathrm{Open}(X)$ is said to be *open* in X. If there exists an open subset U such that $A \subset U \subset V \subset X$, then V is called a *neighborhood* of A. An $U \subset X$ is open if and only if it is a neighborhood of all its points.

Topologies are stable under intersection inside $\mathcal{P}(X)$. In particular, we may always consider the the topology generated by any set of subsets of X. A topology is *finer* (resp. *coarser*) than another if it contains (resp. is contained in) the other: it has more (resp. less) open subsets. There exists a topology on X which is finer (resp. coarser) than any other: the *discrete* (resp. *coarse*) topology.

The complement of an open subset is said to be *closed*. There always exists a smallest closed (resp. biggest open) subset $\overline{Y}$ (resp. $\mathring{Y}$) containing (resp. contained in) a given subset Y which is called its *closure* (resp. its *interior*). A subset Y is said to be *dense* in X if $\overline{Y} = X$.

Assume X and Y are two topological spaces. If $A \subset X$, then a *limit* for a map (remember that this is the same thing as a family) $f : A \to Y$ at $x \in \overline{A}$ is a $y \in Y$ such that, for all neighborhood V of y in Y, there exists a neighborhood U of x in X such that $f(A \cap U) \subset V$. We then write $y = \lim_x f$ (but we should write $\in$ because the limit may not be unique). A map $f : X \to Y$ is said to be *continuous* at $x \in X$ if $f(x)$ is a limit of f at x.

A map $f : X \to Y$ between two topological spaces is (everywhere) continuous if and only if

$$f^{-1}(\mathrm{Open}(Y)) \subset \mathrm{Open}(X).$$

We shall denote by $\mathcal{C}(X, Y)$ the set of all continuous maps $X \to Y$. On the other hand, a map f is said to be *open* (resp. *closed*) if it sends open (resp. closed) subsets to open (resp. closed) subsets. A *subspace* of a topological space X is a subset Y endowed with the *induced* topology:

$$V \in \mathrm{Open}(Y) \Leftrightarrow \exists U \in \mathrm{Open}(X), V = U \cap Y.$$

If Y is a quotient of a topological space and $p : X \twoheadrightarrow Y$ denotes the quotient map, then the *quotient* topology on Y is defined by

$$V \in \mathrm{Open}(Y) \Leftrightarrow p^{-1}(V) \in \mathrm{Open}(X).$$

Unless otherwise specified, we will always implicitly endow a subset (resp. a quotient) with the induced (resp. quotient) topology. A topological space has *locally* a property P if any neighborhood V of a point x contains a neighborhood V' of x having property P. We shall also encounter the notion of a *local homeomorphism* $f : X' \to X$. It means that there exists for all $x' \in X'$, an open neighborhood U' of x' (resp. U of $x := f(x')$) such that f induces a homeomorphism $U' \simeq U$.

A topological space X is said to be

- *Fréchet* if all points are closed,
- *Hausdorff* if any two distinct points have disjoint neighborhoods,
- *normal* if any two disjoint closed subsets have disjoint neighborhoods.

One also say T_1 for Frechet, T_2 for Hausdorff and T_4 for normal Hausdorff (or equivalently normal Frechet).

A *covering* of a topological space X is simply a set (or a family) E of subsets of X such that $X = \bigcup_{Y \in E} Y$. A *refinement* of E is a subset $E' \subset E$. A topological space X is said to be *compact* (resp. *paracompact*) if any open covering R of X has a finite (resp. locally finite) refinement $R' \subset R$ which is also a covering of X. A paracompact Hausdorff space is normal (and therefore T_4). Actually, if $X = \bigcup_{i \in I} U_i$ is a locally finite covering of a paracompact Hausdorff space, then there exists another covering $X = \bigcup_{j \in J} V_j$ such that $\forall j \in J, \exists i \in I, \overline{V}_j \subset U_i$.

A subset of a topological space X is said to be *clopen* if it is both open and closed. The space X is said to be *connected* if only $\emptyset$ and X are clopen in X. The *connected components* in X are the maximal non-empty connected subspaces. The set of connected components of X is denoted by $\pi_0(X)$. The image of a connected space by a continuous map is always connected. The closure of a connected subset is connected. A connected component is closed (but not necessarily open). Connected components are disjoints. A clopen subset is a union of connected components.

A *semidistance* on a set X is a map $d : X \times X \to \mathbb{R}$ satisfying $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,z) \leq d(x,y) + d(y,z)$ for $x, y, z \in E$. It is a *distance* if moreover $d(x,y) = 0 \Rightarrow x = y$. A set X endowed with a (semi) distance is called a *(semi) metric space.* A subset U of X is said to be *open* if

$$\forall x \in U, \exists \epsilon > 0, \forall y \in X, d(x,y) \leq \epsilon \Rightarrow y \in U.$$

This turns X into a topological space and d is a distance if and only if X is Hausdorff.

A *seminorm* on a real vector space E is a map

$$E \to \mathbb{R}, \quad x \mapsto \|x\|$$

satisfying

$$\|0\| = 0, \quad \|\lambda x\| = |\lambda|\|x\| \quad \text{and} \quad \|x + y\| \leq \|x\| + \|y\|$$

for $x, y \in E$ and $\lambda \in \mathbb{R}$. It is a *norm* if moreover $\|x\| = 0 \Rightarrow x = 0$. Formula $d(x,y) := \|y - x\|$ defines a (semi) distance on E. In the case of finite dimensional vector spaces, all norms are equivalent and all linear maps are continuous. Finally, if $f : A \to B$ is a map between subsets of finite dimensional vector spaces, we write $f(h) = o(h)$ (we should write $\in$ instead of $=$) if there exists $\epsilon : A \to \mathbb{R}$ such that $\|f(h)\| = \|h\|\epsilon(h)$ for $h \in A$ and $\lim_0 \epsilon = 0$.

0.6 Some algebraic topology

A *homotopy* is a continuous map $s : X \times [0,1] \to Y$. One then sets $s_t(x) := s(x,t)$ for $x \in X$ and $t \in [0,1]$, says that s is a homotopy between s_0 and s_1 and writes $s : s_0 \sim s_1$. A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $\mathrm{Id}_X \sim g \circ f$ and $f \circ g \sim \mathrm{Id}_Y$. A topological space said to be *contractible* if it is homotopy equivalent to a point.

For $n \in \mathbb{N}$, the *standard n-simplex*[7] is the set

$$[n] := n + 1 := \{0, \ldots, n\}$$

endowed with its natural order: $0 \leq 1 \leq \cdots \leq n$. For $0 \leq i \leq n$, the (order preserving) map

$$\delta_i^n : [n-1] \to [n] \quad (\text{resp. } \sigma_i^n : [n+1] \to [n])$$

that forgets (resp. repeats) the i-th term is called the ith *face* (resp. *degeneracy*) map. We have the following relations

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad \text{for } 0 \leq i < j \leq n+1,$$
$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad \text{for } 0 \leq i \leq j \leq n,$$
$$\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & \text{for } 0 \leq i < j \leq n \\ \mathrm{Id}_{[n]} & \text{for } 0 \leq j \leq i \leq j+1 \leq n+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & \text{for } 0 \leq j < i-1 \leq n. \end{cases}$$

We now turn to the topological version. For $n \in \mathbb{N}$, the *standard topological n-simplex* is the subset:

$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}_{\geq 0}^{n+1}, \sum_{i=0}^n t_i = 1 \right\}.$$

For $0 \leq i \leq n$, the map ith *face* (resp. *degeneracy*) map is (same notation as above)

$$\delta_i^n : |\Delta^{n-1}| \to |\Delta^n|, (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1}) \quad (\text{resp.}$$

$$\sigma_i^n : |\Delta^{n+1}| \to |\Delta^n|, (t_0, \ldots, t_{n+1}) \mapsto (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1})).$$

All above formulas still hold. More generally, any order preserving map $u : [m] \to [n]$ yeilds an analogous map (written in the same way) $u : |\Delta^m| \to |\Delta^n|$. This is described as follows. When $X \subset \mathbb{R}^N$ is a (convex) subset, and $v_0, \ldots, v_n \in X$, one writes

$$[v_0, \ldots, v_n] \; : |\Delta^n| \; \to X, (t_0, \ldots, t_n) \mapsto \sum_{i=0}^n t_i v_i.$$

If $(e_0, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^{n+1}$, then this is the unique linear map sending e_i to v_i. Then the map induced by u on topological simplices is $[e_{u(0)}, \ldots, e_{u(n)}]$.

[7] Sometimes also written Δ^n.

A *semisimplicial set*

$$S: \quad \cdots S_2 \rightrightarrows S_1 \rightrightarrows S_0$$

is a family of maps $d_n^i : S_n \to S_{n-1}$ for $n > 0$ satisfying

$$d_n^i \circ d_{n+1}^j \circ = d_n^{j-1} \circ d_{n+1}^i \quad \text{for } 0 \leq i < j \leq n+1.$$

The *augmentation* map is the unique map $\epsilon : S_0 \to \{0\}$. A *simplicial set*

$$S: \quad \cdots S_2 \rightrightarrows S_1 \rightleftarrows S_0$$

is the data of two families of maps $d_n^i : S_n \to S_{n-1}$ for $n > 0$ and $s_n^i : S_n \to S_{n+1}$ for $n \geq 0$ with $0 \leq i \leq n$ satisfying

$$\begin{aligned} &d_n^i \circ d_{n+1}^j = d_n^{j-1} \circ d_{n+1}^i \quad \text{for } 0 \leq i < j \leq n+1, \\ &s_{n+1}^i \circ s_n^j = s_{n+1}^{j+1} \circ s_n^i \quad \text{for } 0 \leq i \leq j \leq n, \\ &d_{n+1}^i \circ s_n^j = \begin{cases} s_{n-1}^{j-1} \circ d_n^i & \text{for } 0 \leq i < j \leq n \\ \mathrm{Id}_{S_n} & \text{for } 0 \leq j \leq i \leq j+1 \leq n+1 \\ s_{n-1}^j \circ d_n^{i-1} & \text{for } 0 \leq j < i-1 \leq n. \end{cases} \end{aligned}$$

If T is another (semi) simplicial set, then a *map of (semi) simplicial sets* $f : S \to T$ is a family of compatible maps $f_n : S_n \to T_n$.

When S is a (semi) simplicial set, we let $\mathrm{C}_n(S) := \mathbb{Z} \cdot S_n$ and, when M is an abelian group, $\mathrm{C}^n(S, M) := \mathcal{F}(S_n, M)$. The map d_n^i extends uniquely to a group homomorphism (still written) $d_n^i : \mathrm{C}_n(S) \to \mathrm{C}_{n-1}(S)$. On the other hand, the map d_{n+1}^i induces by composition a group homomorphism $d_n^i : \mathrm{C}^n(S, M) \to \mathrm{C}^{n+1}(S, M)$. In both cases, we shall be concerned with the alternating sum

$$d_n := \sum_{i=0}^{n-1} (-1)^i d_n^i.$$

Note that the augmentation map provides a homomorphism $e : C_0(S) \to \mathbb{Z}$ (sending S_0 to 1) and a homomorphism $e : M \to C^0(S)$ (sending m to the constant map m).

If X is a topological space, there exists a simplicial set $\mathrm{S}_\bullet(X)$ with

$$\mathrm{S}_n(X) := \mathcal{C}(|\Delta^n|, X), \quad d_i^n(\sigma) := \sigma \circ \delta_n^i, \quad \text{et} \quad s_i^n(\sigma) := \sigma \circ \sigma_n^i$$

and we shall write $C_n(X) := C_n(S_\bullet(X))$ as well as $C^n(X, M) := C^n(S_\bullet(X), M)$.

0.7 Some analytic geometry

Let $U \subset E$ and $V \subset F$ be open subsets of finite dimensional real vector spaces. A map $f : U \to V$ is said to be *differentiable* at $x \in U$ if there exists $f'(x) \in \mathrm{L}(E, F)$ such that

$$f(x+h) - f(x) - f'(x)(h) = o(h).$$

When f is everywhere differentiable, this defines a map $f' : U \to \mathrm{L}(E, F)$. One can then define $f^{(k)}$ and the notion of a *k-differentiable* map by induction on $k \in \mathbb{N}$. We should mention now *Schwarz theorem* that states that $f'' \in L(E, L(E, \mathbb{R})) \simeq B(E, E; \mathbb{R})$ is a symmetric map. The map f is said to be *$\mathcal{C}^k$-differentiable* (resp. *$\mathcal{C}^\infty$-differentiable* or *smooth*, resp. *$\mathcal{C}^{\mathrm{an}}$-differentiable* or *analytic*) if moreover $f^{(k)}$ is continuous (resp. if f is k-differentiable for all $k \in \mathbb{N}$, resp. and if moreover

$$f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)(h, \ldots, h)$$

in a neighborhood of any $x \in U$ – seeing $f^{(k)}(x)$ as a multilinear map). There exists an obvious complex analog but any complex differentiable map is then automatically $\mathcal{C}^{\mathrm{an}}$ and said to be *holomorphic* in which case we may write $\mathcal{C}^{\mathrm{hol}}$. Everything below has an equivalent in this situation.

Let us fix some $k \in \{0, 1, \ldots, \infty, \mathrm{an}\}$.

A *chart* on a set X is a bijection $\varphi : U \xrightarrow{\sim} U'$ where $U \subset X$ and U' is an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. The components $x_1, \ldots, x_n$ of the composite map $U \xrightarrow{\sim} U' \rightarrowtail \mathbb{R}^n$ are called *coordinates* on U or *local coordinates* on X. A *$\mathcal{C}^k$-atlas* on X is a family of charts $\varphi_i : U_i \xrightarrow{\sim} U'_i$ for $i \in I$ such that $X = \bigcup_{i \in I} U_i$ and for all $i, j \in I$, if we let $U_{ij} := U_i \cap U_j$, then $U'_{ij} := \varphi_i(U_{ij})$ is open and the map

$$\varphi_{j|U_{ji}} \circ \varphi^{-1}_{i|U'_{ij}} : U'_{ij} \simeq U'_{ji}$$

is $\mathcal{C}^k$-differentiable. Then, there exists a unique topology on X making each U_i open and φ_i a homeomorphism. Any $\mathcal{C}^k$-atlas is contained in a unique maximal $\mathcal{C}^k$-atlas and a *$\mathcal{C}^k$-differentiable manifold* is a set X endowed with a maximal $\mathcal{C}^k$-atlas. For convenience, one also makes the extra topological assumption that X is Hausdorff and *countable at infinity* (a countable union of compact subsets). A map $f : X \to Y$ between $\mathcal{C}^k$-differentiable manifolds is said to be *$\mathcal{C}^k$-differentiable* if for all charts $\varphi : U \xrightarrow{\sim} U'$ and $\psi : V \xrightarrow{\sim} V'$ of X and Y respectively, if we set $W := U \cap f^{-1}(V)$, then $W' = \varphi(W)$ is open and the map

$$\psi \circ f_{|W} \circ \varphi^{-1}_{|W'} : W' \to V'$$

is $\mathcal{C}^k$-differentiable. It is called a *$\mathcal{C}^k$-diffeomorphism* it is bijective and f^{-1} also is $\mathcal{C}^k$-differentiable. We shall denote by $\mathcal{C}^k(X, Y)$ the set of all $\mathcal{C}^k$-differentiable maps from X to Y and by $\mathcal{O}(X)$ the ring of $\mathcal{C}^k$-differentiable maps $f : X \to \mathbb{R}$. Note that, if $f \in \mathcal{O}(X)$ does not vanish, then $1/f \in \mathcal{O}(X)$. Finally, *Urysohn's lemma* holds when $k \leq \infty$: if $Z_1, Z_2 \subset X$ are two disjoint closed subsets, then there exists a $\mathcal{C}^k$-differentiable map f on X such that $f_{|Z_1} = 1$ and $f_{|Z_2} = 0$.

Again, there exists a complex analog of the above theory leading to the notion of a *complex manifold*. This is however independent of k when $k \neq 0$ in which case we say *holomorphic* and write $\mathcal{C}^{\mathrm{hol}}$ (and Urysohn's lemma then does not hold).

A *vector bundle* on a $\mathcal{C}^k$-differentiable manifold X is a $\mathcal{C}^k$-differentiable manifold E endowed with a $\mathcal{C}^k$-differentiable map $p : E \twoheadrightarrow X$ and the structure of a vector space on each fiber $E_x := p^{-1}(x)$ such that there exists a covering of X by open subsets and for each such open subset U, a $\mathcal{C}^k$-diffeomorphism $E_{|U} := p^{-1}(U) \simeq U \times \mathbb{R}^n$

inducing an isomorphism of vector spaces $E_x \simeq \{x\} \times \mathbb{R}^n$ for all $x \in U$. As a set, E is simply the disjoint union of the vector spaces E_x for $x \in X$ (but they vary smoothly). A first example is given by the *trivial* vector bundle $X \times V$ if V is a finite dimensional vector space. It may be convenient to write $\mathcal{O}_X := X \times \mathbb{R}$.

A *morphism of vector bundles* is a $\mathcal{C}^k$-differentiable map $u : E \to F$ inducing a linear map $u_x : E_x \to F_x$ for all $x \in X$. We shall denote by $\mathrm{Hom}(E, F)$ the set of all morphisms of vector bundles from E to F. One defines a subbundle as well as an isomorphism of vector bundles in the usual way. One can generalize the definitions of $E \oplus F$, $\mathrm{L}(E, F)$ (and in particular $\mathrm{L}(E)$ and $E^\vee$) and $E \otimes F$ from vector spaces to vector bundles so that they specialize to fibers (for example $(E \oplus F)_x = E_x \oplus F_x$). One can also extend the notions of tensor, symmetric and exterior powers and define $\mathrm{T}^n(E)$, $\mathrm{S}^n(E)$ and $\Lambda^n(E)$ when E is a vector bundle. If $f : X \to Y$ is a $\mathcal{C}^k$-differentiable map, then any vector bundle E on Y defines a vector bundle f^*E on X with $(f^*E)_x = E_{f(x)}$ for $x \in X$. A *section* of a vector bundle E on an open subset U of X is a $\mathcal{C}^k$-differentiable map $s : U \to E$ such that the composite $p \circ s$ is merely the inclusion $U \rightarrowtail X$. The set $E(U)$ (or $\Gamma(U, E)$) of all sections of E on U is an $\mathcal{O}(U)$-module (note that $\mathcal{O}_X(U) \simeq \mathcal{O}(U)$ showing that our notations are compatible). Finally, if E, F are two vector bundles on X then there exists a bijection

$$\mathrm{Hom}(E, F) \simeq \Gamma(X, \mathrm{L}(E, F)), \quad u \mapsto (x \mapsto u_x \in \mathrm{L}(E_x, F_x) = \mathrm{L}(E, F)_x).$$

We assume from now on that $k > 0$ (and henceforth say *topological manifold* in the case $k = 0$).

A *curve* on a $\mathcal{C}^k$-manifold X is a $\mathcal{C}^k$-differentiable map $\gamma : I \to X$ defined on an open neighborhood I of $0 \in \mathbb{R}$. Two curves γ_1, γ_2 are said to be *tangent* if there exists a chart $\varphi : U \xrightarrow{\sim} U'$ of X such that[8]

$$\gamma_1(0) = \gamma_2(0) \in U \quad \text{and} \quad (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

This is an equivalence relation on the set of all curves γ on X and the set $\mathrm{T}X$ of equivalence classes $[\gamma]$ is the *tangent bundle*: there exists an obvious map

$$p : \mathrm{T}X \to X, \quad [\gamma] \mapsto \gamma(0)$$

and if $\varphi : U \xrightarrow{\sim} U'$ is a chart of X, a bijection

$$\mathrm{T}X_{|U} \simeq U' \times \mathbb{R}^n, \quad [\gamma] \mapsto ((\phi \circ \gamma)(0), (\varphi \circ \gamma)'(0)).$$

Any morphism $f : X \to Y$ provides a morphism of vector bundles

$$\mathrm{T}f : \mathrm{T}X \to f^*\mathrm{T}Y, \quad [\gamma] \mapsto [f \circ \gamma]$$

(on charts, it corresponds to $(x, h) \mapsto (x, f'(x)(h))$).

We assume from now on that $k = \infty$ or $k = \mathrm{an}$ (or $k = \mathrm{hol}$ in the complex case).

A *differential form* of degree p on X is a section of the vector bundle $\Omega^p_X := \Lambda^p(\mathrm{T}X)^\vee$ and we may sometimes simply write Ω^p. The maps $\Omega^p_X \times \Omega^q_X \to \Omega^{p+q}_X$ coming from the ring structure of exterior powers provide maps

$$\Omega^p(X) \times \Omega^q(X) \to \Omega^{p+q}(X), \quad (\omega, \eta) \mapsto \omega \wedge \eta.$$

[8] After replacing γ_i by its restriction to $\gamma_i^{-1}(U)$.

There exists an $\mathbb{R}$-linear map

$$\mathrm{d} : \mathcal{O}(X) = \mathcal{C}^k(X,\mathbb{R}) \xrightarrow{\mathrm{T}} \mathrm{Hom}(\mathrm{T}X, \mathcal{O}_X) \simeq \Omega^1(X).$$

If $x_1, \ldots, x_n$ are coordinates on $U \subset X$, then $\Omega^1(U)$ is a free $\mathcal{O}(U)$-module with basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ and (by definition)

$$\mathrm{d}f_{|U} =: \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n.$$

The map d extends uniquely to a sequence of maps $\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X)$ for all $p = 1, \ldots, n-1$ through *Liebnitz's rule*

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge \mathrm{d}\eta \tag{1}$$

for $\omega \in \Omega^p(X)$ and $\eta \in \Omega^q(X)$. The sequence of maps

$$\mathcal{O}(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(X)$$

is called[9] the de Rham complex of X.

0.8 Some algebraic geometry

All rings are assumed to be *commutative*.

If $\mathfrak{a}$ is an ideal in a ring A, then its *radical* is

$$\sqrt{\mathfrak{a}} = \{f \in A, \exists n \in \mathbb{N}, f^n \in \mathfrak{a}\}.$$

An $f \in \sqrt{(0)}$ is then said to be *nilpotent*. The ring A is said to be *reduced* if 0 is the only nilpotent element. It is called a *domain* (resp. a *field*) if $A \setminus \{0\}$ is a multiplicative sub-monoid (resp. group). If $\mathfrak{a}$ is an ideal in a ring A and M is an A-module, then then $A/\mathfrak{a}$ becomes a ring and $M/\mathfrak{a}M$ an $A/\mathfrak{a}$ module. The ideal $\mathfrak{a}$ is said to be *radical* (resp. *prime*, resp. *maximal*) if $A/\mathfrak{a}$ is reduced (resp. a domain, resp. a field). One can check that the radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$. The ring A is called a *local ring* if there exists a unique maximal ideal $\mathfrak{m}$ in which case $k := A/\mathfrak{m}$ is called the *residue field* of A.

If $S \subset A$ is a multiplicative submonoid and M is an A-module, then the *localization of M at S* is

$$S^{-1}M = (M \times S)/\sim \quad \text{with} \quad (a,s) \sim (b,t) \Leftrightarrow \exists u \in S, u(ta - sb) = 0.$$

The set $S^{-1}A$ is actually a ring and $S^{-1}M$ is an $S^{-1}A$-module. If S is generated by G, then there exists isomorphisms

$$S^{-1}A \simeq A[\{T_s\}_{s\in G}]/(sT_s - 1)_{s\in G} \quad \text{and} \quad S^{-1}M \simeq S^{-1}A \otimes_A M.$$

As a special case, if $f \in A$ (resp. $\mathfrak{p} \subset A$ is a prime ideal), then the *localization of M at f* (resp. $\mathfrak{p}$) is M_f (resp. $M_\mathfrak{p}$) $:= S^{-1}M$ where S is the submonoid generated by

[9] Be careful however that de Rham cohomology is the cohomology of this complex only in the smooth case.

f (resp. $S = A \setminus \mathfrak{p}$). If A is a domain, then $\mathrm{Frac}(A) := A_{(0)}$ is called the *fraction field* of A. In general, when $\mathfrak{p}$ is a prime ideal of A, $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

The *(prime) spectrum* of A is the set $X := \mathrm{Spec}(A)$ of prime ideals of A. Most results below also hold when $X = \mathrm{Spm}(A)$ is the set of maximal ideals (the *maximal spectrum*). If $\mathfrak{p} \in X$, we let $\kappa(\mathfrak{p}) := \mathrm{Frac}(A/\mathfrak{p})$. If $f \in A$, we shall denote by $f(\mathfrak{p})$ the image of f in $\kappa(\mathfrak{p})$:

$$\begin{array}{ccccc} A & \twoheadrightarrow & A/\mathfrak{p} & \rightarrowtail & \kappa(\mathfrak{p}) \\ f & \longmapsto & f \mod \mathfrak{p} & \longmapsto & f(\mathfrak{p}) \end{array}$$

so that $f(\mathfrak{p}) = 0 \Leftrightarrow f \in \mathfrak{p}$. If $S \subset A$, we may consider the *zero locus*

$$\mathrm{V}(S) := \{\mathfrak{p} \in X \ / \ \forall f \in S, f(\mathfrak{p}) = 0\}$$

of S, so that $\mathfrak{p} \in \mathrm{V}(S) \Leftrightarrow S \subset \mathfrak{p}$. We always have

$$\mathrm{V}(\emptyset) = X, \quad \mathrm{V}(A) = \emptyset, \quad \mathrm{V}(ST) = \mathrm{V}(S) \cup \mathrm{V}(T) \quad \text{and} \quad \mathrm{V}\left(\bigcup_{i\in I} S_i\right) = \bigcap_{i \in I} \mathrm{V}(S_i).$$

We define the *Zariski topology* on X by requiring a set to be closed if and only if it is the zero locus of some $S \subset A$.

On the other hand, if $E \subset X$, then we set

$$\mathrm{I}(E) := \{f \in A \ / \ \forall \mathfrak{p} \in E, f(\mathfrak{p}) = 0\}.$$

The maps $S \mapsto \mathrm{V}(S)$ et $E \mapsto \mathrm{I}(E)$ induce inverse bijections between radical ideals of A and closed subsets of X. More precisely,

$$\forall S \subset A, \quad \mathrm{I}(\mathrm{V}(S)) = \sqrt{(S)} \quad \text{and} \quad \forall E \subset X, \quad \mathrm{V}(\mathrm{I}(E)) = \overline{E}.$$

A basis for the Zariski topology of X is given by the *special domains*

$$\mathrm{D}(f) := \{\mathfrak{p} \in X \ / \ f(\mathfrak{p}) \neq 0\}$$

for $f \in A$ and we have $\mathrm{D}(f) \cap \mathrm{D}(g) = \mathrm{D}(fg)$. One easily checks that X is compact (but not Hausdorff in general).

If $\varphi : A \to B$ is a ring homomorphism and $Y := \mathrm{Spec}(B)$, then there exists[10] a continuous map $u := \varphi^{-1} : Y \to X$. If $\mathfrak{a}$ is an ideal of A (resp. $f \in A$), then the "canonical" map $A \to A/\mathfrak{a}$ (resp. $A \to A[1/f]$) provides a homeomorphism

$$\mathrm{Spec}(A/\mathfrak{a}) \simeq \mathrm{V}(\mathfrak{a}) \quad (\text{resp. } \mathrm{Spec}(A[1/f]) \simeq \mathrm{D}(f)).$$

If $R \to A$ is a morphism of commutative rings and $I := \ker(A \otimes_R A \to A)$, then $\Omega_{A/R} := I/I^2$ is the *module of relative differential forms* of A/R. The R-linear map $A \to A \otimes A, f \mapsto 1 \otimes f - f \otimes 1$ induces an R-linear map $d : A \to \Omega_{A/R}$. If we set $\Omega^p_{A/R} := \Lambda^p \Omega_{A/R}$, then d extends through formula (1) to give the *de Rham complex* $\Omega^\bullet_{A/R}$ of A/R.

[10] This is not true in general for maximal ideals.

1. Categories and functors

For those who might worry about set-theoretic issues (see [Shu08] for example), we shall stay in a fixed *universe* (some large set, see definition 1.1.1 of [KS06] for example). We shall then call *set* only those sets that belong ($\in$) to our universe[1] and rename *collection* (or call it *large*) a set that is only contained ($\subset$) in our universe.

1.1 Category

1.1.1 Definition/Examples

Definition 1.1.1 A *category*[a] consists in the following data:

1. a collection $\mathcal{C}$ of *objects*,
2. for all $X, Y \in \mathcal{C}$, a set $\mathrm{Hom}(X, Y)$ of *morphisms*,
3. for all $X \in \mathcal{C}$, an *identity* morphism $\mathrm{Id}_X \in \mathrm{End}(X) := \mathrm{Hom}(X, X)$,
4. for all $X, Y, Z \in \mathcal{C}$, a composition rule

$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), \quad (g, f) \mapsto g \circ f$$

such that

(a) $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$,
(b) if $h \in \mathrm{Hom}(Z, T)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

[a] Some authors allow collections (and not merely sets) of morphisms in the definition of a category and call *locally small category* what we named a category.

The category is said to be *small* if all its objects (or equivalently all its morphisms) form a set and not merely a collection and *finite* if there exists only a finite number of morphisms (and consequently of objects). It is called *discrete* if the only morphisms are the identities. It is *empty* if there is no objects at all (and no morphisms).

[1] They are usually called *small sets* but we do not want to keep this epithet everywere.

We will usually write $f : X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$ and call X (resp. Y) the *domain* (resp. the *codomain*) of f. Note that Id_X is uniquely determined by condition (4a). A morphism from X to itself is called an *endomorphism*. In practice, we shall simply say that the collection $\mathcal{C}$ *is* a category[2] but we must not forget that it involves some extra structure: morphisms and composition. One may write $\mathrm{Ob}(\mathcal{C})$ (resp. $\mathrm{Mor}(\mathcal{C})$) for the collection of all objects (resp. all morphisms) of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (with a subscript $\mathcal{C}$) for the set of morphisms from X to Y.

Examples

1. If G is a monoid, one can then consider the category $\mathbf{G}$ with a unique object $\bullet$, $\mathrm{End}(\bullet) := G$ and composition given by multiplication on G ($g \circ f = gf$). This way, we get essentially all categories with a unique object.
2. If $\leq$ is a preorder on a set X, we will then denote by $\mathbf{X}$ the category whose objects are the elements $x \in X$ and morphisms are pairs (x, y) for $x \leq y$ (composition is then uniquely defined). This way, we get essentially all the small categories whose Hom have at most one element.
3. As a particular case, one will always endow an unordered set with the minimal (also called trivial) preorder "=" and consider any set as a category (small discrete category). This way, we get all small discrete categories.
4. We shall consider the category $\mathbf{n}$ associated to the (trivial) set $n := \{0, 1, \ldots, n-1\}$ (with only identities as morphisms). But we shall also consider the category $[\mathbf{n}]$ associated to the set $[n] := \{0, \ldots, n\}$ with respect to usual order $0 \leq 1 \leq \ldots \leq n$. For example, $[\mathbf{0}] = \mathbf{1}$ is the category that has exactly one object and one morphism, $[\mathbf{1}]$ is a category with two distinct objects and a unique morphism between them (plus the identities).
5. If X is a topological space, then $\mathrm{Open}(X)$ is ordered by inclusion and we shall denote by $\mathbf{O}\mathrm{pen}(X)$ the corresponding category.
6. There exists a (small) category $\mathbf{\Delta}$ with positive integers $[n] := n+1 = \{0, \ldots, n\}$ as objects and order preserving maps as morphisms.
7. We shall denote by $\mathbf{S}\mathrm{et}$ the (large) category whose objects are sets and morphisms are maps between them. We shall write $\mathcal{F}(X, Y)$ for the set of all maps between two sets.
8. In the same way, we shall consider the (large) category $\mathbf{T}\mathrm{op}$ whose objects are topological spaces and morphisms are continuous maps. We shall write $\mathcal{C}(X, Y)$ for the set of continuous maps between two topological spaces.
9. Finally, we will denote by $\mathbf{A}\mathrm{b}$ the (large) category whose objects are abelian groups and morphisms are homomorphisms.

Exercise 1.1.2 Define the categories $\mathbf{O}\mathrm{rd}$, $\mathbf{M}\mathrm{on}$, $\mathbf{G}\mathrm{rp}$, $\mathbf{R}\mathrm{ing}$, $\mathbf{CR}\mathrm{ing}$, G-$\mathbf{S}\mathrm{et}$ (resp. $\mathbf{S}\mathrm{et}$-G), A-$\mathbf{M}\mathrm{od}$ (resp. $\mathbf{M}\mathrm{od}$-A) and k-$\mathbf{A}\mathrm{lg}$ of preordered sets, monoids, groups, rings, commutative rings, left (resp. right) G-sets, left (resp. right) A-modules and k-algebras.

The *opposite category* to a category $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$ but $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ (and composition going in the reverse direction). We have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. The *product* $\mathcal{C} \times \mathcal{C}'$ of two categories $\mathcal{C}$ and $\mathcal{C}'$ is made into a

[2]As one usually denotes a group by G without explicitly mentioning the multiplication rule.

category in the obvious way: everything is done termwise. This extends to arbitrary products $\prod_{i\in I}\mathcal{C}_i$ (families of objects and families of morphisms). If $\mathcal{C}$ is a category, then the category $\mathbf{Mor}(\mathcal{C})$ of *morphisms of* $\mathcal{C}$ is defined as follows: an object is a morphism of $\mathcal{C}$ and a morphism between $f : X \to Y$ and $g : X' \to Y'$ is a pair of morphisms in $\mathcal{C}$, $\varphi : X \to X'$ and $\psi : Y \to Y'$, such that $g\circ\varphi = \psi\circ f$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

Also, if $X \in \mathcal{C}$, then the category ${}_{X\backslash}\mathcal{C}$ of *objects of* $\mathcal{C}$ *under* X is defined as follows: an object of ${}_{X\backslash}\mathcal{C}$ is a morphism $f : X \to Y$ and a morphism from $f : X \to Y$ to $g : X \to Z$ is a morphism $h : Y \to Z$ such that $h \circ f = g$:

$$\begin{array}{ccc} & & Y \\ & \nearrow^{f} & \\ X & & \downarrow{\scriptstyle h} \\ & \searrow_{g} & \\ & & Z. \end{array}$$

Also, $\mathcal{C}_{/X} := ({}_{X\backslash}\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is the category of *objects of* $\mathcal{C}$ *over* X. This is our first example of *dual* construction.

Exercise 1.1.3 Show that, if G is a monoid, then the set of objects of $\mathbf{Mor}(\mathbf{G})$ is G and that, if X is a preordered set, then the set of objects of $\mathbf{Mor}(\mathbf{X})$ is the graph Γ of the relation.

Exercise 1.1.4 Make $\mathcal{C}_{/X}$ explicit when X is an object of a category $\mathcal{C}$.

1.1.2 Isomorphism

Definition 1.1.5 In a category $\mathcal{C}$,

1. a *section* (resp. a *retraction*) of a morphism $f : X \to Y$ is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ (resp. $\mathrm{Id}_X = g \circ f$):

$$X \underset{f}{\overset{g}{\rightleftarrows}} Y \circlearrowleft \qquad (\text{resp.} \circlearrowright X \underset{f}{\overset{g}{\rightleftarrows}} Y\ ).$$

2. an *isomorphism* is a morphism that has at the same time a section and a retraction. When there exists an isomorphism $X \xrightarrow{\sim} Y$, one says that X and Y are *isomorphic* and writes $X \simeq Y$. An isomorphism between X and itself is called an *automorphism*.

One denotes by $\mathrm{Isom}(X, Y)$ the set of all isomorphisms $X \xrightarrow{\sim} Y$ and by $\mathrm{Aut}(X)$ the group of all automorphisms of X.

Sections and retractions may also be called *right* and *left inverses* respectively. Clearly, f is a section of g if and only if g is a retraction of f. Also retraction (resp.

section) in $\mathcal{C}$ is the same thing as a section (resp. retraction) in $\mathcal{C}^{\mathrm{op}}$ (they are *dual* notions).

Examples
1. A map $f : X \to Y$ (between sets) has a section (resp. retraction) if and only if it is surjective (resp. injective unless $X = \emptyset$).
2. A *retract* A of a topological space X is a subspace such that the inclusion map $A \rightarrowtail X$ has a retraction.
3. A *direct summand* M' of a module M is a submodule such that the inclusion map $M' \rightarrowtail M$ has a retraction.

Proposition 1.1.6 If f is an isomorphism, then it has a unique section and a unique retraction and they are the same.

Proof. If $\mathrm{Id}_Y = f \circ g$ and $h \circ f = \mathrm{Id}_X$, then

$$h = h \circ \mathrm{Id}_Y = h \circ f \circ g = \mathrm{Id}_X \circ g = g.$$ ■

The unique section/retraction of an isomorphism f is called its *inverse* and denoted by f^{-1}.

Exercise 1.1.7 What is an isomorphism in $\mathbf{Set}$, in $\mathbf{Top}$, in $\mathbf{Ab}$, etc. ? In $\mathbf{G}$ if G is a monoid ? In $\mathbf{X}$ if X is a preordered set ?

1.1.3 Subcategory

Definition 1.1.8 A *subcategory* of a category $\mathcal{C}$ is the data of
1. a subcollection $\mathcal{C}' \subset \mathcal{C}$,
2. for all $X, Y \in \mathcal{C}'$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that
   (a) if $X \in \mathcal{C}'$, then $\mathrm{Id}_X \in \mathrm{End}_{\mathcal{C}'}(X) := \mathrm{Hom}_{\mathcal{C}'}(X, X)$,
   (b) if $X, Y, Z \in \mathcal{C}'$, $f \in \mathrm{Hom}_{\mathcal{C}'}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}'}(Y, Z)$, then $g \circ f \in \mathrm{Hom}_{\mathcal{C}'}(X, Z)$.

It is a *full* subcategory if actually

$$\forall X, Y \in \mathcal{C}', \quad \mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

A subcategory becomes a category with the induced composition. A full subcategory is uniquely determined by its objects. The subcategory is said to be *wide* if $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C})$.

Examples
1. $\mathbf{Ab}$ is a full subcategory of $\mathbf{Grp}$ which itself is a full subcategory of $\mathbf{Mon}$ (which itself is a *non-full* subcategory of the category of semigroups or magmas for example).
2. If X is a topological space, then $\mathbf{Open}(X)$ is a subcategory of $\mathbf{Set}$ which is *not* full.
3. $\mathbf{Top}$ is *not* a subcategory of $\mathbf{Set}$ and neither is $\mathbf{Ab}$ (but see the notion of a faithful functor below).
4. If X is any topological space, then an *espace étalé* over X is a local homeomorphism $X' \to X$. They form a full subcategory $\mathbf{Et}(X) \subset \mathbf{Top}_{/X}$.
5. $\mathbf{\Delta}$ is a full subcategory of $\mathbf{Ord}$.

6. We may consider the wide subcategory $\mathbf{\Delta}_{\text{inj}}$ of $\mathbf{\Delta}$ with the same objects but only order preserving *injective* maps as morphism.
7. Conversely, we may consider $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}_{\text{inj}}$) as a full subcategory of $\mathbf{\Delta}^+$ (resp. $\mathbf{\Delta}^+_{\text{inj}}$) which is defined by adding $[-1] = \emptyset$.

If $\mathcal{C}$ is a category, we can consider the relation "$\to$" on the collection $\mathcal{C}$ given by

$$X \to Y \Leftrightarrow \text{Hom}(X,Y) \neq \emptyset$$

as well as the quotient

$$\pi_0(\mathcal{C}) := \mathcal{C}/\to .$$

Definition 1.1.9 A *connected component* of $\mathcal{C}$ is an element of $\pi_0(\mathcal{C})$. The category $\mathcal{C}$ is said to be *connected* if there exists a unique connected component.

The relation "$\to$" is not an equivalence relation: actually, X and Y will be connected to each other (in the same connected component) when there exists a zigzag of morphisms

$$X = X_0 \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_{n-1} \to X_n = Y.$$

The category $\mathcal{C}$ is the disjoint union of its connected components that will all be viewed as full subcategories of $\mathcal{C}$.

Examples
1. The categories **S**et, **A**b, **T**op, etc. are all connected.
2. If a set X is endowed with a preorder relation R, then $\pi_0(\mathbf{X}) = X/R$ and a connected component is the same thing as an equivalence class.

1.2 Functor

1.2.1 Definition/Examples

Definition 1.2.1
1. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{C}'$ between two categories is the data, for all $X \in \mathcal{C}$, of $F(X) \in \mathcal{C}'$ and, for all $f : X \to Y$, of $F(f) : F(X) \to F(Y)$, in such a way that we always have $F(\text{Id}_X) = \text{Id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.
2. If $G : \mathcal{C}' \to \mathcal{C}''$ is another functor, then their *composite* is the functor $G \circ F$ given by $(G \circ F)(X) = G(F(X))$ and $(G \circ F)(f) = G(F(f))$.

We will denote by $\text{Hom}(\mathcal{C},\mathcal{C}')$ the *collection* of all functors $\mathcal{C} \to \mathcal{C}'$. We will often describe the functors by their action on the objects and let the reader guess what happens for morphisms.

There always exists an identity functor $\text{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ that doesn't change anything. Also, a category $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if and only if all objects (resp. morphisms) of $\mathcal{C}'$ are objects (resp. morphisms) of $\mathcal{C}$ and inclusion is functorial. A functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *isomorphism* it there exists a functor G such that $\text{Id}_\mathcal{C} = G \circ F$ and $F \circ G = \text{Id}_{\mathcal{C}'}$ (but this is not a very interesting notion). A functor $F : \mathcal{C}^{\text{op}} \to \mathcal{C}'$ is also called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{C}'$. Any functor $F : \mathcal{C} \to \mathcal{C}'$ provides a functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{C}'^{\text{op}}$ and this construction is "functorial" : $\text{Id}_{\mathcal{C}^{\text{op}}} = \text{Id}_\mathcal{C}^{\text{op}}$ and $(G \circ F)^{\text{op}} = G^{\text{op}} \circ F^{\text{op}}$.

Examples
1. There exists a functor that forgets topology (and continuity) **T**op $\to$ **S**et (underlying set). In the other direction, there exits a functor $X \mapsto X^{\text{disc}}$ (resp. $X \mapsto X^{\text{coarse}}$) that endow a set X with the discrete (resp. coarse) topology.

2. There also exists a (contravariant) functor

$$\mathbf{Top}^{\mathrm{op}} \to \mathbf{Ord}, \quad X \mapsto \mathrm{Open}(X), \quad f \mapsto f^{-1}.$$

3. There exists a functor that forgets the algebraic structure $\mathbf{Ab} \to \mathbf{Set}$ (underlying set) and, in the other direction, a functor $X \mapsto \mathbb{Z} \cdot X$ (or $\mathbb{Z}^{(X)}$) which sends a set to the free *free abelian group* generated by X.
4. There exists an inclusion functor $\mathbf{Grp} \rightarrowtail \mathbf{Mon}$ and two functors $G \mapsto G^{\times}$ and

$$G \mapsto G^{\mathrm{gr}} := \langle x_g, g \in G \;/\; x_{gh} = x_g x_h, g, h \in G, x_1 = 1\rangle$$

in the reverse direction.
5. Small categories and functors between them form a category that we shall denote $\mathbf{Cat}$ and there exists functors

$$\mathbf{Mon} \to \mathbf{Cat}, \quad G \mapsto \mathbf{G} \quad \text{and} \quad \mathbf{Ord} \to \mathbf{Cat}, \quad X \mapsto \mathbf{X}.$$

6. A (contravariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is also called a *presheaf* on $\mathcal{C}$ with values in $\mathcal{D}$. If X is a topological space, then a presheaf F on $\mathbf{Open}(X)$ with values in $\mathbf{Set}$ (also called a *presheaf of sets on* X) is the following data:
   (a) a set $F(U)$ of *sections* on any open subset U of X, and
   (b) *restriction maps* $F(U) \to F(V), s \mapsto s_{|V}$ for $V \subset U$ satisfying $(s_{|V})_{|W} = s_{|W}$ when $W \subset V$ and $s_{|U} = s$.

Exercise 1.2.2 What is a functor $\mathbf{G} \to \mathbf{H}$ between categories associated to monoids ? What is a functor $\mathbf{X} \to \mathbf{Y}$ between categories associated to preordered sets ?

Exercise 1.2.3 What are the analogs of the "free abelian group" functor $X \mapsto \mathbb{Z} \cdot X$ for the categories $\mathbf{Mon}$, $\mathbf{Grp}$, A-$\mathbf{Mod}$ and k-$\mathbf{Alg}$?

Hint. We only describe the first one. The free monoid generated by a set X is $G := \{g : n \to X, n \in \mathbb{N}\}$ endowed with

$$gh : n + m \to X, \quad i \mapsto \begin{cases} g(i) & \text{if } i < n \\ h(i-n) & \text{otherwise.} \end{cases}$$

In other words, X represents the alphabet, G denotes the set of words formed using this alphabet, and the operation is concatenation. ■

Exercise 1.2.4 Show that, besides the inclusion functor $\mathbf{Ab} \rightarrowtail \mathbf{Grp}$, there exists an *abelianization* functor $G \mapsto G^{\mathrm{ab}} = G/[G, G]$ in the other direction. Show that the center $\mathbb{Z}$ is *not* functorial in the sense that a group homomorphism $\varphi : G \to H$ does not necessarily induce a morphism of abelian groups $\mathrm{Z}(G) \to \mathrm{Z}(H)$.

Exercise 1.2.5 Show that the categories $\mathbb{Z}$-$\mathbf{Mod}$ and $\mathbf{Ab}$ are isomorphic. Same thing with the categories $\mathbb{Z}$-$\mathbf{Alg}$ and $\mathbf{Ring}$, and, more generally, k-$\mathbf{Alg}$ and a full subcategory of $_{k\backslash}\mathbf{Ring}$ (the image of k must be in the center).

Exercise 1.2.6 Show that the image of a section (resp. a retraction, resp. an inverse) by a functor is a section (resp. a retraction, resp. an inverse).

If we are given two categories $\mathcal{C}$ and $\mathcal{C}'$, then the projections $\mathcal{C} \times \mathcal{C}'$ on $\mathcal{C}$ and $\mathcal{C}'$ are functorial. The same holds for the obvious partial functors $\mathcal{C}' \rightarrowtail \mathcal{C} \times \mathcal{C}'$ or $\mathcal{C} \rightarrowtail \mathcal{C} \times \mathcal{C}'$ associated to a fixed object $X \in \mathcal{C}$ or $X' \in \mathcal{C}'$. If $\mathcal{C}$ is any category, then there exists domain and codomain functors $\mathbf{Mor}(\mathcal{C}) \rightrightarrows \mathcal{C}$ as well as forgetful functors ${}_{X\backslash}\mathcal{C} \to \mathcal{C}$ and $\mathcal{C}_{/X} \to \mathcal{C}$.

Exercise 1.2.7 Let us denote by $\mathbf{Op}(\mathcal{C}) \subset \mathbf{Mor}(\mathcal{C})$ the subcategory whose objects are morphisms with codomain identical to the domain and morphisms have same component on domain and codomain (objects with operator). Show that, if k is a commutative ring, then $\mathbf{Op}(k\text{-}\mathbf{Mod})$ is isomorphic to $k[t]\text{-}\mathbf{Mod}$.

Solution. If E is a $k[t]$-module, then it is a k-module by restriction of scalars and the map $u : E \to E, x \mapsto tx$ is k-linear. Conversely, if E is a k-module and $u \in L(E)$, then there exists a unique morphism of k-algebras $k[t] \to L(E), t \mapsto u$. This defines the structure of a $k[t]$-module on E. Concretely, $fx := f(u)(x)$ for $f \in k[t]$ and $x \in E$. Thus, we have an explicit bijection on objects and we now turn to morphisms. First of all, functoriality is automatic. Now, if we are given a morphism of $k[t]$-modules $\varphi : E \to F$ and we denote by $u \in L(E)$ and $v \in L(F)$ the corresponding endomorphisms, we then have $(\varphi \circ u)(x) = \varphi(tx) = t\varphi(x) = (v \circ \varphi)(x)$ for $x \in E$ so that $\varphi \circ u = v \circ \varphi$. It remains to show that any k-linear map $\varphi : E \to F$ satisfying $\varphi \circ u = v \circ \varphi$ is a morphism of $k[t]$-modules. We give ourselves $f \in k[t]$ and $x \in E$ and we need to prove that $\varphi(fx) = f\varphi(x)$, or equivalently $\varphi(f(u)(x)) = f(v)(\varphi(x))$. In other words, we have to show that $\varphi \circ f(u) = f(v) \circ \varphi$. By linearity, we may assume that $f = t^n$ for some $n \in \mathbb{N}$ and the assertion reduces to $\varphi \circ u^n = v^n \circ \varphi$ which is easily obtained by induction. ■

If $\mathcal{C}$ is any category, then there exists a very important functor[3]

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}, \quad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

that sends a pair (f, g) of morphisms to the map $\varphi \mapsto g \circ \varphi \circ f$. If we compose with partial functors, we get the (fundamental) functors

$$h^X_{\mathcal{C}} : \mathcal{C} \to \mathbf{Set}, \quad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

and

$$h^{\mathcal{C}}_Y := h^Y_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

When there is no ambiguity, we shall drop the reference to $\mathcal{C}$ and simply write h^X and h_Y. We may also write $g^*(\varphi) := h^g(\varphi) = \varphi \circ g$ and $f_*(\varphi) = h_f(\varphi) = f \circ \varphi$.

[3]This is why we only consider *locally small categories* in this course.

1.2.2 Natural transformation

Definition 1.2.8 1. If $F, G : \mathcal{C} \to \mathcal{C}'$ are two functors, then a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of morphisms $\alpha_X : F(X) \to G(X)$ for all $X \in \mathcal{C}$ such that, for all $f : X \to Y$, we have $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

We shall say *natural isomorphism* and write $F \simeq G$ if all α_X are isomorphisms.

2. If $\beta : G \Rightarrow H$ is another natural transformation, then their *composition* $\beta \circ \alpha : F \Rightarrow H$ is the natural transformation defined by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for $X \in \mathcal{C}$.

We shall denote by $\mathrm{Hom}(F, G)$ the collection of all natural transformations from F to G. If $\mathcal{C}$ is a small category, then the functors $F : \mathcal{C} \to \mathcal{C}'$ make a (large) category $\mathbf{Hom}(\mathcal{C}, \mathcal{C}')$ with natural transformations as morphisms. One can check that the isomorphisms are exactly the natural isomorphisms considered above.

Examples 1. With

$$\begin{array}{rcl} F : \mathbf{CRing} & \longrightarrow & \mathbf{Grp} \\ k & \longmapsto & \mathrm{GL}_n(k) \end{array} \quad \text{and} \quad \begin{array}{rcl} G : \mathbf{CRing} & \longrightarrow & \mathbf{Grp} \\ k & \longmapsto & k^\times, \end{array}$$

there exists a natural transformation $\alpha : F \to G$ by considering

$$\alpha_k = \det_k : \mathrm{GL}_n(k) \to k^\times.$$

This is a natural isomorphism for $n = 1$.

2. If $\mathcal{C}$ is any category, then there exists isomorphisms of categories

$$\mathbf{Hom}(\mathbf{0}, \mathcal{C}) \simeq \mathbf{1}, \quad \mathbf{Hom}(\mathbf{1}, \mathcal{C}) \simeq \mathcal{C},$$

$$\mathbf{Hom}(\mathbf{2}, \mathcal{C}) \simeq \mathcal{C} \times \mathcal{C} \text{ and } \mathbf{Hom}([1], \mathcal{C}) \simeq \mathbf{Mor}(\mathcal{C}).$$

3. If G is a monoid, then there exists an isomorphism $\mathbf{Hom}(\mathbf{G}, \mathbf{Set}) \simeq G\text{-}\mathbf{Set}$.
4. Presheaves on a small category $\mathcal{C}$ with values in a category $\mathcal{D}$ form a category $\widehat{\mathcal{C}}(\mathcal{D}) := \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ simply denoted $\widehat{\mathcal{C}}$ in the case $\mathcal{D} = \mathbf{Set}$. In the case of presheaves of sets on a topological space X, a morphism $\alpha : F \to G$ is given by a family of maps $\alpha_U : F(U) \to G(U)$ for U open in X satisfying $\alpha_V(s_{|V}) = (\alpha_U(s))_{|V}$ whenever $V \subset U$ and $s \in \mathcal{F}(U)$.

Exercise 1.2.9 Show that

1. if $\mathcal{C}$ is a small category, then there exists an isomorphism of categories $\mathbf{Hom}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$,
2. if $\mathcal{C}$ and $\mathcal{C}'$ are two small categories, then there exists an isomorphism of

categories

$$\mathbf{Hom}(\mathcal{C} \times \mathcal{C}', \mathcal{D}) \simeq \mathbf{Hom}(\mathcal{C}, \mathbf{Hom}(\mathcal{C}', \mathcal{D})).$$

By composition, if $\mathcal{C}$ is a small category, then any functor $G : \mathcal{D} \to \mathcal{D}'$ induces a functor

$$\mathbf{Hom}(\mathcal{C}, \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D}'), \quad F \mapsto G \circ F.$$

Also, if $\mathcal{D}$ is any category, then any functor $G : \mathcal{C} \to \mathcal{C}'$ beween small categories induces a functor

$$\mathbf{Hom}(\mathcal{C}', \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D}), \quad F \mapsto F \circ G.$$

Exercise 1.2.10 Show that there exists a functor

$$\mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}, \quad (\mathcal{C}, \mathcal{D}) \mapsto \mathbf{Hom}(\mathcal{C}, \mathcal{D}).$$

Definition 1.2.11 Assume given two functors $S : \mathcal{S} \to \mathcal{C}$ and $T : \mathcal{T} \to \mathcal{C}$ with same target $\mathcal{C}$. An object in the *comma category* $(S \downarrow T)$ is a triple $(X \in \mathcal{S}, Y \in \mathcal{T}, f : S(X) \to T(Y))$. A morphism $(X, Y, f) \to (X', Y', f')$ is a pair of morphisms $u : X \to X'$, $v : Y \to Y'$ such that $F(v) \circ f = f \circ F(u)$.

Examples
1. In case $\mathcal{C} = \mathbf{1}$, we have $(S \downarrow T) \simeq \mathcal{S} \times \mathcal{T}$.
2. We have $(S \downarrow T)^{\mathrm{op}} \simeq (T^{\mathrm{op}} \downarrow S^{\mathrm{op}})$.
3. $(\mathrm{Id}_{\mathcal{C}} \downarrow \mathrm{Id}_{\mathcal{C}}) \simeq \mathbf{Mor}(\mathcal{C}) \simeq \mathbf{Hom}([\mathbf{1}], \mathcal{C})$.
4. $(\mathrm{Id}_{\mathcal{C}} \downarrow 1 \overset{X}{\to} \mathcal{C}) \simeq \mathcal{C}_{/X}$ and $(1 \overset{X}{\to} \mathcal{C} \downarrow \mathrm{Id}_{\mathcal{C}}) \simeq {}_{X\backslash}\mathcal{C}$.
5. One may aslo write $\mathcal{C}_{/T} := (\mathrm{Id}_{\mathcal{C}} \downarrow T)$ and ${}_{S\backslash}\mathcal{C} := (S \downarrow \mathrm{Id}_{\mathcal{C}})$.
6. As well as $S_{/X} := (S \downarrow 1 \overset{X}{\to} \mathcal{C})$ and ${}_{X\backslash}T := (1 \overset{X}{\to} \mathcal{C} \downarrow T)$.

Recall that the notion of a simplicial set was introduced in section 0.6.

Exercise 1.2.12 Show that the category $\mathbf{SSet}$ of simplicial sets is isomorphic to the category $\widehat{\mathbf{\Delta}}$ of presheaves of sets on $\mathbf{\Delta}$. Analog with semi-simplicial sets and $\widehat{\mathbf{\Delta}}_{\mathrm{inj}}$.

Hint. This is based on the following observations:
1. Any morphism in $\mathbf{\Delta}$ splits as a surjective morphism followed by an injective morphism,
2. any injective (resp. surjective) morphism $[n] \hookrightarrow [m+1]$ with $n \leq m$ factors through an injective (resp. surjective) morphism $[m] \to [m+1]$ ($[n] \to [n-1]$),
3. an injective (resp. surjective) morphism $[m] \to [m+1]$ (resp. $[n] \to [n-1]$) is the same thing as a face (resp. degeneracy) map. ■

It is straightforward to generalize the notion of a simplicial set:

Definition 1.2.13 A *simplicial* object of a category $\mathcal{D}$ is a presheaf

$$X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{D}, \quad [n] \mapsto X_n.$$

Dually, a *cosimplicial* object is a functor $X^\bullet : \mathbf{\Delta} \to \mathcal{D}, [n] \mapsto X^n$.

Exercise 1.2.14 Show that there exists a cosimplicial object $|\Delta^\bullet|$ of $\mathbf{Top}$ sending $[n]$ to $|\Delta^n|$ and $u : [n] \to [m]$ to the unique linear map sending e_i to $e_{u(i)}$ if $(e_0, \ldots, e_n)$ denotes the usual basis of $\mathbb{R}^{n+1}$.

Exercise 1.2.15 Show that the simplicial set $S_\bullet(X)$ associated to a topological space X corresponds to $h_X \circ |\Delta^\bullet|$ under $\mathbf{SSet} \simeq \widehat{\boldsymbol{\Delta}}$ and that this provides a functor $\mathbf{Top} \to \widehat{\boldsymbol{\Delta}}$ sending a topological space X to its associated simplical set $S_\bullet(X)$.

Definition 1.2.16 If $\mathcal{C}$ is a small category, then the *nerve* of $\mathcal{C}$ is the simplicial set

$$\mathcal{C}_\bullet : [n] \mapsto \mathrm{Hom}([\mathbf{n}], \mathcal{C}).$$

This provides a functor

$$\mathbf{Cat} \to \widehat{\boldsymbol{\Delta}}, \quad \mathcal{C} \mapsto \mathcal{C}_\bullet.$$

Concretely, $\mathcal{C}_n$ may be seen as the set of

$$((X_i)_{i=0}^n, (f_i \in \mathrm{Hom}(X_{i-1}, X_i)_{i=1}^n)$$

depicted as $X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} X_n$ and such an object is sent by $u = [m] \to [n]$ to

$$((X_{u(i)})_{i=0}^m, (f_{u(i)} \circ f_{u(i)-1} \circ \cdots \circ f_{u(i-1)} \in \mathrm{Hom}(X_{u(i)-1}, X_{u(i)})_{i=1}^m)$$

(with the convention that the empty composition is the identity). In low degree, we see that $\mathcal{C}_0 = \mathcal{C}$ is the set of objects X, $\mathcal{C}_1$ is the set of morphisms $X \to Y$, $\mathcal{C}_2$ is the set of ordered pairs of morphisms $X \to Y \to Z$, etc. with

$$s^0(X) = \mathrm{Id}_X, \quad \begin{cases} d^0(X \overset{f}{\to} Y) = Y \\ d^1(X \overset{f}{\to} Y) = X, \end{cases} \quad \begin{cases} s^0(X \overset{f}{\to} Y) = X \overset{\mathrm{Id}_X}{\to} X \overset{f}{\to} Y \\ s^1(X \overset{f}{\to} Y) = X \overset{f}{\to} Y \overset{\mathrm{Id}_Y}{\to} Y, \end{cases}$$

$$\begin{cases} d^0(X \overset{f}{\to} Y \overset{g}{\to} Z) = Y \overset{g}{\to} Z \\ d^1(X \overset{f}{\to} Y \overset{g}{\to} Z) = X \overset{g \circ f}{\to} Z \\ d^2(X \overset{f}{\to} Y \overset{g}{\to} Z) = X \overset{f}{\to} Y, \end{cases} \quad \begin{cases} s^0(X \overset{f}{\to} Y \overset{g}{\to} Z) = X \overset{\mathrm{Id}_X}{\to} X \overset{f}{\to} Y \overset{g}{\to} Z \\ s^1(X \overset{f}{\to} Y \overset{g}{\to} Z) = X \overset{f}{\to} Y \overset{\mathrm{Id}_Y}{\to} Y \overset{g}{\to} Z \\ s^2(X \overset{f}{\to} Y \overset{g}{\to} Z) = X \overset{f}{\to} Y \overset{g}{\to} Z \overset{\mathrm{Id}_Z}{\to} Z. \end{cases}$$

1.2.3 Equivalence

Definition 1.2.17 A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be

1. *faithful* (resp. *full*, resp. *fully faithful*) if for all $X, Y \in \mathcal{C}$, the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y)), \quad f \mapsto F(f)$$

   is injective (resp. surjective, resp. bijective).
2. *essentially surjective* if for all $X' \in \mathcal{C}'$, there exists $X \in \mathcal{C}$ such that $X' \simeq F(X)$.
3. an *equivalence of categories* if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that $\mathrm{Id}_{\mathcal{C}} \simeq G \circ F$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{C}'}$ (and G is then called a *quasi-inverse*).

One can also define the *essential image* of a functor F as the collection of all $X' \in \mathcal{C}'$ such that there exists $X \in \mathcal{C}$ with $X' \simeq F(X)$. The functor F is then essentially surjective when the essential image is equal to $\mathcal{C}'$.

The inclusion of a (full) subcategory is a (fully) faithful functor that we shall call an *embedding*. An isomorphism of categories is an equivalence (but not conversely).

We shall use the notation $\mathcal{C} \simeq \mathcal{C}'$ for the wider notion of *equivalence* (and not merely isomorphism) of categories. One sometimes says that two categories $\mathcal{C}$ and $\mathcal{C}'$ are *anti-equivalent* if $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}'$.

Exercise 1.2.18 Show that the forgetful functors $\mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Ab} \to \mathbf{Set}$ are faithful but not fully faithful.

Exercise 1.2.19 Show that there exists a fully faithful functor

$$\mathbf{Mon} \rightarrowtail \mathbf{Cat}, \quad G \mapsto \mathbf{G}, \quad (\text{resp. } \mathbf{Ord} \rightarrowtail \mathbf{Cat}, \quad X \mapsto \mathbf{X}).$$

What is the essential image ?

Exercise 1.2.20 Show that there exists a fully faithful functor $\mathbf{Cat} \rightarrowtail \widehat{\boldsymbol{\Delta}}$.

Exercise 1.2.21 Show that if $F \simeq F'$, then F is faithful (resp. full, resp. fully faithful, essentially surjective, an equivalence) if and only if F' is.

Theorem 1.2.22 A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Proof. To show that the condition is necessary, we first remark that our functor $F : \mathcal{C} \to \mathcal{C}'$ will be essentially surjective since we will always have $X' \simeq F(X)$ with $X := G(X')$ if G is a quasi-inverse for F. Then, we consider the following sequence of maps

$$\mathrm{Hom}(X,Y) \xrightarrow{F} \mathrm{Hom}(F(X),F(Y))$$

$$\xrightarrow{G} \mathrm{Hom}(G(F(X)),G(F(Y))) \xrightarrow{F} \mathrm{Hom}(F(G(F(X))),F(G(F(Y))))$$

Since $G \circ F$ (resp. $F \circ G$) is fully faithful, the composition of the first (resp. last) two arrows is bijective. It follows that all arrows are actually bijections and, in particular, F is fully faithful.

To show that the condition is sufficient, we *choose* for all $X' \in \mathcal{C}'$ an object $X \in \mathcal{C}$ and an isomorphism $\beta_{X'} : F(X) \simeq X'$. We set $G(X') := X$ so that $\beta_{X'} : F(G(X')) \simeq X'$. Since F is fully faithful, there exists for each $f' : X' \to Y'$ in $\mathcal{C}'$ a unique $f : G(X') \to G(Y')$ such that $F(f) = \beta_{Y'}^{-1} \circ f' \circ \beta_{X'}$. We then set $G(f') = f$ so that $\beta_Y \circ F(G(f')) = f' \circ \beta_{X'}$ One easily checks that G is a functor and we obtain by construction a natural isomorphism $\beta : F \circ G \simeq \mathrm{Id}_{\mathcal{C}'}$. In particular, if $X \in \mathcal{C}$, then there exists a natural isomorphism $\beta_{F(X)} : F(G(F(X))) \simeq F(X)$ and, since F is fully faithful, there exists a unique morphism $\alpha_X : X \simeq (G \circ F)(X)$ such that $F(\alpha_X) = \beta_{F(X)}^{-1}$. One easily checks that α is indeed a natural isomorphism. ∎

Exercise 1.2.23 Show that if X is a preordered set and Y denotes its ordered quotient, then the categories $\mathbf{X}$ and $\mathbf{Y}$ are equivalent.

Exercise 1.2.24 Show that, if A is a ring, then $\mathbf{Mat}(A) := \mathbb{N}$, endowed with $\mathrm{Hom}(m,n) = \mathrm{M}_{n\times m}(A)$ and multiplication of matrices, is a small category. Show that if A is a field k, then $\mathbf{Mat}(k)$ is equivalent, but not isomorphic, to the category of finite dimensional k-vector spaces (which is large).

1.2.4 Universal property

Definition 1.2.25 Given a functor $F : \mathcal{C} \to \mathbf{Set}$, an object $X \in \mathcal{C}$ and $s \in F(X)$, we say that (X, s) *represents* F, or that it is *universal* for F, if

$$\forall Y \in \mathcal{C}, \forall t \in F(Y), \exists! f : X \to Y, \quad F(f)(s) = t. \tag{1.1}$$

This may be pictured as follows:

$$\begin{array}{ccccc} X & \longmapsto & F(X) & & s \\ {\scriptstyle \exists! f}\,\downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow \\ Y & \longmapsto & F(Y) & & t. \end{array}$$

We may also say that "the pair $(X \in \mathcal{C}, s \in F(X))$ is universal among all pairs $(Y \in \mathcal{C}, t \in F(Y))$", or that "$s \in F(X)$ is universal for $t \in F(Y)$" or that "X represents F" (not mentioning s explicitlty).

Examples
1. $(\mathbb{Z}, 1)$ represents the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$.
2. If k is a commutative ring then $(k[t], t)$ represents the forgetful functor k-$\mathbf{Alg} \to \mathbf{Set}$ (universal property of polynomials).
3. $(\mathbb{Z}, 0)$ represents the constant functor $\mathbf{Ring} \twoheadrightarrow \mathbf{1} \rightarrowtail \mathbf{Set}$ ($\mathbb{Z}$ is initial for rings).
4. The real line $\mathbb{R}$ (together with $\mathrm{Id}_{\mathbb{R}}$) represents the (contravariant) functor

   $$\mathcal{O} : X \mapsto \mathcal{C}^k(X, \mathbb{R})$$

   on $\mathcal{C}^k$-manifolds.
5. The inclusion $Y \rightarrowtail X$ of a subspace into a topological space is universal for continuous maps $f : Z \to X$ such that $f(Z) \subset Y$ (contravariant F).

Exercise 1.2.26 Show that, if F is represented by both X and X', then $X \simeq X'$. More precisely, show that if both (X, s) and (X', s') represent F, then there exists a unique isomorphism $f : X \simeq X'$ such that $F(f)(s) = s'$.

Exercise 1.2.27 Show that usual forgetful functors are representable.

Exercise 1.2.28 Show that $M \otimes_k N$ is universal for (the functor that sends P to the set of) bilinear maps $M \times N \to P$ when k is a commutative ring.

Exercise 1.2.29 Let k be a commutative ring and $f_1, \ldots, f_r \in k[t_1, \ldots, t_n]$. Show that the functor that sends a commutative k-algebra A (make the morphisms explicit) to the set

$$\mathcal{S}(A) := \{(a_1, \ldots, a_n) \in A^n \ / \ f_1(a_1, \ldots, a_n) = \cdots = f_r(a_1, \ldots, a_n) = 0\}$$

of all solutions with values in A, is representable.

Lemma 1.2.30 — Yoneda. If $F : \mathcal{C} \to \mathbf{Set}$ is any functor and $X \in \mathcal{C}$, then there exists a natural bijection

$$\mathrm{Hom}(h^X, F) \simeq F(X), \quad \alpha \mapsto \alpha_X(\mathrm{Id}_X).$$

With a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, it simply reads $\mathrm{Hom}(h_X, F) \simeq F(X)$.

Proof. Given $s \in F(X)$, if $Y \in \mathcal{C}$ and $f : X \to Y$, then we set $\alpha_Y(f) := F(f)(s)$. This defines a map $\alpha_Y : h^X(Y) := \mathrm{Hom}(X, Y) \to F(Y)$ and we shall show that this is natural, meaning that

$$\alpha_Z \circ h^X(g) = F(g) \circ \alpha_Y$$

if $g : Y \to Z$. Indeed, we do have

$$(\alpha_Z \circ h^X(g))(f) = \alpha_Z(h^X(g)(f)) = \alpha_Z(g \circ f) = F(g \circ f)(s)$$
$$= (F(g) \circ F(f))(s) = (F(g)(F(f)(s)) = F(g)(\alpha_Y(f)) = (F(g) \circ \alpha_Y)(f).$$

It only remains to check that we did define a natural inverse as well as the naturality (exercise). ■

Proposition 1.2.31 A functor $F : \mathcal{C} \to \mathbf{Set}$ is represented by $X \in \mathcal{C}$ if and only if $h^X \simeq F$.

Proof. We simply apply Yoneda lemma: the condition means that there exists a natural transformation $\alpha : h^X \to F$ which is an isomorphism. This α corresponds to some $s \in F(X)$ and property 1.1 exactly means that α_Y is always bijective since, necessarily, $F(f)(s) = \alpha_Y(f)$. ■

In other words, F is represented by X if and only if there exists a natural bijection $\mathrm{Hom}(X, Y) \simeq F(Y)$. For a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, the condition reads $\mathrm{Hom}(Y, X) \simeq F(Y)$ or, equivalently, $h_X \simeq F$.

Exercise 1.2.32 Show that if $\mathcal{C}$ is a small category, then there exists a fully faithful *Yoneda* functor

$$\text{よ} : \mathcal{C} \rightarrowtail \widehat{\mathcal{C}}, \quad X \mapsto h_X.$$

Solution. If $X, Y \in \mathcal{C}$, then Yoneda's lemma on $\mathcal{C}^{\mathrm{op}}$ shows that the map

$$\mathrm{Hom}(h_X, h_Y) \simeq h_Y(X) = \mathrm{Hom}(X, Y), \quad \alpha \mapsto \alpha_X(\mathrm{Id}_X)$$

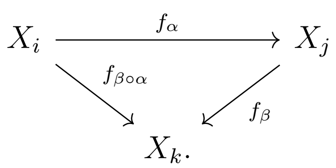
is bijective. It is therefore sufficient to notice that, for $f : X \to Y$, we have $h^X_f(\mathrm{Id}_X) = f \circ \mathrm{Id}_X = f$. ■

1.3 Limit

1.3.1 Diagram, cone and limit

Definition 1.3.1 Let I be a small category and $\mathcal{C}$ any category. A *commutative diagram* of shape I in $\mathcal{C}$ is a functor $D : I \to \mathcal{C}$.

A commutative diagram on I in $\mathcal{C}$ is therefore the data of an object X_i for all $i \in I$ and a morphism $f_\alpha : X_i \to X_j$ for all $\alpha : i \to j$ satisfying $f_{\mathrm{Id}_i} = \mathrm{Id}_{X_i}$ and $f_{\beta\circ\alpha} = f_\beta \circ f_\alpha$:



We shall denote such a diagram as $(f_\alpha : X_i \to X_j)$ or (X_i, f_α), and the category of all commutative diagrams on I in $\mathcal{C}$ by $\mathcal{C}^I := \mathbf{H}\mathrm{om}(I, \mathcal{C})$. A morphism $(X_i, f_\alpha) \to (Y_i, g_\alpha)$ is simply a family of morphisms $u_i : X_i \to Y_i$ satisfying $g_\alpha \circ u_i = u_j \circ f_\alpha$ for all $\alpha : i \to j$.

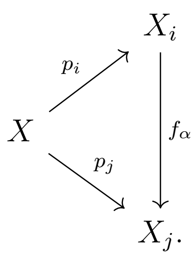
A commutative diagram is said to be *empty*, *discrete* or *finite* when I has this property. In particular, a discrete diagram is the same thing as a family of objects.

By composition, any functor $\lambda : I \to J$ between small categories will provide a functor $\lambda^* : \mathcal{C}^J \to \mathcal{C}^I$ between diagrams (and this is functorial). As a particular case, the unique functor $I \to \mathbf{1}$ induces the *constant diagram* functor

$$\mathcal{C} \simeq \mathcal{C}^{\mathbf{1}} \to \mathcal{C}^I, \quad X \mapsto \underline{X}.$$

Definition 1.3.2 A *cone* for a diagram D in $\mathcal{C}$ is an object $X \in \mathcal{C}$ together with a morphism $\underline{X} \to D$.

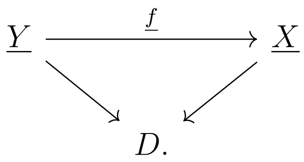
In more down to earth terms, a cone for (X_i, f_α) is an object X together with a family of *structural morphisms* $p_i : X \to X_i$ satisfying $p_j = f_\alpha \circ p_i$ whenever $\alpha : i \to j$:



The cone is said to be *empty*, *discrete* or *finite* when the diagram has this property. The dual notion is that of a *cocone*.

Definition 1.3.3 A *limit* X of a commutative diagram D of shape I in $\mathcal{C}$ is a universal cone for D.

In other words, X is a limit for D if and only if X is a cone for D and, given any cone Y for D, there exists a unique morphism $f : Y \to X$ making commutative the diagram

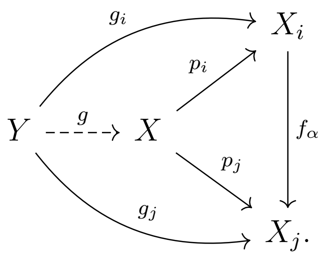


Formally, it means that the composite (contravariant) functor $h_D \circ \_$ is represented by X. Equivalently, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}^I}(\underline{Y}, D) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

in $Y \in \mathcal{C}$.

In down to earth terms, a commutative diagram (X_i, f_α) has X as a limit if and only if we are given for each $i \in I$ a morphism $p_i : X \to X_i$ such that for all $\alpha : i \to j$, we have $p_j = f_\alpha \circ p_i$ with the following universal property: if we are given some $Y \in \mathcal{C}$ endowed for all $i \in I$ with a morphism $g_i : Y \to X_i$ such that for each $u : i \to j$, we have $g_j = f_\alpha \circ g_i$, then there exists a unique morphism $g : Y \to X$ such that for all $i \in I$, we have $g_i = p_i \circ g$:



(1.2)

A limit is unique up to a unique isomorphism and we may sometimes say *the* limit and write $X = \varprojlim D$ or $X = \varprojlim_I D$. This should however been understood in the sense that X (together with all structural maps) is *a* limit of D. A limit X of a diagram D in $\mathcal{C}^{\mathrm{op}}$ is also called a *colimit* in $\mathcal{C}$ and we shall write $X = \varinjlim D$. Some authors call a limit (resp. colimit) an *inverse* (resp. a *direct*) limit, a *projective* (resp. an *inductive*) limit or a *left* (resp. a *right*) limit. We have the following important formulas:

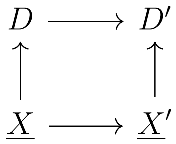
$$\mathrm{Hom}(\underline{Y}, D) \simeq \mathrm{Hom}(Y, \varprojlim D) \quad \text{et} \quad \mathrm{Hom}(\varinjlim D, Y) \simeq \mathrm{Hom}(D, \underline{Y}).$$

When we write a limit or a colimit, we implicitly assume that it exists.

Exercise 1.3.4 Write down a diagram for the colimit as in (1.2).

Exercise 1.3.5 Show that in a preordered set, a limit (resp. colimit) is a least upper bound or *inf* or *join* (resp. greatest lower bound or *sup* or *meet*). What about cone and cocone ?

Exercise 1.3.6 Show that, if $D \to D'$ is a morphism of diagrams of shape I in $\mathcal{C}$ with respective limits X and X', then there exists a unique morphism $X \to X'$ making commutative the diagram



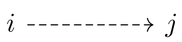
Definition 1.3.7 A functor $\lambda : J \to I$ between small categories is said to be *final*[a] if the comma category ${}_{i\backslash}\lambda$ is non empty connected whenever $i \in I$.

[a]This used to be called cofinal.

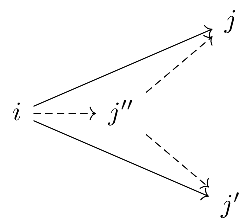
For the sake of completeness, recall that an object of ${}_{i\backslash}\lambda$ is a pair (j, α) with $j \in J$ and $\alpha : i \to \lambda(j)$. A morphism $(j, \alpha) \to (j', \alpha')$ is a morphism $\beta : j \to j'$ in J such that $\lambda(\beta) \circ \alpha = \alpha'$.

We may also say that J is final in I via λ. The dual notion is that of an *initial* functor (λ^{op} is final). An order preserving map $\lambda : I \to J$ is a between two preordered sets is said to be final/initial if the corresponding functor is

Examples 1. $I \to 1$ is final if and only if I is non empty connected.
2. A functor $1 \to \mathcal{C}, 0 \mapsto X$ is final if and only if X is a final object (see below) of $\mathcal{C}$.
3. A subset of $\mathbb{N}$ is final (for the usual order) if and only if it is infinite.
4. Assume $J \subset I$ and we can complete (on the right) any of the following diagrams:



and



(it means that

$$\begin{aligned}&\forall i \in I, \exists j \in J, \exists \alpha : i \to j \text{ and}\\ &\forall i \in I, \forall j, j' \in J, \forall \alpha : i \to j, \alpha' : i \to j',\\ &\exists j'' \in J, \exists \gamma : i \to j'', \beta' : j'' \to j, \beta' : j'' \to j', \beta \circ \gamma = \alpha, \beta' \circ \gamma = \alpha').\end{aligned}$$

Then J is final in I.

Exercise 1.3.8 Show that, if X is a metric space and $\mathcal{V}$ denotes the set of neighborhoods of $x \in X$, then the map

$$\mathbb{R}_{>0} \to \mathcal{V}, \quad \epsilon \mapsto \mathbb{B}(x, \epsilon)$$

is initial[a].

[a]This maybe seen as the origin of the surge of ϵ in analysis.

We may always replace the shape of a diagram along a final (or initial) functor:

Proposition 1.3.9 The following are equivalent (and dual):
1. the functor $\lambda : J \to I$ is final,
2. if D is a diagram of shape I, then $\varinjlim \lambda^*(D) = \varinjlim D$.

Proof. To do. ∎

1.3.2 Specific limits

Definition 1.3.10 A empty limit in $\mathcal{C}$ is called a *final* object and denoted by $1_{\mathcal{C}}$. Dually, we get the notion of an *initial* object $0_{\mathcal{C}}$.

If $X \in \mathcal{C}$, there exists a unique morphism $X \to 1_{\mathcal{C}}$ (resp. $0_{\mathcal{C}} \to X$).

Examples
1. In **Set**, the initial object is $\emptyset$ and 1 is a final object (defined up to a unique bijection).
2. Same thing in **Top**.
3. In **Ab**, $\{0\}$ is both a final and an initial object.

In general, cones for a diagram D in $\mathcal{C}$ form a category $\mathcal{C}_{/D}$ and a limit of D is the same thing as a final object of $\mathcal{C}_{/D}$.

Definition 1.3.11 A discrete limit is called a *product* and denoted by $\prod_{i\in I} X_i$. The dual notion is that of a *coproduct* denoted by $\coprod_{i\in I} X_i$.

If $X = \prod_{i\in I} X_i$, then there exists structural maps $p_i : X \to X_i$. It is equivalent to give a morphism $f : Y \to X$ or its *components* $f_i := p_i \circ f$ and we shall then write $f = (f_i)_{i\in I}$ (and dual).

When there are only a finite number of objects $X_1, \ldots, X_n$, we shall write $X_1 \times \cdots \times X_n$ (resp. $X_1 \sqcup \cdots \sqcup X_n$). When all X_i are equal to the same X, we shall write X^I (resp. $X^{(I)}$). Note that a final object is nothing else but the empty product (and dual).

Examples
1. In **Set**, the cartesian product is a product and the disjoint union is a coproduct.
2. In **Top**, this is the same thing with the coarser (resp. finer) topology making the projections (resp. injections) continuous.
3. In **Ab**, the cartesian product with termwise addition is a product and the direct sum is a coproduct[4] (product equals coproduct when I is finite).

The morphism $\delta = (\mathrm{Id}_X, \mathrm{Id}_X) : X \to X \times X$ is called the *diagonal morphism*. The morphism $\tau = (p_2, p_1) : X \times Y \to Y \times X$ is called the *flip morphism*. The morphism $(p_1, p_2, p_1, p_3) : X \times Y \times Z \to X \times Y \times X \times Z$ (resp. (p_1, p_3, p_2, p_3)) is called the *left* (resp. *right*) *distribution morphism*.

Exercise 1.3.12 Show that there exists “canonical” isomorphisms (and dual)
1. $X \times 1_{\mathcal{C}} \simeq X$ for $X \in \mathcal{C}$,
2. $X \times Y \simeq Y \times X$ for $X, Y \in \mathcal{C}$ and
3. $(X \times Y) \times Z \simeq X \times (Y \times Z)$ for $X, Y, Z \in \mathcal{C}$.

Solution. We only do the last one. There exists obvious morphisms $p_1 : (X \times Y) \times Z \to X \times Y \to X$, $p_2 : (X \times Y) \times Z \to X \times Y \to Y$ and $p_3 : (X \times Y) \times Z \to Z$ defining a cone for the discrete diagram (X, Y, Z). Conversely, given such a cone $(f_1 : T \to X, f_2 : T \to Y, f_3 : T \to Z)$, there exists a unique morphism $f' : T \to X \times Y$ whose components are f_1 and f_2. Then, there exists a unique morphism $f : T \to (X \times Y) \times Z$

[4] Be careful that what is called *free product* is a coproduct in the category of (non abelian) groups.

whose components are f' and f_3. In other words, there exists a unique morphism satisfying $p_i \circ f = f_i$ for $i = 1, 2, 3$. This shows that $(X \times Y) \times Z$ is a limit for (X, Y, Z). Following the same lines, on sees that $X \times (Y \times Z)$ also is a limit for this diagram. They are therefore isomorphic. ■

Definition 1.3.13 A category is said to be *cartesian* if all finite products exist.

Exercise 1.3.14 Show that, if $\mathcal{C}$ is a cartesian category, then there exists a functor

$$\mathcal{C} \to \widehat{\mathbf{\Delta}}(\mathcal{C}) := \mathbf{Hom}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}), \quad X \mapsto X_\bullet$$

where $X_n := X^{n+1}$ and $u : [m] \to [n]$ is sent to the unique morphism $X^{n+1} \to X^{m+1}$ whose i-th component is the j-th projection with $j := u(i-1)$.

Definition 1.3.15 A limit X of a diagram $(X_1 \overset{f_1}{\to} X_0 \overset{f_2}{\leftarrow} X_2)$ is called a *fibered product* of X_1 and X_2 over X_0 and denoted by $X_1 \times_{X_0} X_2$. We shall then also say that the diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & X_1 \\ \downarrow{\scriptstyle p_2} & \lrcorner & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow{f_2} & X_0 \end{array} \tag{1.3}$$

is *cartesian* or that p_2 is the *pullback* of f_1 along f_2 (and p_1 is the *pullback* of f_2 along f_1). Dually, there exists the notions of a *fibered coproduct* denoted by $X_1 \sqcup_{X_0} X_2$ associated to a diagram $(X_1 \overset{f_1}{\leftarrow} X_0 \overset{f_2}{\to} X_2)$, a *cocartesian* square and a *pushout*.

Thus, by definition, diagram (1.3) is cartesian if, given $g_1 : Y \to X_1$ and $g_2 : Y \to X_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique $g : Y \to X$ such that $p_1 \circ g = g_1$ and $p_2 \circ g = p_2$. Note that a product of two objects is nothing but a fibered product over a final object (and dual). Conversely, a fibered product as in the definition is the same thing as the product in $\mathcal{C}_{/X_0}$ (and dual).

Examples 1. In **Set**, we have

$$X_1 \times_{X_0} X_2 = \{(x_1, x_2) \ / \ f_1(x_1) = f_2(x_2)\} \subset X_1 \times X_2$$

and

$$X_1 \sqcup_{X_0} X_2 = (X_1 \sqcup X_2)/\sim$$

where $\sim$ is the relation $f_1(x_0) \sim f_2(x_0)$ when $x_0 \in X_0$.

2. In **Top**, this is the same thing with the induced (resp. quotient) topology.
3. In **Ab** we have with $(M_1 \overset{f_1}{\to} M_0 \overset{f_2}{\leftarrow} M_2)$

$$M_1 \times_{M_0} M_2 = \ker(M_1 \oplus M_2 \to M_0, (x_1, x_2) \mapsto f(x_2) - f(x_1))$$

and with $(M_1 \overset{f_1}{\leftarrow} M_0 \overset{f_2}{\to} M_2)$

$$M_1 \sqcup_{M_0} M_2 = \mathrm{coker}(M_0 \to M_1 \oplus M_2 \to M_0, x_0 \mapsto (f_1(x_0), f_2(x_0)).$$

Exercise 1.3.16 Show that the fibered coproduct in the category of commutative rings is tensor product.

Exercise 1.3.17 Show that[a], in a diagram

$$\begin{array}{ccccc} Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ X_2 & \longrightarrow & X_1 & \longrightarrow & X_0, \end{array}$$

if the right hand square is cartesian, then the left hand square is cartesian if and only if the full rectangle is cartesian.

[a] This is an illustration of the more general proposition 1.3.42 below.

Definition 1.3.18 A limit X of a pair $Y \rightrightarrows Z$ is called a *kernel* (or *equalizer*) and denoted by $X = \ker(f, g)$. We shall also say that the sequence

$$X \xrightarrow{i} Y \overset{f}{\underset{g}{\rightrightarrows}} Z \tag{1.4}$$

is *left exact*. Dually, there exists the notion of a *cokernel* (or *coequalizer*) coker (f, g) and *right exact* sequence.

Thus, by definition, diagram (1.4) is left exact if, given $h : T \to Y$ such that $f \circ h = g \circ h$, there exists a unique $k : T \to X$ such that $h = i \circ k$.

Examples 1. In **Set**, we have

$$\ker(f, g) = \{y \in Y \ / \ f(y) = g(y)\} \quad \text{and} \quad \text{coker}\,(f, g) = Z/\sim$$

where $\sim$ is the relation $f(y) \sim g(y)$ for $y \in Y$.

2. Same thing in **Top** with induced and quotient topology respectively.
3. In **Ab**, $\ker(f, g) = \ker(g - f)$ (and dual).

Exercise 1.3.19 Show that if $\mathcal{C}$ is a small category, then $\mathbf{Op}(\mathcal{C})$ is the kernel of the domain and codomain functors $\mathbf{Mor}(\mathcal{C}) \rightrightarrows \mathcal{C}$ in **Cat**.

Exercise 1.3.20 Make explicit specific limits (final object, products, fibered product and kernel) and colimits (initial object, coproduct, fibered coproduct and cokernel) in **Mon**, **Grp**, G-**Set**, A-**Mod** or **Cat**.

1.3.3 Monomorphism/epimorphism

These notions are usually introduced before that of limits/colimits but are better understood as particular instances of limits/colimits.

Definition 1.3.21 A morphism $i : X \to Y$ is called a

1. a *monomorphism* if the diagram

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{i} & Y, \end{array} \tag{1.5}$$

is cartesian and we shall then write $i : X \rightarrowtail Y$.
2. a *regular monomorphism* if there exists an exact sequence $X \xrightarrow{i} Y \rightrightarrows Z$.
3. a *split monomorphism* if it has a retraction.

The dual notion is that of a *(regular, split) epimorphism* and we shall then write $Y \twoheadrightarrow X$.

Examples 1. A morphism in **Set** (resp. **Ab**, resp. **Top**) is a monomorphism/epimorphism if and only if it is injective/surjective.
2. A regular monomorphism/epimorphism in **Top** is a homeomorphism with a subspace/from a quotient map.
3. The inclusion map $\mathbb{Q} \rightarrowtail \mathbb{R}$ is at the same time a monomorphism and an epimorphism in the category of *Hausdorff* topological spaces.
4. Same for the inclusion map $\mathbb{Z} \rightarrowtail \mathbb{Q}$ in the category of rings.

Proposition 1.3.22 The following are equivalent (and dual):
1. The morphism $i : X \to Y$ is a monomorphism,
2. if $f, g : Z \to X$ satisfy $i \circ f = i \circ g$, then $f = g$,
3. for all $Z \in \mathcal{C}$, the map $i_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is injective,
4. any commutative diagram

$$\begin{array}{ccc} Y' & = & Y' \\ \downarrow{\scriptstyle j} & & \downarrow \\ X & \xrightarrow{i} & Y \end{array} \tag{1.6}$$

is cartesian.

Proof. By definition, the morphism i is a monomorphism if and only if, given $f, g : Z \to X$ satisfying $i \circ f = i \circ g$, there exists a unique h such that $f = g = h$. This is clearly the same thing as the second assertion which in turn is an explicit form of the third one. Assume now that we are given a commutative diagram (1.6) as well as two morphisms $f, g : Z \to X'$ satisfying $i \circ j \circ f = i \circ g$. If i is a monomorphism, then $j \circ f = g$ and the diagram is therefore cartesian. Finally, in the last assertion, if we put i on the right hand side, then we fall back onto the definition of a monomorphism. ■

Exercise 1.3.23 Show that a split monomorphism is a regular monomorphism and that a regular monomorphism is a monomorphism (and dual).

Solution. Assume $i : X \to Y$ is a split monomorphism. By definition, it admits a retraction $r : Y \to X$ (so that $r \circ i = \mathrm{Id}_X$) and we will prove that the diagram

$$X \xrightarrow{i} Y \underset{\mathrm{Id}_Y}{\overset{i \circ r}{\rightrightarrows}} Y$$

is left exact. If we are given a morphism $f : Z \to Y$ such that $i \circ r \circ f = f$, and we set $g = r \circ f : Z \to X$, then $i \circ g = f$. Assume conversely, that g satisfies $i \circ g = f$. Then, $g = r \circ i \circ g = r \circ f$ which shows uniqueness.

Assume now that $i : X \to Y$ is a regular monomorphism so that there exists a left exact diagram

$$X \xrightarrow{i} Y \underset{v}{\overset{u}{\rightrightarrows}} Y.$$

We give ourselves $g_1, g_2 : Z \to X$ such that $i \circ g_1 = i \circ g_2$ and we denote this common map by f so that $f = i \circ g_k$ for $k = 1, 2$. We have $u \circ f = u \circ i \circ g_k = v \circ i \circ g_k = v \circ f$. Therefore, there exists a *unique* $g : Z \to X$ such that $f = g \circ i$. It follows that $g = g_k$ and then $g_1 = g_2$.

By duality, a split epimorphism is a regular epimorphism and a regular epimorphism is an epimorphism. ■

Exercise 1.3.24 Show that a morphism $f : X \to Y$ is a monomorphism (resp. an epimorphism) if and only if the induced functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ (resp. ${}_{Y\backslash}\mathcal{C} \to {}_{X\backslash}\mathcal{C}$) is fully faithful.

Solution. If we denote the morphism by $i : X \to Y$, then an object in $\mathcal{C}_{/X}$ is a morphism $f : Z \to X$ and the induced functor sends f to the composite $i \circ f : Z \to Y$. A morphism between $f : Z \to X$ and $f' : Z' \to X$ is a morphism $g : Z \to Z'$ such that $f' \circ g = f$. It is sent to the same morphism g but the condition now reads $i \circ f' \circ g = i \circ f$. To make it clear, we have

$$\mathrm{Hom}_{\mathcal{C}_{/X}}(f, f') \subset \mathrm{Hom}_{\mathcal{C}_{/Y}}(i \circ f, i \circ f') \subset \mathrm{Hom}_{\mathcal{C}}(Z, Z').$$

The induced functor is fully faithful when the first inclusion is an equality. It means that condition "$i \circ f' \circ g = i \circ f \Rightarrow f' \circ g = f$" is always satisfied. By definition, this is automatic when i is a monomorphism. Conversely, the condition in the case $Z = Z'$ and $g = \mathrm{Id}_Z$ will tell us that i is a monomorphism.

By duality, $\pi : X \to Y$ is an epimorphism in $\mathcal{C}$ if and only it is a monomorphism (from Y to X) in $\mathcal{C}^{\mathrm{op}}$ if and only if $\mathcal{C}^{\mathrm{op}}_{/Y} \to \mathcal{C}^{\mathrm{op}}_{/X}$ is fully faithful, or equivalently since it boils down to the same condition, if and only if ${}_{Y\backslash}\mathcal{C} = (\mathcal{C}^{\mathrm{op}}_{/Y})^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}}_{/X})^{\mathrm{op}} = {}_{X\backslash}\mathcal{C}$ is fully faithful. ■

Exercise 1.3.25 Show that[a] if both $f : X \to Y$ and $g : Y \to Z$ are monomorphisms, then so is $g \circ f$ (and dual). Show that, conversely, if $g \circ f$ is a monomorphism, then so is f (and dual).

[a]Be careful that this is not the case for regular monomorphisms in general.

Exercise 1.3.26 Show that a regular epimorphism which is also a monomorphism is automatically an isomorphism (and dual).

Exercise 1.3.27 Show that, if we are given $f, g : X \to Y$ and $i : Y \rightarrowtail Z$ is a monomorphism, then[a] $\ker(i \circ f, i \circ g) = \ker(f, g)$ (and dual). Analog for fibered products (and dual) ?

[a] If one of them exists, then so does the other and...

Exercise 1.3.28 Show that, if we are given a commutative diagram of monomorphisms

$$\begin{array}{ccc} X_1 & \rightleftarrows & X_2 \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

then, the upper arrows are inverse isomorphisms to each other (and dual).

Exercise 1.3.29 Show that, in a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & \ulcorner & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

if f is a monomorphism, then f' also is monomorphism (and dual).

Exercise 1.3.30 Show that if $i : X \rightarrowtail Y$ is an injective map and the diagram of *sets*

$$\begin{array}{ccc} X & \overset{i}{\rightarrowtail} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is cocartesian, then it is also cartesian.

Recall that we always implicitly assume that all limits and colimits that we write do exist. For safety, the reader can assume up to the end of the section that all finite limits and all finite colimits exist.

Definition 1.3.31 The *(regular) image* of a morphism $f : X \to Y$ is

$$\mathrm{im}(f) := \ker(Y \rightrightarrows Y \sqcup_X Y).$$

The dual notion is that of a *(regular) coimage*

$$\mathrm{coim}(f) := \mathrm{coker}(X \times_Y X \rightrightarrows X).$$

Examples 1. We already know $\mathrm{im}(f)$ in **Set** and $\mathrm{coim}(f) = X/\sim$ with $x \sim x' \Leftrightarrow f(x) = f(x')$.

2. In **Top**, this is the same thing with induced and quotient topology respectively.
3. In **Ab** also, we know $\operatorname{im}(f)$ for $f : M \to N$ and we have $\operatorname{coim}(f) = M/\ker(f)$.

Exercise 1.3.32 Show that, if $f : X \to Y$ is an epimorphism, then $\operatorname{im}(f) = Y$ (and dual).

Proposition 1.3.33 The factorization $f : X \to \operatorname{im}(f) \rightarrowtail Y$ is universal for factorizations of a morphism $f : X \to Y$ through a *regular* monomorphism (and dual).

Proof. We have to show that, when $Y' \rightarrowtail Y$ is a regular monomorphism, then any commutative diagram may be uniquely completed as follows:

$$\begin{array}{ccccc} X & \longrightarrow & Y' & \rightarrowtail & Y \\ & \searrow & \uparrow & \nearrow & \\ & & \operatorname{im} f & & \end{array}$$

By definition, there exists a left exact sequence $Y' \to Y \rightrightarrows T$ and we may then contemplate the commutative diagram

$$\begin{array}{ccccccc} Y' & & \rightarrowtail & & Y & \rightrightarrows & T \\ \uparrow & \nwarrow & & & \| & & \uparrow \\ X & \longrightarrow & \operatorname{im} f & \rightarrowtail & Y & \rightrightarrows & Y \sqcup_X Y. \end{array}$$

■

If we remove the regular condition in proposition 1.3.33, we then obtain the definition of a "non-regular" image. In general, there always exists a factorization of a morphism $f : X \to Y$ as follows:

$$\begin{array}{ccccccc} X \times_Y X & \rightrightarrows & X & \xrightarrow{f} & Y & \rightrightarrows & Y \sqcup_X Y \\ & & \downarrow & & \uparrow & & \\ & & \operatorname{coim}(f) & \dashrightarrow & \operatorname{im}(f). & & \end{array}$$

Definition 1.3.34 A morphism f is said to be *strict* if $\operatorname{coim}(f) \simeq \operatorname{im}(f)$.

Example Any morphism is strict in **Set** or **Ab**[5] but not in **Top**.

Exercise 1.3.35 Show that a strict epimorphism is regular (and dual).

Hint. Use exercise 1.3.26. ■

[5] This is Noether's first isomorphism.

Definition 1.3.36 A category is said to be *balanced* if a morphism which is at the same time a monomorphism and an epimorphism is automatically an isomorphism.

Note that the converse is always true.

Example The categories **Set** and **Ab** are balanced but **Top** and **Ring** are not.

Proposition 1.3.37 If $\mathcal{C}$ is balanced, then a strict morphism factors uniquely up to an isomorphism as an epimorphism followed by a monomorphism.

Proof. Factorization is clear since a strict morphism $f : X \to Y$ splits as $f : X \twoheadrightarrow \operatorname{coim}(f) = \operatorname{im}(f) \rightarrowtail Y$. Conversely, any such decomposition fits into a commutative diagram[6]

$$\begin{array}{ccccccccc} X \times_Y X & \rightrightarrows & X & \twoheadrightarrow & Z & \rightarrowtail & Y & \rightrightarrows & Y \sqcup_X Y \\ & & \downarrow & \nearrow & & \searrow & \uparrow & & \\ & & \operatorname{coim}(f) & & = & & \operatorname{im}(f). & & \end{array}$$

This implies that $\operatorname{coim}(f) \simeq Z \simeq \operatorname{im}(f)$ when the category is balanced. ■

1.3.4 Constructions of limits

Lemma 1.3.38 We give ourselves a commutative diagram (X_i, f_α) and we assume that both $X' := \prod_i X_i$ and $X'' := \prod_{\alpha:i\to j} X_j$ exist. Denote by p (resp. q) the morphism $X' \to X''$ induced by the projections p_j onto the codomain (resp. the composites of the projection p_i onto the domain and f_α). Then, $X = \varprojlim(X_i, f_\alpha)$ if and only if there exists a left exact sequence

$$X \longrightarrow X' \overset{p}{\underset{q}{\rightrightarrows}} X''.$$

In particular, $X \rightarrowtail X'$ is then a regular monomorphism.

Hint. By definition, a cone for our diagram is object Y together with a family of morphisms $g_i : Y \to X_i$ satisfying $g_j = f_\alpha \circ g_i$ whenever $\alpha : i \to j$. This is the same thing as a morphism $g : Y \to X'$ satisfying $p \circ g = q \circ g$. Same cones, same limits. ■

Example If $(f_\alpha : X_i \to X_j)$ is a commutative diagram of *sets*, then

$$\varprojlim X_i = \left\{(x_i) \in \prod X_i, \forall \alpha : i \to j, f_\alpha(x_i) = x_j\right\}$$

and

$$\varinjlim X_i = \coprod X_i / \sim$$

where $\sim$ denotes the (smallest equivalence relation) such that by $x_i \sim x_j$ whenever $f_\alpha(x_i) = x_j$.

[6] Be careful that we cannot use the universal property of image or coimage here because it is not assumed that the monomorphism nor epimorphism is regular in the factorization.

Exercise 1.3.39 Assume given $f, g : X \to Y$ such that $Y \times Y$ exists. Show that $\ker(f, g)$ exists if and only if there exists a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle (f,g)} \\ Y & \xrightarrow{\ \delta\ } & Y \times Y \end{array}$$

in which case $Z = \ker(f, g)$.

Proposition 1.3.40 Let $\mathcal{C}$ be a category.
1. If all (finite) products and all kernels exist, then all (finite) limits exist (and dual).
2. If all fibered products exist and there is a final object, then all finite limits exist (and dual).

Proof. Immediate consequence of exercises 1.3.38 and 1.3.39. ∎

When all (finite) limits exist, then $\mathcal{C}$ is said to be *(finitely) complete*. The dual terminology is *cocomplete*. When both properties hold, we say bicomplete. Be careful that a complete (or cocomplete) *small* category is automatically (associated to) a preorder:

$$|\mathrm{Mor}(\mathcal{C})| \geq \left|\mathrm{Hom}(X, Y^{\mathrm{Mor}(\mathcal{C})})\right| = |\mathrm{Hom}(X, Y)|^{|\mathrm{Mor}(\mathcal{C})|} \Rightarrow |\mathrm{Hom}(X, Y)| \leq 1.$$

Exercise 1.3.41 Show that $\mathbf{S}$et, $\mathbf{T}$op, $\mathbf{A}$b, etc. are bicomplete.

The next statement is a consequence of coming corollary 1.4.15:

Proposition 1.3.42 If all limits of shape I exist in $\mathcal{C}$, then all limits of shape I exist in $\mathcal{C}^J$ and, if $D \in (C^J)^I \simeq \mathcal{C}^{I\times J}$, then
1. $\forall j \in J, \left(\varprojlim_I D\right)(j) = \varprojlim_I D(-, j)$ and
2. $\varprojlim_J \left(\varprojlim_I D\right) \simeq \varprojlim_{I\times J} D \simeq \varprojlim_I \left(\varprojlim_J D\right).$

And dually for colimits.

Proof. Let us write $D'_j := \varprojlim_I D(-, j) \in \mathcal{C}$ for $j \in J$. Any morphism $j \to k$ in J will provide a compatible family of morphism $D(i, j) \to D(i, k)$ for $i \in I$, and taking limits on I, a morphism $D'_j \to D'_k$. This is functorial and provides $D' \in \mathcal{C}^J$. By construction, this is a cone on D: there exists morphisms $D'_j \to D(i, j)$ for all $(i, j) \in I \times J$ providing a morphism $\underline{D'} \to D$ in $\mathcal{C}^{I\times J}$. If we are given another cone $\underline{E} \to D$ with $E \in \mathcal{C}^J$, then there exists for each j, a morphism

$$E_j := \varprojlim_I \underline{E}(-, j) \to \varprojlim_I D(-, j) = D'_j.$$

Again, they are compatible and provide $E \to D'$. This shows that $D' = \varprojlim_I D$. The first assertion then follows directly from the definition of D' and it remains to show that $\varprojlim_J D' \simeq \varprojlim_{I\times J} D$. We give ourselves a cone $\underline{X} \to D$ with $X \in \mathcal{C}$. It induces a cone $\underline{X} \to D(-,j)$ in $\mathcal{C}^I$ for all $j \in J$ and taking limits on I, a morphism $X \to D'_j$. Again, they are compatible on J and combine (uniquely) into a morphism $\underline{X} \to D'$ in $\mathcal{C}^J$. Taking limits, we obtain $X \to \varprojlim_J D'$ showing that this is a universal cone. ■

This assertion essentially says that limits and colimits (of diagrams) are computed termwise.

Note that there also exists a morphism

$$\varinjlim_J \left(\varprojlim_I D\right) \longrightarrow \varprojlim_I \left(\varinjlim_J D\right)$$

which is not an isomorphism in general.

Definition 1.3.43 Given a cocone $D \to \underline{Y}$, we shall say that the colimit of D is *stable under pullback* along $Y' \to Y$ if

$$\varinjlim(D \times_{\underline{Y}} \underline{Y'}) \simeq \left(\varinjlim D\right) \times_Y Y'.$$

The dual notion is that of limit being *stable under pushout*.

Exercise 1.3.44 Show that colimits of *sets* are stable under pullback.

Solution. Using exercise 1.3.38, it is sufficient to consider the cases of a coproduct or a cokernel.

If $X := \coprod_{i\in I} X_i$ is a disjoint union of sets and we are given two maps $f : X \to Y$ and $g : Z \to Y$, then clearly

$$\coprod_{i\in I} (X_i \times_Y Z) = X \times_Y Z.$$

Actually, on both sides, an élément is a pair (x, z) with $x \in X$, $z \in Z$ and $f(x) = g(z)$.

We consider now a pair of maps $u, v : X' \to X$, a cocone which is another pair of maps $f' : X' \to Y, f : X \to Y$ satisfying $f \circ u = f \circ v = f'$ and finally a map $g : Z \to Y$. We have coker $(u, v) = X/\widetilde{R}$ where $\widetilde{R}$ is the equivalence relation generated by $u(x')Rv(x')$ whenever $x' \in X'$. Since $f \circ u = f \circ v$, the map f factors through the surjection $X \twoheadrightarrow X/\widetilde{R}, x \mapsto \overline{x}$ and provides $\overline{f} : X/\widetilde{R} \to Y$. We can then consider the canonical map

$$\Pi : X \times_Y Z \to X/\widetilde{R} \times_Y Z, \quad (x, z) \mapsto (\overline{x}, z).$$

It is well defined and surjective: this follows from the fact that, since $\overline{f}(\overline{x}) = f(x)$, we have

$$\forall x \in X, z \in Z, \quad (x, z) \in X \times_Y Z \Leftrightarrow (\overline{x}, z) \in X/\widetilde{R} \times_Y Z.$$

The equivalence relation $\widetilde{S}$ on $X \times_Y Z$ generated by $(u(x'), z)S(v(x'), z)$ whenever $(x', z) \in X' \times_Y Z$ is given by

$$(x_1, z_1)\widetilde{S}(x_2, z_2) \Leftrightarrow x_1\widetilde{R}x_2 \text{ and } z_1 = z_2.$$

Thus, Π provides a bijection

$$\text{coker } (U, V) = (X \times_Y Z)/\widetilde{S} \simeq X/\widetilde{R} \times_Y Z = \text{coker } (u, v) \times_Y Z.$$ ■

As a particular example, epimorphisms are stable under pullback if, given any cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & \ulcorner & \downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

if f is an epimorphism, then f' also is an epimorphism.

Examples 1. Epimorphisms are stable under pull back in **S**et, **A**b or **T**op (and dual).
2. This is not the case in **H**aus or **R**ing :

$$\begin{array}{ccc} \emptyset & \rightarrowtail & \mathbb{R}\setminus\mathbb{Q} \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{Q} & \rightarrowtail & \mathbb{R} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Q} & \rightarrowtail & \mathbb{Q}[s] \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{Q}[t] & \rightarrowtail & \mathbb{Q}[t,s]/(ts-1). \end{array}$$

3. Be careful that, in the category of sets, we have

$$\varinjlim_{I\times J}(X_i\times Y_j) = \left(\varinjlim_I X_i\right)\times\left(\varinjlim_J Y_j\right) \text{ but } \varinjlim_I(X_i\times Y_i) \neq \left(\varinjlim_I X_i\right)\times\left(\varinjlim_I Y_i\right)$$

in general.

1.3.5 Preservation of limit

Recall that any functor $F : \mathcal{C} \to \mathcal{D}$ provides by composition a functor

$$F^I : \mathcal{C}^I \to \mathcal{D}^I, \quad D \mapsto F^I(D) := F \circ D.$$

We shall usually simply write F instead of F^I so that $F(X_i, f_\alpha) = (F(X_i), F(f_\alpha))$.

Definition 1.3.45 If D is a commutative diagram in $\mathcal{C}$, then a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve* (or *commute* with) the limit of D, if

$$F(\varprojlim D) \simeq \varprojlim F(D).$$

Of course, it is assumed here that the limit of D exists and it implies that the limit of $F(D)$ also does. There exists an obvious analog for colimits. Be careful that a (contravariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ preserves a limit when it turns a *colimit* in $\mathcal{C}$ into limit in $\mathcal{D}$.

Definition 1.3.46 A functor is said to be *left exact* (resp. *right exact*) if it preserves all finite limits (resp. colimits). Il is said to be *exact* if it is both left and right exact[a].

[a] We should say exact, coexact and biexact respectively but we will follow the mainstream terminology.

A functor that preserves all limits (resp. colimits) is sometimes said to be *continuous* (resp. *cocontinuous*) but this may conflict with the vocabulary of topos theory.

Examples 1. The forgetful functor **Top** $\to$ **Set** preserves all limits and colimits.

2. The functor **Set** $\to$ **Top** that endows a set with the discrete topology is exact and preserves all colimits. It does not preserve infinite limits however: with $\overline{n} = \{0, 1, \ldots, n-1, \infty\}$ (disccrete), we have $\varprojlim \overline{n} \simeq \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ infinite and compact (not discrete).
3. The functor **Set** $\to$ **Top** that endows a set with the coarse topology preserves all limits but is not exact: we have $1 \sqcup 1 = 2$.
4. The forgetful functor **Ab** $\to$ **Set** preserves all limits and the free abelian group functor **Set** $\to$ **Ab** preserves all colimits.
5. The functors h^X and h_X preserve all limits (see exercise 1.3.49). Be careful that the second one is contravariant: it turns colimits into limits.
6. The forgetful functor A-**Mod** $\to$ **Ab** preserves all limits and colimits.
7. For fixed $M \in$ **Mod**-A, the functor

$$A\text{-}\mathbf{Mod} \to \mathbf{Ab}, \quad N \mapsto M \otimes_A N$$

preserves all colimits.

Exercise 1.3.47 Show that, if F is left exact, then F preserves (regular, strict) monomorphisms (and dual).

Exercise 1.3.48 Show that, if $\mathcal{C}$ is a small category and all limits of shape I exist in $\mathcal{D}$, then all limits of shape I also exist in the category $\widehat{\mathcal{C}}(\mathcal{D})$ of presheaves and they are preserved by the functor

$$\widehat{\mathcal{C}}(\mathcal{D}) \to \mathcal{D}, \quad F \mapsto F(X)$$

for fixed $X \in \mathcal{C}$ (and dual).

Hint. This is a variant of proposition 1.3.42. ∎

Exercise 1.3.49 Show that a representable functor $F : \mathcal{C} \to$ **Set** preserves all limits[a].

[a]There exists no dual statement and the notion of a limit plays a special role.

Solution. We may assume that $F = h^X$ with $X \in \mathcal{C}$. It is then sufficient to check that if D is a commutative diagram in $\mathcal{C}$, then we have a sequence of bijections

$$h^X(\varprojlim D) \simeq \mathrm{Hom}(X, \varprojlim D) \simeq \mathrm{Hom}(\underline{X}, D)$$

$$\simeq \mathrm{Hom}(\underline{\{0\}}, h^X(D)) \simeq \mathrm{Hom}(\{0\}, \varprojlim h^X(D)) \simeq \varprojlim h^X(D).$$

Only the middle one needs to be checked by hand: if we write $D =: (X_i, f_\alpha)$, then, giving a morphism $\underline{\{0\}} \to h^X(D)$ is equivalent to give a compatible family of maps $\{0\} \to \mathrm{Hom}(X, X_i)$, or in other words, to give for each $i \in I$, a morphism $g_i : X \to X_i$ such that $f_\alpha \circ g_i = g_j$, or finaly a morphism $\underline{X} \to D$. ∎

Proposition 1.3.50
1. The following are equivalent:
   (a) The functor F preserves all limits,
   (b) F preserves all products and all kernels.
2. The following are equivalent:
   (a) The functor F is left exact,
   (b) F preserves all finite products and all kernels,
   (c) F preserves all fibered products and the final object.

Analogs for colimits.

Proof. Immediate consequence of exercises 1.3.38 and 1.3.39. ■

Definition 1.3.51
1. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* the limit of a diagram D in $\mathcal{C}$ if any cone $\underline{X} \to D$ satisfying $F(X) \simeq \varprojlim F(D)$ also satisfies $X \simeq \varprojlim D$.
2. The functor F is said to be *conservative* if a morphism f of $\mathcal{C}$ is automatically an isomorphism when $F(f)$ is an isomorphism.

One defines dually the property of reflecting colimits. Also, one says that a functor *reflects* monomorphisms (resp. epimorphisms, resp. isomorphisms) if a morphism f of $\mathcal{C}$ is automatically a monomorphism (resp. an epimorphism, resp. an isomorphism) when $F(f)$ is so. Thus, the functor F is conservative if and only if it reflects isomorphisms if and only if it reflects limits of diagrams of shape $\mathbf{1}$.

Examples
1. A functor that reflects fibered products reflects monomorphisms (and dual)
2. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ reflects limits but the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ does not.

Exercise 1.3.52 Show that
1. a fully faithful functor reflects all limits and colimits,
2. a conservative functor reflects all limits and colimits that it preserves,
3. a faithful functor reflects all monomorphisms and epimorphisms.

Exercise 1.3.53 Show that if $\mathcal{C}$ is balanced, then any faithful functor $F : \mathcal{C} \to \mathcal{D}$ is conservative.

1.3.6 Filtered colimit

Definition 1.3.54 A small category I is said to be *filtered* if any finite diagram in I has a cocone.

A *filtered diagram* is a diagram $I \to \mathcal{C}$ with I filtered. A *filtered colimit* is a colimit of a filtered diagram (and dual).

Example The category associated to a preordered set is filtered if and only if the preordered set is directed (any finite subset has an upper bound).

Exercise 1.3.55 Show that a small category I is filtered if and only if
1. $I \neq \emptyset$,
2. $\forall i, j \in I, \exists k \in I, i \to k, j \to k$,
3. $\forall u, v : i \to j, \exists k \in I, c : j \to k \;/\; c \circ u = c \circ v$.

In the computation of colimits, one may always replace a filtered category with a directed set:

Proposition 1.3.56 If I is a filtered category, then there exists a directed set J and a functor $u : J \to I$ such that, for all diagram $D : I \to \mathcal{C}$, if $\varinjlim(D \circ u)$ exists, then $\varinjlim D$ exists and $\varinjlim(D \circ u) \simeq \varinjlim D$.

Proof. To do (hard). ■

Exercise 1.3.57 Show that

1. a set (resp. a small category) is the filtered colimit of its finite subsets (resp. finite subcategories),
2. a category with finite colimits (resp. finite coproducts) and filtered colimits has all colimits (resp. all coproducts).
3. a functor that preserves finite colimits (resp. finite coproducts) and filtered colimits preserves all colimits (resp. coproducts).

Filtered colimits of *sets* are exact:

Proposition 1.3.58 If I is a *filtered* category, J is a *finite* category and D is a diagram of *sets* of shape $I \times J$, then

$$\varinjlim_I \varprojlim_J D \simeq \varprojlim_J \varinjlim_I D$$

Proof. According to proposition 1.3.56, we may assume that I is a directed set. Moreover, thanks to proposition 1.3.50 it is sufficient to treat the cases of a final object, a product of two objects or a kernel. We shall use the fact that colimits of diagrams are computed termwise (proposition 1.3.42). In particular, the case of a final object is trivial (colimit of a constant diagram). We consider now two families of morphisms $(f_{ij} : X_i \to X_j)$ and $(g_{ij} : Y_i \to Y_j)$ defined for $i < j$ with $f_{jk} \circ f_{ij} = f_{ik}$ and $g_{jk} \circ g_{ij} = g_{ik}$ when $i < j < k$. Since I is directed, we have

$$\varinjlim X_i = \coprod X_i / \sim \quad \text{with} \quad x_i \sim x_j \Leftrightarrow \exists k \geq i, j, f_{ik}(x_i) = f_{jk}(x_j)$$

for $x_i \in X_i$ and $x_j \in X_j$. We shall denote by $\overline{x}_i$ the class of $x_i \in X_i$ and use analogous notations for all diagrams of shape I. By considering colimits on the projections, we have the obvious morphisms

$$\varinjlim(X_i \times Y_i) \to \varinjlim X_i \quad \text{and} \quad \varinjlim(X_i \times Y_i) \to \varinjlim Y_i.$$

The universal property of products provides us with a map

$$\varinjlim(X_i \times Y_i) \to \varinjlim X_i \times \varinjlim Y_i, \quad \overline{(x_i, y_i)} \mapsto (\overline{x}_i, \overline{y}_i)$$

and we have to show that it is bijective. Assume that $(\overline{x}_i, \overline{y}_i) = (\overline{x}'_j, \overline{y}'_j)$. As explained above, there exists $k \geq i, j$ such that $f_{ik}(x_i) = f_{jk}(x'_j)$ and $\ell \geq i, j$ such that $g_{i\ell}(y_i) = g_{j\ell}(y'_j)$. Since I is directed, there exists $m \geq k, \ell$ and we will have $(f_{im}(x_i), g_{im}(y_i)) = (f_{jm}(x'_j), g_{jm}(y'_j))$ so that $\overline{(x_i, y_i)} = \overline{(x'_j, y'_j)}$. This shows injectivity. Surjectivity is proven in the same way. This takes care of the product of two sets and we will now treat the case of kernels. We give ourselves two families of maps $(\varphi_i, \psi_i : X_i \to Y_i)$ compatible with all f_{ij}'s and g_{ij}'s. The universal property of kernels provides us with a map

$$\varinjlim \ker(\varphi_i, \psi_i) \to \ker(\varinjlim \varphi_i, \varinjlim \psi_i).$$

Injectivity and surjectivity are then shown as before, using the explicit description of filtered colimits of sets. ■

Exercise 1.3.59 Show that filtered colimits are exact in **Ab**, **Top**, etc. and that they are preserved by the forgetful functors to **Set** (one may show the second assertion first).

Exercise 1.3.60 Show that **Ab** satisfies AB4* extra condition: arbitrary products are exact.

Exercise 1.3.61 Show that **Ab** satisfies AB6 extra condition: filtered colimits commute with arbitrary products:

$$\prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j} \simeq \varinjlim_{\prod_{j\in J} I_j} \prod_{j\in J} M_{i_j}$$

when each I_j is filtered.

Definition 1.3.62 An *ind-object* "$\varinjlim X_i$" of a category $\mathcal{C}$ is a filtered diagram $(X_i)_{i\in I}$. They form a category $\mathrm{Ind}(\mathcal{C})$ with

$$\mathrm{Hom}(``\varinjlim X_i", ``\varinjlim Y_j") = \varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}(X_i, Y_j).$$

The dual notion is that of a *pro-object* "$\varprojlim X_i$"and they form a category $\mathrm{Pro}(\mathcal{C}) := \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

One gets an equivalent category by considering only directed sets instead of filtered categories.

Example The category of profinite sets (pro-objects of the category of finite sets) is equivalent to the category of totally disconnected compact Hausdorff spaces.

Exercise 1.3.63 Show that the obvious functor $\mathcal{C} \rightarrowtail \mathrm{Ind}(\mathcal{C})$ is fully faithful, exact and preserves all limits.

1.4 Adjointness

1.4.1 Definition

Definition 1.4.1 A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be *adjoint* to a functor $G : \mathcal{C}' \to \mathcal{C}$ if there exists a natural isomorphism

$$\forall X \in \mathcal{C}, X' \in \mathcal{C}', \quad \Phi_{X,X'} : \mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, G(X')).$$

One sometimes writes $F \dashv G$ or $F : \mathcal{C} \leftrightarrows \mathcal{C}' : G$ for short. The dual notion is that of a *coadjoint* so that G is coadjoint to F if and only if F is adjoint to G. It may be useful to write down explicitly what it means for $\Phi_{X,X'}$ and its inverse to be natural: given $g : Y \to X, g' : X' \to Y'$, then

$$\forall f : F(X) \to X', \quad \Phi_{Y,Y'}(g' \circ f \circ F(g)) = G(g') \circ \Phi_{X,X'}(f) \circ g \quad \text{and}$$

$$\forall f' : X' \to G(X), \quad \Phi^{-1}_{Y,Y'}(G(g') \circ f \circ g) = g' \circ \Phi^{-1}_{X,X'}(f) \circ F(g).$$

Examples 1. The forgetful functor **Top** → **Set** has both an adjoint (discrete topology) and a coadjoint (coarse topology):

$$\mathcal{C}(X^{\text{disc}}, Y) \simeq \mathcal{F}(X, Y) \quad \text{et} \quad \mathcal{F}(X, Y) \simeq \mathcal{C}(X, Y^{\text{coarse}}).$$

2. The forgetful functor **Ab** → **Set** has an adjoint (free abelian group):

$$\text{Hom}(\mathbb{Z} \cdot X, M) \simeq \mathcal{F}(X, M).$$

3. The inclusion **Grp** ↣ **Mon** has both an adjoint and a coadjoint: for a group G and a monoid H, we have

$$\text{Hom}(H^{\text{gr}}, G) \simeq \text{Hom}(H, G) \quad \text{et} \quad \text{Hom}(G, H) \simeq \text{Hom}(G, H^{\times}).$$

Exercise 1.4.2 Show that most forgetful and inclusion functors we have already met have an adjoint (and sometimes a coadjoint) and make them explicit.

Exercise 1.4.3 Show that the adjoint to the forgetful functor k-**Alg** → k-**Mod** is the tensor algebra functor $M \mapsto \text{T}(M)$. Same thing with $\text{S}(M)$ when we restrict to commutative algebras.

Exercise 1.4.4 Sow that, if $f : A \to B$ is a morphism of rings, then the forgetful functor B-**Mod** → A-**Mod** has both an adjoint $M \mapsto B \otimes_A M$ and a coadjoint $M \mapsto \text{Hom}_A(B, M)$.

Exercise 1.4.5 Show that (for fixed Y) the functor $X \mapsto X \times Y$ from **Set** to itself is adjoint to the functor $Z \mapsto \mathcal{F}(Y, Z)$:

$$\mathcal{F}(X \times Y, Z) \simeq \mathcal{F}(X, \mathcal{F}(Y, Z)).$$

This is called *Currying*. Write down the analogous statements for **Cat** and **Ab**.

Exercise 1.4.6 Show that if both F_1 and F_2 are adjoint to G, then $F_1 \simeq F_2$ (and dual).

Solution. Both $F_1(X)$ and $F_2(X)$ represent the same functor $X' \mapsto \text{Hom}(X, G(X'))$ and there exists therefore an isomorphism $F_1(X) \simeq F_2(X)$ which is easily seen to be natural. ■

Exercise 1.4.7 Show that if $F_1 : \mathcal{C} \leftrightarrows \mathcal{C}' : G_1$ and $F_2 : \mathcal{C}' \leftrightarrows \mathcal{C}'' : G_2$ then, $F_2 \circ F_1 : \mathcal{C} \leftrightarrows \mathcal{C}'' : G_1 \circ G_2$.

1.4.2 Unit and counit

Proposition 1.4.8 A functor $F : \mathcal{C} \to \mathcal{C}'$ is adjoint to $G : \mathcal{C}' \to \mathcal{C}$ if and only if there exists $\alpha : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\beta : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{C}'}$ such that $\beta_F \circ F(\alpha) = \mathrm{Id}_F$ and $G(\beta) \circ \alpha_G = \mathrm{Id}_G$.

The condition means that the diagrams

$$\circlearrowright F(X) \underset{\beta_{F(X)}}{\overset{F(\alpha_X)}{\rightleftarrows}} F(G(F(X))) \quad \text{and} \quad G(F(G(X'))) \underset{\alpha_{G(X')}}{\overset{G(\beta_{X'})}{\rightleftarrows}} G(X') \circlearrowleft$$

are always commutative.

Proof. Assume first that there exists a natural isomorphism

$$\Phi_{X,X'} : \mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, G(X')).$$

We may then set

$$\alpha_X := \Phi_{X,F(X)}(\mathrm{Id}_{F(X)}) \text{ and } \beta_{X'} := \Phi^{-1}_{G(X'),X'}(\mathrm{Id}_{G(X')}).$$

Then, we have

$$\begin{aligned} \beta_{F(X)} \circ F(\alpha_X) &= \Phi^{-1}_{G(F(X)),F(X)}(\mathrm{Id}_{G(F(X))}) \circ F(\Phi_{X,F(X)}(\mathrm{Id}_{F(X)})) \\ &= \Phi^{-1}_{X,F(X)}(\Phi_{X,F(X)}(\mathrm{Id}_{F(X)})) \\ &= \mathrm{Id}_{F(X)} \end{aligned}$$

and dually.

Conversely, given $f : F(X) \to X'$, we set $\Phi_{X,X'}(f) = G(f) \circ \alpha_X$ and define dually for $f' : X \to G(X')$, $\Psi_{X,X'}(f') := \beta_{X'} \circ F(f')$. We shall then have

$$(\Psi_{X,X'} \circ \Phi_{X,X'})(f) = \beta_{X'} \circ F(G(f)) \circ F(\alpha_X) = f \circ \beta_{F(X)} \circ F(\alpha_X) = f$$

and symmetrically by duality. ∎

Definition 1.4.9 The natural transformations α and β are then called *unit* and *counit* for the adjunction.

Examples

1. In the case of topological spaces and sets, for the discrete/forget (resp. forget/coarse) adjunction, we get the identity for both unit and counit, the counit (resp. unit) being the canonical continuous map $X^{\mathrm{disc}} \to X$ (resp. $X \to X^{\mathrm{coarse}}$).
2. In the case of abelian groups and sets, for the free/forget adjunction, we have unit $E \rightarrowtail \mathbb{Z} \cdot E, x \mapsto 1 \cdot x$ and counit $\mathbb{Z} \cdot M \twoheadrightarrow M, \sum a_i \cdot x_i \mapsto \sum a_i x_i$.
3. In the case of groups and monoids, for the gr/forget (resp. forget/$\times$) adjunction, we get inclusion $G \rightarrowtail G^{\mathrm{gr}}$ and identity (resp. identity and inclusion $G \rightarrowtail G^{\times}$) for unit and counit respectively.

Proposition 1.4.10 If F is adjoint to G with unit α and counit β, then F is faithful (resp. fully faithful) if and only if α_X is always a monomorphism (resp. an isomorphism). And dually.

Proof. For $X, Y \in \mathcal{C}$, there exists a commutative diagram

$$\begin{array}{ccc} & \mathrm{Hom}(Y,X) & \\ {\scriptstyle F}\swarrow & & \searrow{\scriptstyle \alpha_{X*}} \\ \mathrm{Hom}(F(Y),F(X)) & \xrightarrow{\sim} & \mathrm{Hom}(Y,G(F(X))). \end{array}$$

This follows from from the fact that α is a natural transformation so that

$$\Phi_{Y,F(X)}(F(f)) = G(F(f)) \circ \alpha_Y = \alpha_X \circ f.$$

Thus we see that the map F is injective (resp. bijective) for all X, Y in $\mathcal{C}$ if and only tif this is the case for α_{X*}, which means that α_X is a monomorphism (resp. an isomorphism) for all $X \in \mathcal{C}$.

Now, we have G^{op} is adjoint to F^{op} and the unit for this adjunction is β^{op}. Moreover, $\beta^{\mathrm{op}}_{X'}$ is a monomorphism (resp. an isomorphism) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism). Therefore, G is faithful (resp. fully faithful) if and only if G^{op} is faithful (resp. fully faithful) if and only if $\beta_{X'}$ is an epimorphism (resp. an isomorphism) for all X'. ■

Exercise 1.4.11 Describe unit and counit in all the examples studied so far. Deduce in each case faithfulness or full faithfulness of the functors.

Exercise 1.4.12 Show that, if a small category $\mathcal{C}$ has (self) coproducts, then all representable functors F on $\mathcal{C}$ have an adjoint.

Hint. We may assume that $F = h^X$, consider the functor

$$\mathrm{Set} \to \mathcal{C}, \quad I \mapsto X^{(I)} = \coprod_I X$$

define unit $I \to \mathrm{Hom}(X, X^{(I)})$ and counit $X^{(\mathrm{Hom}(X,Y))} \to Y$ and check the properties. ■

1.4.3 Adjoint and limit

Proposition 1.4.13 All limits of shape I exist in a category $\mathcal{C}$ if and only if the functor $X \mapsto \underline{X}$ has a coadjoint which is then given by[a] $D \mapsto \varprojlim D$.

[a] Once a specific choice of the limit is made for each diagram D.

Proof. We have indeed a natural isomorphism in X given by

$$\mathrm{Hom}(\underline{X}, D) \simeq \mathrm{Hom}(X, \varprojlim D)$$

and it follows from exercise 1.3.6 that it is also natural in D. ■

Exercise 1.4.14 Show that any adjunction between two functors F and G extends to an adjunction on diagrams of a given shape I:

$$\mathrm{Hom}(F(D), E) \simeq \mathrm{Hom}(D, G(E)).$$

Corollary 1.4.15 A functor that has an adjoint preserves all limits (and dual).

Proof. We have to show that, if G has an adjoint F and D is a diagram, then $G(\varprojlim D) \simeq \varprojlim G(D)$. This follows by adjunction from $F(\underline{X}) = \underline{F(X)}$. ■

As a consequence, we recover the result from proposition 1.3.42: limits commute with limits (and dual).

Example As we already noticed, the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ preserves all limits and all colimits, the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ preserves all limits and the inclusion functor $\mathbf{Mon} \rightarrowtail \mathbf{Grp}$ preserves all limits and colimits.

Theorem 1.4.15 has a partial converse:

Theorem 1.4.16 — Freyd adjunction theorem. If $\mathcal{C}'$ be a complete category and $G : \mathcal{C}' \to \mathcal{C}$ a functor that preserves all limits and satisfies the *solution-set condition* below, then it has an adjoint (and dual).

Solution-set condition: given any X in $\mathcal{C}$, there exists a *set* of morphisms $X \to G(Y_i)$ such that any morphism $X \to G(Y)$ factors through some $G(Y_i)$.

Proof. (Sketch) It is sufficient to set

$$F(X) := \varprojlim_{X \to G(Y_i)} Y_i.$$ ■

Theorem 1.4.17 Assume $F : \mathcal{C} \to \mathcal{C}'$ is adjoint to a fully faithful functor G. If D' is a diagram in $\mathcal{C}'$ and $X = \varprojlim G(D')$, then $X' := F(X) = \varprojlim D'$ and $X \simeq G(X')$.

Proof. To do (corollary 5.6.6 of [Riehl16]). ■

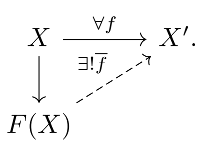
1.4.4 Reflective subcategory

Definition 1.4.18 A full subcategory $\mathcal{C}' \subset \mathcal{C}$ is said to be *reflective* if the inclusion functor has an adjoint, called *reflection*. The dual terminology is *coreflective*.

In other words, a functor $F : \mathcal{C} \to \mathcal{C}'$ is a reflection if there exists a natural bijection

$$\mathrm{Hom}(F(X), X') \simeq \mathrm{Hom}(X, X')$$

when $X' \in \mathcal{C}'$. It means that there exists a natural morphism $X \to F(X)$ with the following universal property:



Examples
1. The category of preordered sets is a reflective subcategory of the category of posets.
2. The category of abelian groups is a reflective subcategory of the category of all groups.
3. The category of groups is both a reflective and a coreflective subcategory of the category of all monoids.
4. The category of compact Hausdorff spaces is a reflective subcategory of the category of all topological spaces.
5. The category of sets is isomorphic to the reflective (resp. coreflective) subcategory of discrete (resp. chaotic) topological spaces.

Exercise 1.4.19 Show that if $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$, then any diagram D' in $\mathcal{C}'$ that has a limit (resp. a colimit) in $\mathcal{C}$ has a limit (resp. a colimit) in $\mathcal{C}'$. Show also that $\mathcal{C}'$ is stable under limits that exist in $\mathcal{C}$. Finally, show that, if a colimit in $\mathcal{C}$ of objects of $\mathcal{C}'$ is an object of $\mathcal{C}'$, then this is a colimit in $\mathcal{C}'$.

Exercise 1.4.20 Show that, if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$, then a functor $F : \mathcal{C} \to \mathcal{C}'$ is a reflection if and only if there exists a natural morphism $\alpha_X : X \to F(X)$ for $X \in \mathcal{C}$ such that α_X is an isomorphism when $X \in \mathcal{C}'$.

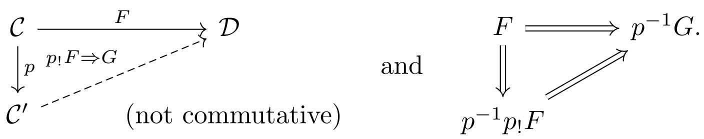
Solution. The subcategory $\mathcal{C}'$ is reflective with reflection F if and only if there exists a natural morphism $\alpha_X : X \to F(X)$ for $X \in \mathcal{C}$ and a natural isomorphism $\beta_X : F(X) \simeq X$ when $X \in \mathcal{C}'$ such that $\beta_{F(X)} \circ F(\alpha_X) = \mathrm{Id}_{F(X)}$ when $X \in \mathcal{C}$ and $\beta_X \circ \alpha_X = \mathrm{id}_X$ when $X \in \mathcal{C}'$. This is clearly equivalent to our condition with $\beta_X := \alpha_X^{-1}$. ■

Exercise 1.4.21 Show that, if filtered colimits exist in $\mathcal{C}$, then $\mathcal{C}$ is a reflective subcategory of $\mathrm{Ind}(\mathcal{C})$ with adjoint "$\varinjlim X_i$" $\mapsto \varinjlim X_i$.

1.4.5 Kan extension

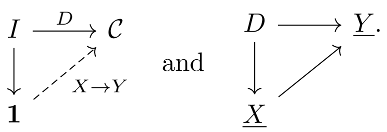
Definition 1.4.22 Let $p : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. The *(left) Kan extension* of a functor $F : \mathcal{C} \to \mathcal{D}$ along p is a functor $p!F$ which is universal for all functors $G : \mathcal{C}' \to \mathcal{D}$ and natural transformations $F \Rightarrow p^{-1}G := G \circ p$.

In other words, $p!F$ represents the functor $G \mapsto \mathrm{Hom}(F, G \circ p)$ on the category $\mathbf{Hom}(\mathcal{C}', \mathcal{D})$. It means that $p!F : \mathcal{C}' \to \mathcal{D}$ is endowed with a natural transformation $\alpha : F \Rightarrow p^{-1}p_!F$ such that, given any natural transformation $\gamma : F \Rightarrow p^{-1}G$, there exists a unique natural transformation $\widetilde{\gamma} : p_!F \Rightarrow G$ such that $\gamma = p^{-1}(\widetilde{\gamma}) \circ \alpha$:

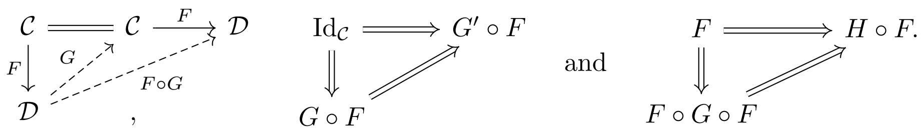


There exists the dual notion of a *right Kan extension* p_*F with $\gamma : p^{-1}G \Rightarrow F$ this time.

Examples
1. A diagram $D : I \to \mathcal{C}$ has colimit X if and only if the constant functor $\mathbf{1} \to \mathcal{C}, 0 \mapsto X$ is the Kan extenstion of D along the projection $I \to \mathbf{1}$:



2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories has a coadjoint G if and only if the Kan extension of $\mathrm{Id}_\mathcal{C}$ along F exists and $F \circ F_!\mathrm{Id}_\mathcal{C} = F!F$, in which case $G = F_!\mathrm{Id}_\mathcal{C}$:



Proposition 1.4.23 Let $p : \mathcal{C} \to \mathcal{C}'$ be functor between small categories. Then the functor

$$p^{-1} : \mathbf{Hom}(\mathcal{C}', \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D}), \quad G \mapsto G \circ p.$$

has an adjoint $p_!$ if and only if all Kan extensions along p with values in $\mathcal{D}$ exist (and dual).

Proof. Follows immediately from the definition. ■

Proposition 1.4.24 If all colimits exist in $\mathcal{D}$, then the Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $p : \mathcal{C} \to \mathcal{C}'$ always exists (and dual).

Proof. (Sketch) We set

$$(p_! F)(X') := \varinjlim_{p(X) \to X'} F(X)$$

and check. ■

Exercise 1.4.25 Show that if $\mathcal{C}$ is small, all colimits exist in $\mathcal{D}$ and $F : \mathcal{C} \rightarrowtail \mathcal{C}'$ is fully faithful, then (the above diagram commutes:) $F \simeq p!F \circ p$ (and dual).

Exercise 1.4.26 Show that, if $g : \mathcal{C} \to \mathcal{C}'$ is a functor between small categories and all colimits exist in $\mathcal{D}$, then the functor

$$\widehat{g}^{-1} : \widehat{\mathcal{C}'}(\mathcal{D}) \to \widehat{\mathcal{C}}(\mathcal{D}), \quad T' \mapsto T' \circ g.$$

has an adjoint $\widehat{g}_!$ (resp. a coadjoint $\widehat{g}_*$) $: \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$.

Solution. Follows from propositions 1.4.23 and 1.4.24. ■

1.5 Miscellaneous

1.5.1 Algebraic structure

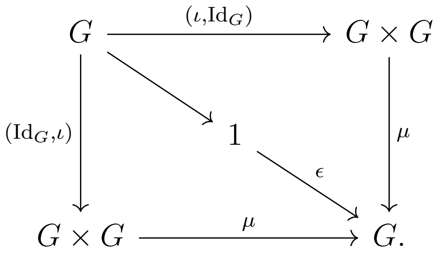
Definition 1.5.1 Let $\mathcal{C}$ be a *cartesian* category. A *monoid* of $\mathcal{C}$ is an object G endowed with a *multiplication morphism* $\mu : G \times G \to G$ and a *unit morphism*[a] $\epsilon : 1 \to G$ making commutative the following diagrams:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \mathrm{Id}_G} & G \times G \\ \downarrow{\scriptstyle \mathrm{Id}_G \times \mu} & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\epsilon \times \mathrm{Id}_G} & G \times G \\ \downarrow{\scriptstyle \mathrm{Id}_G \times \epsilon} & \searrow\!\!\searrow & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

(under the identification $1 \times G \simeq G \simeq G \times 1$). A *morphism of monoids* $G \to G'$ of $\mathcal{C}$ is a morphism $f : G \to G'$ in $\mathcal{C}$ making the following commutative:

$$\begin{array}{ccc} G \times G & \xrightarrow{f \times f'} & G' \times G' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ G & \xrightarrow{f} & G' \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{f} & G'. \\ {\scriptstyle \epsilon}\nwarrow & & \nearrow{\scriptstyle \epsilon'} \\ & 1 & \end{array}$$

It is a *group* if there exists an *inversion morphism* $\iota : G \to G$ making commutative



Il is *abelian* if $G^{\mathrm{op}} = G$ where G^{op} denotes the monoid obtained by exchanging factors in $G \times G$ (composing μ with the flip morphism).

[a]Automatically unique.

Monoids (resp. groups, resp. abelian groups) of $\mathcal{C}$ make a category $\mathbf{Mon}(\mathcal{C})$ (resp. $\mathbf{Grp}(\mathcal{C})$, resp. $\mathbf{Ab}(\mathcal{C})$). We shall mostly concentrate on abelian groups.

Examples

1. A group of the category **Set** is nothing but a usual group.
2. A group of **Top** is a topological group (with continuous multiplication and continuous inversion).
3. A group in the category of smooth manifolds is a *Lie group*.
4. An abelian group in the category of abelian groups is an abelian group (!).

Exercise 1.5.2 Show that $k[t]$ (resp. $k[t]_t$) endowed with $t \mapsto t \otimes 1 + 1 \otimes t$ (resp. $t \mapsto t \otimes t$) is an abelian group of the category opposite to the category of k-algebras (it is called a *bialgebra*).

Solution. Commutativity is clear in both cases and makes it faster to check the other properties. To prove associativity, we compute

$$\begin{aligned}((\mathrm{Id} \otimes \mu) \circ \mu)(t) &= (\mathrm{Id} \otimes \mu)(t \otimes 1 + 1 \otimes t)\\ &= t \otimes \mu(1) + 1 \otimes \mu(t)\\ &= t \otimes 1 \otimes 1 + 1 \otimes (t \otimes 1 + 1 \otimes t)\\ &= t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t.\end{aligned}$$

By symmetry, we see that

$$((\mathrm{Id} \otimes \mu) \circ \mu)(t) = (\mu \otimes \mathrm{Id}) \circ \mu)(t).$$

In the same way, we have

$$((\mathrm{Id} \otimes \nu) \circ \nu)(t) = t \otimes t \otimes t = (\nu \otimes \mathrm{Id}) \circ \nu)(t).$$

The unit is $\epsilon : k[t] \;\to k, t \mapsto 0$ (resp. $\eta : k[t]_t \;\to k, t \mapsto 1$) as the following shows:

$$((\epsilon \otimes \mathrm{Id}) \circ \mu)(t) = (\epsilon \otimes \mathrm{Id})(t \otimes 1 + 1 \otimes t) = \epsilon(t) \times 1 + \epsilon(1) \times t = 0 \times 1 + 1 \times t = t.$$

$$(\text{resp. } (\eta \otimes \mathrm{Id}) \circ \nu)(t) = (\eta \otimes \mathrm{Id})(t \otimes t) = \eta(t) \times t = 1 \times t = t).$$

Finally, the inverse is given by $i : t \mapsto -t$ (resp. $j : t \mapsto t^{-1}$): if we to compose μ (resp. ν) with the map

$$\begin{array}{ccc} k[t] \otimes_k k[t] & \longrightarrow & k[t] \\ f \otimes g & \longmapsto & f(-t)g(t) \end{array} \quad \left(\text{resp.} \quad \begin{array}{ccc} k[t]_t \otimes_k k[t]_t & \longrightarrow & k[t]_t \\ f \otimes g & \longmapsto & f(t^{-1})g(t) \end{array} \right),$$

then we get the constant map 0 (resp. 1). ■

Exercise 1.5.3 Show that, if all limits exist in $\mathcal{C}$, then the same holds in $\mathbf{Ab}(\mathcal{C})$ and they are preserved by the forgetful functor $\mathbf{Ab}(\mathcal{C}) \to \mathcal{C}$.

Exercise 1.5.4 Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between cartesian categories. Show that

1. if F preserves finite products, then F induces a functor still written $F : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{C}')$,
2. if moreover, F preserves all limits of $\mathcal{C}$, then it preserves all limits of $\mathbf{Ab}(\mathcal{C})$.

The category of abelian groups in a category of presheaves of sets can be identified with the category of presheaves of abelian groups:

Exercise 1.5.5 Show that if $\mathcal{C}$ is a small category, then there exists an isomorphism of categories $\widehat{\mathcal{C}}(\mathbf{Ab}) \simeq \mathbf{Ab}(\widehat{\mathcal{C}})$.

Solution. For fixed $X \in \mathcal{C}$, the functor $\widehat{\mathcal{C}} \to \mathbf{Set}, F \mapsto F(X)$ preserves finite products (actually all limits and colimits: see for example exercise 1.3.48 below). Therefore, if $\mathcal{M} \in \mathbf{Ab}(\widehat{\mathcal{C}})$ and $X \in \mathcal{C}$, then $\mathcal{M}(X)$ is a usual abelian group and this is clearly functorial. Conversely, if $\mathcal{M} \in \widehat{\mathcal{C}}(\mathbf{Ab})$, then the obvious family of maps

$$\mu_X : \mathcal{M}(X) \times \mathcal{M}(X) \to \mathcal{M}(X), \quad \epsilon_X : \{0\} \to \mathcal{M}(X) \quad \text{and} \quad \iota_X : \mathcal{M}(X) \to \mathcal{M}(X)$$

define a structure of abelian group on the underlying presheaf of sets of $\mathcal{M}$. These two constructions are clearly inverse to each other. ■

Exercise 1.5.6 Show that if $\mathcal{C}$ is a small cartesian category, then there exists a fully faithful functor $\mathbf{Ab}(\mathcal{C}) \rightarrowtail \widehat{\mathcal{C}}(\mathbf{Ab})$.

Definition 1.5.7 Let $\mathcal{C}$ be a cartesian category.

1. Let (G, μ) be a monoid of $\mathcal{C}$ with unit ϵ. An *action* of G on an object E of $\mathcal{C}$ is a *morphism* $\rho : G \times E \to E$ making commutative the following diagrams:

$$\begin{array}{ccc} G \times G \times E & \xrightarrow{\mu \times \mathrm{Id}_E} & G \times E \\ \downarrow{\scriptstyle \mathrm{Id}_G \times \rho} & & \downarrow{\scriptstyle \mu} \\ G \times E & \xrightarrow{\mu} & G \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\epsilon \times \mathrm{Id}_G} & G \times E \\ & \searrow{=} & \downarrow{\scriptstyle \mu} \\ & & E. \end{array}$$

2. A *ring* of $\mathcal{C}$ is an object A endowed with a structure of abelian group (given

by μ) and a structure of monoid (given by ν) making commutative

$$\begin{array}{ccccc} A\times A\times A & \xrightarrow{\mathrm{Id}_A\times\mu} & A\times A & \overset{\nu}{\searrow} & \\ \downarrow & & & & A \\ A\times A\times A\times A & \xrightarrow{\nu\times\nu} & A\times A & \underset{\mu}{\nearrow} & \end{array}$$

with left distribution on the left as well as the analogous diagram for right distribution:

$$\begin{array}{ccccc} A\times A\times A & \xrightarrow{\mu\times\mathrm{Id}_M} & A\times A & \overset{\nu}{\searrow} & \\ \downarrow & & & & A. \\ A\times A\times A\times A & \xrightarrow{\nu\times\nu} & A\times A & \underset{\mu}{\nearrow} & \end{array}$$

3. Let (A,μ,ν) be a ring of $\mathcal{C}$. An A-*module* is an abelian group (M,μ) of $\mathcal{C}$ endowed with an action ρ of (A,ν) making commutative both

$$\begin{array}{ccccc} A\times M\times M & \xrightarrow{\mathrm{Id}_A\times\mu} & A\times M & \overset{\rho}{\searrow} & \\ \downarrow & & & & M \\ A\times M\times A\times M & \xrightarrow{\rho\times\rho} & M\times M & \underset{\mu}{\nearrow} & \end{array}$$

and

$$\begin{array}{ccccc} A\times A\times M & \xrightarrow{\mu\times\mathrm{Id}_M} & A\times M & \overset{\rho}{\searrow} & \\ \downarrow & & & & M. \\ A\times M\times A\times M & \xrightarrow{\rho\times\rho} & M\times M & \underset{\mu}{\nearrow} & \end{array}$$

They form a category G-**S**et$(\mathcal{C})$ (resp. **R**ing$(\mathcal{C})$, resp. A-**M**od$(\mathcal{C})$).

Example We consider the category $\mathcal{C}^k$-**M**an of $\mathcal{C}^k$-differentiable manifolds and $\mathcal{C}^k$-differentiable maps (works with $k = 0, 1, \ldots, \infty$, an and even hol). A group in this category is a $\mathcal{C}^k$-differentiable manifold G endowed with a group structure such that both multiplication $\mu : G \times G \to G$ and inversion $i : G \to G$ are $\mathcal{C}^k$-differentiable. A G^{op}-object is a $\mathcal{C}^k$-differentiable manifold X endowed with a right action $X \times G \to X$ which is $\mathcal{C}^k$-differentiable. For example, the *trivial action* is the (second) projection. Assume that X is endowed with the *trivial* action and consider the category G^{op}-$\mathcal{C}^k$-**M**an$_{/X}$ of G-objects $\pi : Y \to X$ over X. Such an object restricts any open subset U of X by setting $Y_{|U} := \pi^{-1}(U)$ and considering $Y_{|U} \to U$ together with $Y_{|U} \times G \to Y_{|U}$. Such an object is said to be *trivial* if it is isomorphic to the first projection $X \times G \to X$ endowed with $\mathrm{Id}_X \times \mu : X \times G \times G \mapsto X \times G$. It is called a G-*torsor* or a *principal* G-*bundle* if it is *locally trivial*: there exists a covering of X by open subsets U such that Y_U is trivial. We can consider the full subcategory of all G-torsors over X. It is an example of a *groupoid*: any morphism is an isomorphism.

Exercise 1.5.8 Define the category k-**A**lg$(\mathcal{C})$ of k-algebras of a cartesian category $\mathcal{C}$ when k is a commutative ring of $\mathcal{C}$.

Exercise 1.5.9 Show that $k[t]$ endowed with $t \mapsto t \otimes 1 + 1 \otimes t$ and $t \mapsto t \otimes t$ is a commutative ring of the category opposite to the category of k-algebras.

1.5.2 Projective/Injective

Definition 1.5.10 An object X of a category $\mathcal{C}$ is said to be *projective* if h^X preserves epimorphisms.

The dual notion is that of an *injective* object: h_X sends monomorphisms to epimorphisms. Thus X is projective (resp. injective) if whenever $Z \twoheadrightarrow Y$ is an epimorphism (resp. $Y \rightarrowtail Z$ is a monomorphism), then the corresponding map

$$\mathrm{Hom}(X, Z) \twoheadrightarrow \mathrm{Hom}(X, Y) \quad (\text{resp. } \mathrm{Hom}(Z, X) \twoheadrightarrow \mathrm{Hom}(Y, X))$$

is surjective. In other words, X is projective (resp. injective) when any diagram

can be completed with the dotted arrow.

Examples
1. In **Set**, all objects (resp. non empty objects) are projective[7] (resp. injective).
2. In **Ab**, projective objects are free abelian groups and injective objects are divisible groups (proposition 1.5.14 below).
3. In R-**Mod**, projective objects are direct factors of free R-modules ($\mathbb{Z}/2$ is projective – but not free – over $\mathbb{Z}/6$).
4. If E is a fiber bundle on a compact manifold X, then $E(X)$ is a projective $\mathcal{O}(X)$-module (requires an argument).
5. The projective objects of the category of compact Hausdorff spaces are the *Stonean* (meaning extremally disconnected[8]) compact Hausdorff spaces.

Exercise 1.5.11 Show that a coproduct of projectives is projective (and dual).

Exercise 1.5.12 Show that if X is projective, then
1. any epimorphism $Y \twoheadrightarrow X$ is split,
2. if a $Y \rightarrowtail X$ (resp. $X \twoheadrightarrow Y$) is a split monomorphism (resp epimorphism), then Y also is projective (and dual).

Proposition 1.5.13 When epimorphisms are stable under pullback, an object X is projective if and only if any epimorphism $Y \twoheadrightarrow X$ is split (and dual).

Proof. Pulling back an epimorphism $Z \twoheadrightarrow Y$ along a morphism $X \to Y$ provides an epimorphism $X \times_Y Z \twoheadrightarrow X$ that has a section $X \to X \times_Y Z$ that we can compose

[7] This is equivalent to the axiom of choice.
[8] Closure of open is open.

with the projection $X \times_Y Z \to Z$ to get a lifting $X \to Z$ of the original map $X \to Y$ along the epimorphism $Z \twoheadrightarrow Y$:

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & Z \\ \circlearrowleft\downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

■

Proposition 1.5.14 An abelian group Q is injective if and only if it is divisible.

Proof. Assume Q injective. If $a \in Q$ and $n > 0$, then the map $\mathbb{Z} \to Q, 1 \mapsto a$ extends along the injective map $\mathbb{Z} \overset{n}{\rightarrowtail} \mathbb{Z}$ to a map $1 \mapsto b$ with $b = na$. Contradiction.

Assume conversely that Q is divisible and we are given a morphism $f : M \to Q$ and an injective map $M \rightarrowtail N$. By Zorn's lemma, there exists a maximal subgroup M' of N such that f extends to M'. We may assume $M' = M$. If $a \notin M$, then $M \cap \langle a \rangle = \langle na \rangle$. In Q, we can write $f(na) = nx$ and extend f to $M + \langle a \rangle$ by sending a to x (works also for $n = 0$). ■

Definition 1.5.15 A category $\mathcal{C}$ is said to have *enough projectives* if, given any $X \in \mathcal{C}$, there exists a projective Y and an epimorphism $Y \twoheadrightarrow X$ (and dual).

Thus $\mathcal{C}$ has *enough injectives* if there always exists a monomorphism $X \rightarrowtail Y$ with Y injective.

Examples
1. The category **Set** has enough projectives and injectives.
2. The category R-**Mod** has enough projectives and injectives (we shall prove this last statement later).
3. The category of compact Hausdorff spaces has enough projectives.

Exercise 1.5.16 Show that
1. if each X_i is projective, then $(X_i)_{i\in I}$ is projective in $\mathcal{C} := \prod_{i\in I} \mathcal{C}_i$,
2. if each $\mathcal{C}_i$ has enough projectives, so does $\mathcal{C}$.

We shall extend later the next statement to A-modules.

Proposition 1.5.17 The category of abelian groups has enough projectives and injectives.

Proof. We can always write $M = F/N$ with F free and this shows that there are enough projectives. Consider now the embedding $M \hookrightarrow F_{\mathbb{Q}}/N$ with $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} F$. Since a $\mathbb{Q}$-vector space is divisible and a quotient of divisible is automatically divisible, this shows that there are enough injectives too. ■

Definition 1.5.18 An object X of a category $\mathcal{C}$ is said to be *compactly presented* (or *compact*) if h^X preserves filtered colimits.

In other words,

$$\varinjlim \mathrm{Hom}(X, Y_i) \simeq \mathrm{Hom}(X, \varinjlim Y_i)$$

when (Y_i) is filtered. It means that any morphism $X \to \varinjlim Y_i$ factors through some Y_i.

Examples
1. A set is compactly presented if and only if it is finite.
2. A topological space is compactly presented if and only if it is finite discrete.
3. An abelian group is compaclty presented (resp. compactly presented projective) if and only if it is finitely generated (resp. free of finite rank).
4. An open subset U of a topological space X is compact (in the usual sense) if and only if U is a compactly presented object of $\mathrm{Open}(X)$.

Definition 1.5.19 A set $S \subset \mathcal{C}$ is a *set of generators* (or *separators*) for a category $\mathcal{C}$ if the functor $\prod_{G \in S} h^G$ is faithful[a]. When $S = \{G\}$, we say that G is a *generator*.

[a] On should say separators (resp. generators) when the functor is faithful (resp. conservative)

In down to earth terms, it means that if $f_1 \neq f_2 : X \to Y$, then there exists $g : G \to X$ with $G \in S$ such that $f_1 \circ g \neq f_2 \circ g$.

Examples
1. $1 := \{0\}$ is a generator for **Set**.
2. A is a generator for A-**Mod**.
3. If $\mathcal{C}$ is a category, then $\{h_X, X \in \mathcal{C}\}$ is a set of generators for $\widehat{\mathcal{C}}$.

Exercise 1.5.20 Show that, if $\mathcal{C}$ has all coproducts, then S is a set of generators if and only if there exists for all $X \in \mathcal{C}$ an epimorphism $\coprod_{i \in I} G_i \twoheadrightarrow X$ with $G_i \in S$.

Proof. For the converse, choose

$$\coprod_{G \in S, f : G \to X} G \twoheadrightarrow X.$$

■

1.5.3 Localization

In this sketchy section, we sweep set theoretical questions under the rug.

Definition 1.5.21 The *localization* of a (small) category $\mathcal{C}$ with respect to a collection of morphisms W of $\mathcal{C}$ is a category $W^{-1}\mathcal{C}$ which is universal for functors $\mathcal{C} \to \mathcal{D}$ sending W to isomorphisms in $\mathcal{D}$.

It means that there exists a functor $\gamma : \mathcal{C} \to W^{-1}\mathcal{C}$ sending W to isomorphisms such that, given any category $\mathcal{D}$, the functor

$$\gamma^* : \mathbf{Hom}(W^{-1}\mathcal{C}, \mathcal{D}) \to \mathbf{Hom}(\mathcal{C}, \mathcal{D})$$

induces an equivalence with the full subcategory of functors sending W is isomorphisms.

Exercise 1.5.22 An *isogeny* is a homomorphism of abelian groups $f : M \to N$ such that
1. $\forall y \in N, \forall n \in \mathbb{Z} \setminus 0, \exists x \in M, f(nx) = y,$
2. $\forall x \in M, f(x) = 0 \Rightarrow \exists n \in \mathbb{Z} \setminus 0, nx = 0.$

Show that, if W is the collection of all isogenies, then $W^{-1}\mathbf{Ab} \simeq \mathbb{Q}$-$\mathbf{Vec}$ $(:= \mathbb{Q}$-$\mathbf{Mod})$.

Exercise 1.5.23 The category of topological spaces up to homotopy has topological spaces as objects but sets of morphisms $[X, Y] := \mathcal{C}(X, Y)/\sim$ and composition induced by usual composition. Show that it is equivalent to $W^{-1}\mathbf{Top}$ if W denotes the collection of homotopy equivalences.

Proposition 1.5.24 The localization $W^{-1}\mathcal{C}$ of a (small) category $\mathcal{C}$ with respect to a set of morphisms W always exists.

Proof. (Sketch) We may assume that W contains all isomorphisms in $\mathcal{C}$. Then, the objects of $W^{-1}\mathcal{C}$ are the objects of $\mathcal{C}$ and morphisms[9] are finite chains

$$X = X_0 \overset{W}{\leftarrow} X_1 \rightarrow X_2 \overset{W}{\leftarrow} \cdots \rightarrow X_{n-1} \overset{W}{\leftarrow} X_n \rightarrow X_{n+1} = Y$$

up to some equivalence. ■

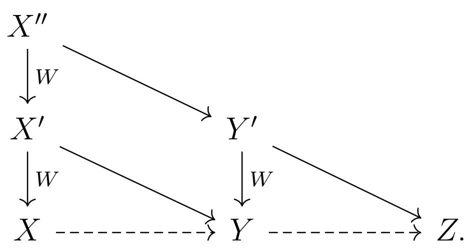
Definition 1.5.25 A category $\mathcal{C}$ admits *right calculus of fractions* with respect to a set of morphisms W if

1. W contains all identities and is stable under composition,
2. given any $f : X \to Y$ in $\mathcal{C}$ and $\varphi : Y' \to Y$ in W, there always exists $f' : X' \to Y'$ and $\varphi' : X' \to X$ in W with $\varphi \circ f' = f \circ \varphi'$,
3. given any $f, g : X \to Y'$ in $\mathcal{C}$ and $\psi : Y' \to Y$ in W such that $\psi \circ f = \psi \circ g$, there exists $\varphi : X' \to X$ in W such that $f \circ \varphi = g \circ \varphi$.

Proposition 1.5.26 If a (small) category $\mathcal{C}$ admits right calculus of fraction with respect to W, then $W^{-1}\mathcal{C}$ is the category having the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{W^{-1}\mathcal{C}}(X, Y) = \varinjlim_{X' \to X \in W} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

Proof. (Sketch) By definition, morphisms and composition are described, up to equivalence, by the following diagram



It is then a matter of checking the various properties. ■

Exercise 1.5.27 Show that if $\mathcal{C}$ admits right calculus of fraction with respect to W, then the localization functor $Q : \mathcal{C} \to W^{-1}\mathcal{C}$ is left exact.

[9] They form a collection and not merely a set.

2. Linear algebra

In the same way as general categories are modeled on the category of sets, there exist the notion of an abelian category that is modeled on the category of abelian groups.

2.1 Additive structure

2.1.1 Preadditive category

Definition 2.1.1 A *pre-additive category* (also called $\mathbf{A}$b-*category*[a] or *ringoid*) is a category $\mathcal{P}$ endowed with a factorization of the Hom functor:

$$\begin{array}{rcl} \mathrm{Hom}: \ \mathcal{P}^{\mathrm{op}} \times \mathcal{P} & \longrightarrow & \mathbf{S}\mathrm{et} \\ & \searrow & \uparrow \\ & & \mathbf{A}\mathrm{b}. \end{array}$$

[a]This is a particular instance of the notion of an *enriched* category.

In other words, $\mathcal{P}$ is preadditive if and only if, for all $M, N \in \mathcal{P}$, $\mathrm{Hom}(M, N)$ is (endowed with the structure of) an abelian group and for all $M, N, P \in \mathcal{P}$, the map

$$\begin{array}{ccc} \mathrm{Hom}(M,N) \times \mathrm{Hom}(N,P) & \longrightarrow & \mathrm{Hom}(M,P) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

is bilinear (check). In other words, distributivity holds. We shall denote by $0_{MN} : M \to N$ (or $0_M : M \to M$) for the zero morphism but we may as well write $0 := 0_{MN}$. If $M \in \mathcal{P}$, then $\mathrm{End}(M)$ is now a ring and not merely a monoid and we may as well write $1 := 1_M := \mathrm{Id}_M$ (which should not be confused with the final morphism).

A preadditive category is not a usual category satisfying some specific property but a usual category *endowed* with some extra structure. Unlike the case of an abelian group versus a set for instance, there is no practical reason to distinguish between the two since, as we shall see, this additional structure is unique when the category is cartesian.

Examples
1. The category **Ab** of abelian groups is preadditive.
2. More generally, if A is a ring, then the category A-**Mod** of left A-modules is preadditive.
3. If $\mathcal{C}$ is a small category, then the category $\widehat{\mathcal{C}}(\mathbf{Ab})$ of presheaves of abelian groups on $\mathcal{C}$ is preadditive.
4. If $\mathcal{C}$ is any cartesian category, then the category $\mathbf{Ab}(\mathcal{C})$ of abelian groups of $\mathcal{C}$ is preadditive. Idem for the category A-$\mathbf{Mod}(\mathcal{C})$ of A-modules of $\mathcal{C}$ (over some ring A of $\mathcal{C}$).
5. As a particular case, the category **AbTop**, **AbHaus** or **AbCHaus** of topological (resp. Hausdorff, resp. compact Hausdorff) abelian groups are preadditive.
6. The category $\mathbf{A}$ with only one object, whose morphisms are the elements of a ring A and composition is given by multiplication ($g \circ f = gf$) is preadditive. Any preadditive category with exactly one object has this form.
7. The category $\mathbf{Mat}_A$ whose morphisms are matrices with coefficients in A (and natural numbers as objects) is preadditive.
8. The category $\mathbf{Vect}(X)$ of vector bundles on a manifold X is preadditive.
9. The categories **Set**, **Top**, **Grp** or **Ring** are *not* preadditive.

Exercise 2.1.2 Show that, if $\mathcal{P}$ is a preadditive category, then $\mathcal{P}^{\mathrm{op}}$ is preadditive too.

Proposition 2.1.3 If $\mathcal{C}$ is a small category and $\mathcal{P}$ is a preadditive category, then $\mathbf{Hom}(\mathcal{C}, \mathcal{P})$ is preadditive.

Proof. Given two natural transformations $\alpha, \beta : F \to G$, on simply sets $(\alpha + \beta)_X = \alpha_X + \beta_X$ for $X \in \mathcal{C}$. Details are left to the reader. For example, if $f : X \to Y$ is a morphism in $\mathcal{C}$, then distributivity in $\mathcal{P}$ implies

$$\begin{aligned}(\alpha + \beta)_Y \circ F(f) &= \alpha_Y \circ F(f) + \beta_Y \circ F(f) \\ &= G(f) \circ \alpha_X + G(f) \circ \alpha_X = G(f) \circ (\alpha + \beta)_X. \end{aligned}$$ ■

It follows that the category $\widehat{\mathcal{C}}(\mathcal{P}) := \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{P})$ of presheaves on a small category $\mathcal{C}$ with values in a preadditive category $\mathcal{P}$ is also is preadditive. Or else, if I is a small category, then the category $\mathcal{P}^I := \mathbf{Hom}(I, \mathcal{P})$ of diagrams of shape I in $\mathcal{P}$ is preadditive.

Definition 2.1.4 A *preadditive subcategory* of a preadditive category $\mathcal{P}$ is a subcategory $\mathcal{P}'$ such that for all $M, N \in \mathcal{P}$, $\mathrm{Hom}_{\mathcal{P}'}(M, N)$ is a subgroup of $\mathrm{Hom}_{\mathcal{P}}(M, N)$.

A preadditive subcategory is preadditive. A full subcategory of a preadditive category is automatically a preadditive category.

If $M \in \mathcal{P}$, we will now write

$$h^M : \mathcal{P} \to \mathbf{A}\mathrm{b} \quad \text{and} \quad h_M : \mathcal{P}^{\mathrm{op}} \to \mathbf{A}\mathrm{b}$$

(replacing henceforth $\mathbf{S}\mathrm{et}$ with $\mathbf{A}\mathrm{b}$).

Exercise 2.1.5 Show that $h^M : \mathcal{P} \to \mathbf{A}\mathrm{b}$ still preserves all limits and is in particular left exact (and dual).

Definition 2.1.6 A functor $F : \mathcal{P} \to \mathcal{Q}$ between two preadditive categories is *additive* if for all $M, N \in \mathcal{P}$, the map

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N)), \quad f \mapsto F(f)$$

is a group homomorphism.

Examples
1. A subcategory $\mathcal{P}'$ of a preadditive category $\mathcal{P}$ is a preadditive subcategory if and only if there exists the structure of a preadditive category on $\mathcal{P}'$ turning the inclusion into an additive functor.
2. If $\mathcal{P}$ is any preadditive category, then the functors h^M and h_M are additive.
3. If A is a ring and M is a right A-module, then the functor $N \mapsto M \otimes_A N$ is additive.
4. If $f : A \to B$ is a ring homomorphism, then the forgetful functor $B\text{-}\mathbf{M}\mathrm{od} \to A\text{-}\mathbf{M}\mathrm{od}$ as well as its adjoint $M \mapsto B \otimes_A M$ and coadjoint $M \mapsto \mathrm{Hom}_A(B, M)$ are all additive.

We shall denote by $\mathrm{Hom}_+(\mathcal{P}, \mathcal{Q})$ the collection of all additive functors.

Proposition 2.1.7 Let $F : \mathcal{P} \to \mathcal{Q}$ be an additive functor. Then,
1. if $G : \mathcal{D} \to \mathcal{E}$ is additive, so is $G \circ F$,
2. $F^{\mathrm{op}} : \mathcal{P}^{\mathrm{op}} \to \mathcal{Q}^{\mathrm{op}}$ also is additive,
3. if I a small category, $F^I : \mathcal{P}^I \to \mathcal{Q}^I$ also is additive.

Proof. Clear. ■

If $\mathcal{P}$ is a small preadditive category and $\mathcal{Q}$ is a preadditive category, then additive functors form a (full) preadditive subcategory $\mathbf{H}\mathrm{om}_+(\mathcal{P}, \mathcal{Q})$ of $\mathbf{H}\mathrm{om}(\mathcal{P}, \mathcal{Q})$. We shall write

$$\mathcal{P}\text{-}\mathbf{M}\mathrm{od} := \mathbf{H}\mathrm{om}_+(\mathcal{P}, \mathbf{A}\mathrm{b}) \quad (\text{resp.} \quad \mathbf{M}\mathrm{od}\text{-}\mathcal{P} = \mathcal{P}^{\mathrm{op}}\text{-}\mathbf{M}\mathrm{od})$$

and call them left (resp. right) $\mathcal{P}$-modules (this is the additive analog to $\widehat{\mathcal{C}}$).

Exercise 2.1.8 Show that
1. if A, B are two rings, then $\mathrm{Hom}_{\mathbf{R}\mathrm{ing}}(A, B) \simeq \mathrm{Hom}_+(\mathbf{A}, \mathbf{B})$,
2. If A is a ring, then $\mathbf{A}\text{-}\mathbf{M}\mathrm{od} \simeq A\text{-}\mathbf{M}\mathrm{od}$.

Exercise 2.1.9 Prove the additive Yoneda lemma: if $\mathcal{P}$ is a small[a] preadditive category, $M \in \mathcal{P}$ and $F \in \mathcal{P}\text{-}\mathbf{M}\mathrm{od}$, then there exists a natural isomorphism of abelian groups $\mathrm{Hom}(h^M, F) \simeq F(M)$ (and dual).

[a]The statement generalizes to large categories.

2.1.2 Additive category

Definition 2.1.10 An initial object in a preadditive category $\mathcal{P}$ is called a *zero* object and denoted by 0. A coproduct of a family $(M_i)_{i\in I}$ is called a *direct sum* and denoted by $\bigoplus_{i\in I} M_i$.

Proposition 2.1.11 Let $\mathcal{P}$ be a preadditive category. Then, for $M, M_1, \dots, M_n$ in $\mathcal{P}$, the following are equivalent

1. $M \simeq M_1 \times \cdots \times M_n$,
2. $M \simeq M_1 \oplus \cdots \oplus M_n$,
3. there exists $p_k : M \twoheadrightarrow M_k$ and $i_k : M_k \rightarrowtail M$ for $k = 1, \dots, n$, such that $p_k \circ i_k = \mathrm{Id}_{M_k}$, $p_\ell \circ i_k = 0_{M_k M_\ell}$ when $k \neq \ell = 1, \dots, n$ and $\sum_{k=1}^n i_k \circ p_k = \mathrm{Id}_M$.

Moreover, the maps $p_k : M \to M_k$ (resp. $i_k : M_k \to M$) are the structural morphisms for the product (resp. direct sum).

Proof. Since (3) is autodual and (2) is dual to (1), it is sufficient to prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

1. (1) $\Rightarrow$ (3) : We already got the structural morphisms p_k and the existence of the morphisms i_k satisfying the commutation rules then come from the universal property of the product. It only remains to check the last equality which formally follows from $p_\ell \circ (\sum_{k=1}^n i_k \circ p_k) = p_\ell$ for $\ell = 1, \dots, n$.
2. (3) $\Rightarrow$ (2) : If we are given $f_k : M_k \to N$ for $k = 1, \dots, n$, we then set $f = \sum_{\ell=1}^n f_\ell \circ p_\ell$. We will have for $k = 1, \dots, n$, $f \circ i_k = \sum_{\ell=1}^n f_\ell \circ p_\ell \circ i_k = f_k$. Assume conversely that $f : M \to N$ satisfies $f \circ i_k = f_k$ for $k = 1, \dots, n$. We will then have

$$\sum_{\ell=1}^n f_\ell \circ p_\ell = \sum_{\ell=1}^n f \circ i_\ell \circ p_\ell = f \circ \sum_{\ell=1}^n i_\ell \circ p_\ell = f. \qquad \blacksquare$$

Examples 1. ($n = 0$) A final object M is the same thing as a zero object (meaning an initial object) and characterized by $\mathrm{Id}_M = 0_M$ or equivalently $\mathrm{End}(M) = 1 = \{0\}$.

2. ($n = 2$) A product M of two objects M_1 and M_2 is the same thing as a direct sum (meaning a coproduct) and characterized by the existence of $p_1 : M \twoheadrightarrow M_1$, $p_2 : M \twoheadrightarrow M_2$, $i_1 : M_1 \rightarrowtail M$ and $i_2 : M_2 \rightarrowtail M$ such that

$$\begin{cases} p_1 \circ i_1 = \mathrm{Id}_{M_1}, \\ p_2 \circ i_2 = \mathrm{Id}_{M_2} \end{cases}, \quad \begin{cases} p_2 \circ i_1 = 0_{M_1 M_2}, \\ p_1 \circ i_2 = 0_{M_2 M_1} \end{cases} \quad \text{and} \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{Id}_M. \tag{2.1}$$

Definition 2.1.12 An *additive* category is a cartesian preadditive category.

It means that the preadditive category admits finite products and this is equivalent to admitting finite direct sums in which case they are the same. Equivalently, by induction, it admits a final object (resp. zero) and products (resp. direct sums) of two objects.

In an additive category the natural morphism $\bigoplus_{i\in I} M_i \to \prod_{i\in I} M_i$ is an isomorphism when I is finite (but not otherwise) and we shall mostly use the first notation.

Examples 1. The categories $\mathbf{A}$b, A-$\mathbf{M}$od, $\mathbf{M}$at$_A$ and $\mathbf{V}$ect(X) are additive.

2. If $\mathcal{C}$ is a small (resp. a cartesian) category, then $\widehat{\mathcal{C}}(\mathbf{A}\mathrm{b})$ (resp. $\mathbf{A}\mathrm{b}(\mathcal{C})$ or A-$\mathbf{M}\mathrm{od}(\mathcal{C})$) is additive.
3. If $\mathcal{P}$ is a small preadditive category, then $\mathcal{P}$-$\mathbf{M}$od is additive.
4. If A is a non-zero ring, then the category $\mathbf{A}$ is *not* an additive category.

Exercise 2.1.13 Show that, if $\mathcal{P}$ is an additive category, then $\mathcal{P}^{\mathrm{op}}$ is additive too. Show also that, if I is a small category, then $\mathcal{P}^I$ is additive.

Lemma 2.1.14 In an additive category, if $M = \bigoplus_{k=1}^m M_k$ and $N = \bigoplus_{j=1}^n N_j$, then

$$\mathrm{Hom}(M,N) \simeq \bigoplus_{1\le j\le n, 1\le k\le m} \mathrm{Hom}(M_k, N_j).$$

Proof. Follows from universal properties of direct sum which is at the same time a product and a coproduct. ∎

We shall use matrix notation and write $f = [f_{jk}]$ with $f_{jk} = p_j \circ f \circ i_k : M_k \to N_j$. Composition is then reduced to matrix multiplication:

Exercise 2.1.15 Show that, if we are given two morphisms $f = M \to N$ and $g : N \to P$ in an additive category, and decompositions $M = \bigoplus_{k=1}^m M_k$ and $N = \bigoplus_{j=1}^n N_j$ and $P = \bigoplus_{i=1}^p N_p$, then $g \circ f = [h_{ik}]$ with $h_{ik} = \sum_j g_{ij} \circ f_{jk}$.

Example 1. In the case $M = M_1 \oplus M_2$, we have $\mathrm{Id}_M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $i_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $i_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $p_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $p_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and we can check identities 2.1 by matrix operations, the last one reading

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. If $p_1, p_2 = M \oplus M \twoheadrightarrow M$ and $i_1, i_2 = M \rightarrowtail M \oplus M$ denote the structural morphisms, and Δ denotes the *diagonal morphism* ($p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_M$), then

$$\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = i_1 + i_2 \quad \text{(and dually } \nabla = \begin{bmatrix} 1 & 1 \end{bmatrix} = p_1 + p_2\text{)}.$$

Proposition 2.1.16 If $\mathcal{P}$ is an additive category, then the preadditive structure is unique.

Proof. If $f_1, f_2 : M \to N$, then

$$f_1 + f_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{2.2}$$

Exercise 2.1.17 Show that the forgetful functor $\mathbf{Ab}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism when $\mathcal{P}$ is additive.

Solution. If an object M of $\mathcal{P}$ is endowed with an abelian group law $\begin{bmatrix} f & g \end{bmatrix}$: $M \oplus M \to M$, then the neutral element for this law is necessarily the unique map $0 : 0 \to M$ and the following diagram is therefore required to commute:

$$\begin{array}{ccc} M & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & M \oplus M \\ {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\downarrow & \overset{1}{\searrow} & \downarrow{\scriptstyle [f\ g]} \\ M \oplus M & \xrightarrow[{[f\ g]}]{} & M. \end{array}$$

If follows that $f = g = 1$ and the law is therefore the codiagonal morphism ∇. One easily checks that, conversely, ∇ defines an abelian group structure on M and that this is functorial. This provides an inverse to the forgetful functor. ■

Exercise 2.1.18 Show that a category $\mathcal{P}$ is additive if and only if there exists a (not unique) cartesian category $\mathcal{C}$ such that $\mathbf{Ab}(\mathcal{C}) \simeq \mathcal{P}$.

Definition 2.1.19 An *additive subcategory* of an additive category is a preadditive subcategory which is additive.

Exercise 2.1.20 Show that

1. a subcategory $\mathcal{P}'$ of an additive category $\mathcal{P}$ is an additive subcategory if and only if it is stable under finite direct sums: if $M_1, \ldots, M_n \in \mathcal{P}'$, then $\bigoplus_{k=1}^n M_k \in \mathcal{P}'$.
2. a functor $F : \mathcal{P} \to \mathcal{Q}$ between additive categories is additive if and only if it preserves finite direct sums: if $M_1, \ldots, M_n \in \mathcal{P}$, then $F(\bigoplus_{k=1}^n M_k) = \bigoplus_{k=1}^n F(M_k)$.

Exercise 2.1.21 Let $\mathcal{Q}$ be an additive category. Show that, if a fully faithful functor $\mathcal{P} \rightarrowtail \mathcal{Q}$ has an adjoint or a coadjoint, then $\mathcal{P}$ also is additive.

Exercise 2.1.22 Show that, if $\mathcal{P}$ is a small preadditive category and $\mathcal{Q}$ is additive, then $\mathbf{Hom}_+(\mathcal{P}, \mathcal{Q})$ is additive.

Proposition 2.1.23 If a functor F between two additive categories is adjoint to a functor G, then both functors are additive and there exists a natural isomorphism of abelian groups

$$\mathrm{Hom}(F(M), N) \simeq \mathrm{Hom}(M, G(N)). \tag{2.3}$$

Proof. Since an adjoint (resp. coadjoint) preserves all colimits (resp. all limits), it preserves finite direct sums and is therefore additive. Now, given two morphisms $f, g : FM \to N$, we have

$$f = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, g = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{and} \quad f + g = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}.$$

Thus, if we denote by Φ the natural isomorphism (2.3), we shall have first

$$\Phi(f) = \begin{bmatrix} 1 & 0 \end{bmatrix} \Phi\left(\begin{bmatrix} f \\ g \end{bmatrix}\right) \quad \text{and} \quad \Phi(g) = \begin{bmatrix} 0 & 1 \end{bmatrix} \Phi\left(\begin{bmatrix} f \\ g \end{bmatrix}\right)$$

from which we derive

$$\Phi\left(\begin{bmatrix} f \\ g \end{bmatrix}\right) = \begin{bmatrix} \Phi(f) \\ \Phi(g) \end{bmatrix}.$$

It follows that

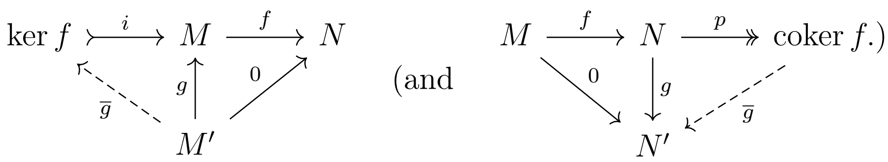
$$\Phi(f+g) = \begin{bmatrix} 1 & 1 \end{bmatrix} \Phi\left(\begin{bmatrix} f \\ g \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \Phi(f) \\ \Phi(g) \end{bmatrix} = \Phi(f) + \Phi(g).$$ ■

2.1.3 Preabelian category

We fix an additive category $\mathcal{P}$.

Definition 2.1.24 The *kernel* of a morphism $f : M \to N$ is $\ker f := \ker(f, 0)$ if it exists. The dual notion is that of a *cokernel* $\operatorname{coker} f$.

Be careful that the kernel actually comes with a morphism $i : \ker f \rightarrowtail M$ and is only defined up to isomorphism (and dual). It has the following universal property: $f \circ i = 0$ and given any $g : M' \to M$ such that $f \circ g = 0$, there exists a unique morphism $\overline{g} : M \to \ker f$ such that $i \circ \overline{g} = g$ (and dual):



Definition 2.1.24 has a kind of converse in the sense that $\ker(f, g) = \ker(g - f)$ when $f, g : M \to N$ (and dual).

Definition 2.1.25 A *preabelian category* is an additive category with all kernels and cokernels.

Equivalently, it is a preaddtive category which is finitely bicomplete. Up to the end of the section, you can assume that $\mathcal{P}$ is preabelian or else follow our convention to implicitly assume that a limit or a colimit exists when we write it down.

Proposition 2.1.26 If $f : M \to N$ is a morphism, then

1. f is a monomorphism if and only if $\ker f = 0$ (and dual),
2. $f = 0$ if and only if $\ker f = M$ (and dual),
3. f is a monomorphism and $f = 0$ if and only if $M = 0$ (and dual).

Proof. If f is a monomorphism and $g : M' \to M$ satisfies $f \circ g = 0$ $(= f \circ 0)$, then $g = 0$ and therefore, g factorizes through 0 (uniquely). Conversely, if $\ker f = 0$ and $g, h : M' \to M$ satisfy $f \circ g = f \circ h$, then $f \circ (g - h) = 0$ and therefore $g - h$ factor through $\ker f = 0$ which implies that $g - h = 0$ and finally $g = h$.

If $f = 0$, then *any* morphism $g : M' \to M$ satisfies $g \circ f = 0$ so that $\ker f = M$. Conversely, if $\ker f = M$, then $f = f \circ \mathrm{Id}_M = 0$.

The last assertion follows from the first two. ■

Definition 2.1.27 1. A sequence

$$0 \to M' \to M \xrightarrow{f} M'' \tag{2.4}$$

is said to be *(left) exact* if the sequence

$$M' \longrightarrow M \overset{f}{\underset{0}{\rightrightarrows}} M''$$

is left exact. The dual notion is that of a *(right) exact sequence* $M' \to M \to M'' \to 0$.

2. A *short exact sequence*[a] is a sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0 \tag{2.5}$$

which is exact both on the left and on the right.

[a] Be careful that there exists alternative weaker definitions at this level.

In other words, a sequence (2.4) is (left) exact if and only if $M' = \ker f$ (and dual) and a sequence (2.5) is exact if and only if $M' = \ker p$ and $\operatorname{coker} i = M''$.

Exercise 2.1.28 Show that

1. a sequence $0 \to M \xrightarrow{f} N \to 0$ is left exact if and only if it is right exact[a] if and only if f is an isomorphism,
2. if $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is left exact (resp. short exact), then f is an isomorphism if and only if $g = 0$ (resp. $M'' = 0$) – and dual.

[a] So that there is no ambiguity in saying that such a sequence is exact.

Solution. 1. By duality, only the case of a left exact sequence requires our attention. Now, this is the kind of situation where we have to take into account the fact that a kernel is only defined up to a unique isomorphism: the sequence is left exact if and only if f induces an isomorphism between M and the kernel of $N \to 0$ which is exactly N.

2. We know that $g = 0$ if and only if $\ker g = M$ which means that f is an isomorphism when the sequence is left exact. If the sequence is moreover right exact, then g is an epimorphism and zero at the same time which means that $M'' = 0$. ∎

Exercise 2.1.29 Show that two sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to N' \to N \to N'' \to 0$ are left (resp. right, resp. short) exact if and only if the (corresponding) sequence $0 \to M' \oplus N' \to M \oplus N \to M'' \oplus N'' \to 0$ is left (resp. right, resp. short) exact.

Exercise 2.1.30 Show that any commutative diagram with exact rows

$$\begin{array}{ccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \end{array}$$

may be uniquely completed as described (and dual).

Left (resp. right, resp. short) exact sequences in $\mathcal{P}$ form a full additive subcategory of the category $\mathcal{P}^{[\mathbf{2}]}$ of all diagrams of the form $M' \to M \to M''$ so that the zero on the left (resp. the right, resp. both sides) is only a decoration to specify what kind of sequence we are considering.

Proposition 2.1.31 Limits of left exact sequences are computed termwise (and dual).

Proof. Limits always preserve kernels since they preserve all limits. ∎

It means that, if we are given a commutative diagram of short exact sequences, then

$$\varprojlim_I (0 \to M_i' \to M_i \to M_i'') = 0 \to \varprojlim_I M_i' \to \varprojlim_I M_i \to \varprojlim_I M_i'',$$

and in particular implies that the left hand side is a left exact sequence. Be careful however that the question of limits or colimits of *short* exact sequences is way more complicated.

Corollary 2.1.32 If all columns and both bottom rows are exact in a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'', \end{array}$$

then the top row also is exact (and dual). ∎

It is important to notice that there is no similar assertion for short exact sequences and that it will be necessary to introduce later long exact sequences to solve this problem.

Exercise 2.1.33 Show that, in A-$\mathbf{M}$od, any short exact sequence is isomorphic to some

$$0 \to M' \to M \to M/M' \to 0$$

where M' is a submodule of M.

Definition 2.1.34 A short exact sequence is said to *split* if it is isomorphic to

$$0 \longrightarrow M_1 \xrightarrow{i_1} M_1 \oplus M_2 \xrightarrow{p_2} M_2 \longrightarrow 0.$$

Proposition 2.1.35 A short exact sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

splits if and only if p is a split epimorphism (and dual).

Proof. The condition is clearly necessary (use i_2). Assume conversely given some $s : M'' \to M$ such that $p \circ s = 1$. Since $p \circ (1 - s \circ p) = 0$, there exists a unique $r : M \to M'$ such that $i \circ r = 1 - s \circ p$. And we have in particular $i \circ r + s \circ p = 1$. Since we already know that $p \circ i = 0$ and $p \circ s = 1$, it only remains to check that $r \circ s = 0$ and $r \circ i = 1$. Since i is a monomorphism, it is sufficient to notice that $i \circ (r \circ s) = (1 - s \circ p) \circ s = 0$ and $i \circ (r \circ i) = (1 - s \circ p) \circ i = i$. ■

It is implicit in the previous argument that we have an isomorphism of short exact sequences with given by $\begin{bmatrix} r \\ p \end{bmatrix} : M \xrightarrow{\sim} M' \oplus M''$ with inverse $\begin{bmatrix} i & s \end{bmatrix}$.

Exercise 2.1.36 Show that a diagram

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

is cartesian (resp. cocartesian, resp. both cartesian and cocartesian) if and only if the sequence

$$0 \longrightarrow M' \xrightarrow{\left[\begin{smallmatrix} f' \\ g' \end{smallmatrix}\right]} M \oplus N' \xrightarrow{[f\ -g]} N \longrightarrow 0$$

is left exact (resp. right exact, resp. short exact).

The following provides a very nice alternative definition for (short) exact sequences:

Exercise 2.1.37 Show that a sequence

$$0 \to M' \to M \to M'' \to 0$$

is left exact (resp. right exact, resp. short exact) if and only if the diagram

$$\begin{array}{ccc} M' & \longrightarrow & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M'' \end{array}$$

is cartesian (resp. cocartesian, resp. both cartesian and cocartesian).

Proposition 2.1.38 If a diagram

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ \downarrow{\scriptstyle g'} & \ulcorner & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

is cartesian, then $\ker f' = \ker f$ and $\ker g' = \ker g$ (and dual).

Solution. There exists a sequence of cartesian diagrams

$$\begin{array}{ccccc} \ker g' & \longrightarrow & M' & \xrightarrow{f'} & N' \\ \downarrow & \ulcorner & \downarrow{\scriptstyle g'} & \ulcorner & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & M & \xrightarrow{f} & N \end{array}$$

showing thanks to exercise 1.3.17 that the rectangle is cartesian and therefore $\ker g' \simeq \ker g$. The original statement is obtained by symmetry. ■

Exercise 2.1.39 Show that if A is a commutative ring, M is an A-module and $f, g \in A$ are comaximal (the ideal generated by f and g is the full ring) then the sequence

$$0 \to M \to M_f \times M_g \xrightarrow{-} M_{fg} \to 0$$

is exact.

Solution. The condition means that there exists $h_1, h_2 \in A$ such that $h_1 f_1 + h_2 f_2 = 1$. Before going any further, note that f^n and g^m are always comaximal when $n, m \geq 0$ and that $M_f = M_{f^n}$ when $n > 0$. We now turn to the proof. First of all, composition is clearly zero. Let us show that the first map is injective: if the image of $s \in M$ is zero in M_{f_i} for $i = 1, 2$, it means that there exists $n_i \in \mathbb{N}$ such that $f_i^{n_i} s = 0$. Up to replacing f_i by $f_i^{n_i}$, we may assume that $f_i s = 0$ and we will have

$$s = 1s = h_1 f_1 s + h_2 f_2 s = 0.$$

We give now ourselves $s_i / f_i^{n_i} \in M_{f_i}$ for $i = 1, 2$ with same image in $M_{f_1 f_2}$ and we have to find $s \in M$ whose image in M_{f_i} is $s_i / f_i^{n_i}$. We may again assume that $n_i = 1$ and our hypothesis means that $(f_1 f_2)^m (f_1 s_2 - f_2 s_1) = 0$ for some m. Actually, up to replacing f_i by f_i^{m+1} and s_i by $f_i^m s_i$, we may assume that $f_1 s_2 - f_2 s_1 = 0$. It is then sufficient to set $s := h_1 s_1 + h_2 s_2$. We will have

$$f_1 s = f_1(h_1 s_1 + h_2 s_2) = (h_1 f_1 + h_2 f_2) s_1 = s_1$$

and symmetrically for s_2. It remains to show that the last map is surjective: we give ourselves $s/(f_1 f_2)^n \in M_{f_1 f_2}$ and we show that it comes from $M_{f_1} \times M_{f_2}$. Again, we may assume that $n = 1$ and consider $(h_2 s/f_1, -h_1 s/f_2)$. We will have

$$\frac{h_2 s}{f_1} - \frac{-h_1 s}{f_2} = \frac{h_2 f_2 s + h_1 f_1 s}{f_1 f_2} = \frac{s}{f_1 f_2}. \qquad \blacksquare$$

2.1.4 Abelian category

Definition 2.1.40 An *abelian category* is a preabelian category where all monomorphisms and all epimorphisms are regular.

The condition means that any monomorphism is a kernel and any epimorphism is a cokernel (the converse being always true).

Examples
1. The category $\mathbf{Ab}$ of abelian groups is abelian.
2. The category A-$\mathbf{Mod}$ of left modules over a ring A is abelian.
3. More generally, if $\mathcal{P}$ is a small preadditive category, then the category $\mathcal{P}$-$\mathbf{Mod}$ of additive functors from $\mathcal{P}$ to $\mathbf{Ab}$ is abelian.
4. Also, if $\mathcal{C}$ is a small category, then the category $\widehat{\mathcal{C}}(\mathbf{Ab})$ of presheaves of abelian goups on $\mathcal{C}$ is abelian.
5. The category $\mathbf{Op}(k\text{-}\mathbf{Mod})$ of k-modules endowed with an operator is abelian.
6. The category $\mathbf{AbCHaus}$ of compact Hausdorff abelian groups is abelian.
7. The category $\mathbf{AbTop}$ is not abelian however because the identity $\mathbb{Z}^{\text{disc}} \to \mathbb{Z}^{\text{coarse}}$ is not a regular monomorphism.
8. The category $\mathbf{AbHaus}$ is not abelian either because the map $\ell^1(\mathbb{R}) \to \ell^2(\mathbb{R})$ is not a regular monomorphism.
9. The category $\mathbf{Mat}_{\mathbb{Z}}$ is not abelian because $[2]$ is not a regular monomorphism.

Exercise 2.1.41 A *representation* of a monoid G (or a *G-module*) is a morphism $\rho : G \to \mathrm{End}_k(M)$ where M is a module on some fixed commutative ring k. A morphism of representations is a k-linear map $M \to N$ which is compatible with the actions of G. Show that they form an abelian category $\mathbf{Rep}_k(G)$.

Hint. One can check that $\mathbf{Rep}_k(G) \simeq A$-$\mathbf{Mod}$ with $A = k \cdot G$. $\blacksquare$

Exercise 2.1.42 Let $\mathcal{O}$ be the ring of holomorphic functions on an open subset U of $\mathbb{C}$. A *derivation* on an $\mathcal{O}$-module M is a group homomorphism $\partial_M : M \to M$ such that $\forall f \in \mathcal{O}, \forall s \in M, \partial_M(fs) = f's + f\partial_M(s)$. An $\mathcal{O}$-linear map $M \to N$ between modules with derivations is sait to be *horizontal* if it commutes with the derivations. Show that modules with derivations and horizontal maps form an abelian category.

Hint. One can check that this category is equivalent to the category $\mathcal{D}$-**Mod** where

$$\mathcal{D} := \left\{ \sum_{k=0}^{d} f_k \partial^k, f_k \in \mathcal{O} \right\}$$

is the (non-commutative) ring of *differential operators* with the commutation rule

$$[\partial, f] := \partial f - f\partial = f'.$$ ∎

Proposition 2.1.43 If $\mathcal{A}$ is an abelian category, then $\mathcal{A}^{\mathrm{op}}$ is also an abelian category, as well as $\mathcal{A}^I$ if I is a small category.

Proof. For the second assertion, use proposition 1.3.42. ∎

Definition 2.1.44 An *abelian subcategory* of an abelian category is a *full* subcategory which is stable under all finite limits and finite colimits.

Exercise 2.1.45 Show that an abelian subcategory is indeed an abelian category.

Exercise 2.1.46 Show that a functor $F : \mathcal{A} \to \mathcal{B}$ between two preabelian categories is left exact if and only if it is additive and preserves left exact sequences (and dual).

Exercise 2.1.47 Show that the category of free abelian groups of finite rank (or equivalently $\mathbf{Mat}_{\mathbb{Z}}$) is preabelian but that the inclusion in the category of all abelian groups is not an exact functor.

Hint. Everything works fine besides the fact that the cokernel Q in **Ab** of a morphism of free abelian groups of finite rank may not be free. However, one can write $Q = F \oplus T$ with F free and T torsion. An then, the cokernel in the original category will be F as one easily checks. ∎

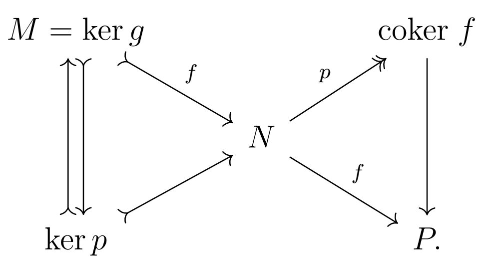
Lemma 2.1.48 In a preabelian category, a morphism $f : M \to N$ is a regular monomorphism if and only if

$$0 \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0$$

is a *short*[a] exact sequence (and dual).

[a]Recall that we mean that it is simultaneously left and right exact.

Proof. The condition is clearly sufficient. Conversely, if f is the kernel of a morphism $g : N \to P$, we can then build the following commutative diagram (using the universal property of the kernel):



As shown in exercise 1.3.28, both left hand morphisms are necessarily inverse to each other, showing that $M = \ker p$. ■

Recall the notion of a balanced category from definition 1.3.36: a morphism which is at the same time a monomorphism and an epimorphism is automatically an isomorphism.

Proposition 2.1.49 An abelian category is balanced.

Proof. Follows from exercise 1.3.26. ■

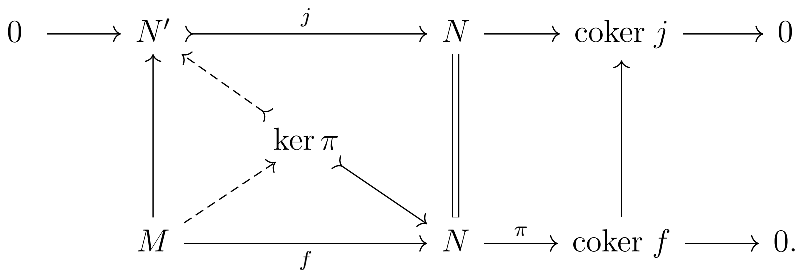
Recall that we introduced the notions of regular image and coimage in definition 1.3.34.

Proposition 2.1.50 If $f : M \to N$ is a morphism in a (*pre*) abelian category, then the sequence

$$0 \to \text{im } f \to N \to \text{coker } f \to 0$$

is (*left*) exact (and dual).

Proof. According to proposition 1.3.33, for left exactness, it is sufficient to show that $\ker(N \xrightarrow{\pi} \text{coker } f)$ is universal for factorization of f through a regular monomorphism $j : N' \rightarrowtail N$. We can then simply contemplate the following commutative diagram:



If the category is abelian, then π is a regular epimorphism with $\ker \pi = \text{im} f$ and the sequence is therefore also right exact. ■

Corollary 2.1.51 A morphism $f : M \to N$ in a preabelian category is an epimorphism if and only if $\text{im } f = N$ (and dual).

Proof. We have $\text{im}(f) = N$ if and only if $\pi : N \to \text{coker } f$ is the zero morphism. But since π is an epimorphism, it means that $\text{coker } f = 0$ or equivalently that f is an epimorphism. ■

Recall that the notion of a *strict morphism* was introduced in definition 1.3.34: it means that regular image and regular coimage coincide.

Proposition 2.1.52 A preabelian category is abelian if and only if all morphisms are strict.

Proof. We already know from exercise 1.3.35 that a strict monomorphism/epimorphism is automatically regular.

Conversely, any morphism $f : M \to N$ splits as $M \xrightarrow{\pi} \operatorname{im}(f) \rightarrowtail N$. Now π splits in turn as $M \to \operatorname{im}(\pi) \rightarrowtail \operatorname{im}(f)$. Since the morphism $\operatorname{im}(\pi) \rightarrowtail N$ is still a monomorphism and all monomorphisms are regular in an abelian category, proposition 1.3.33 implies that $\operatorname{im}(\pi) = \operatorname{im}(f)$. Corollary 2.1.51 implies that π is an epimorphism and we are done. More precisely, this implies that $\operatorname{coim} f \to \operatorname{im} f$ is an epimorphism. The dual argument shows that this is a monomorphism and we know from proposition 2.1.49 that an abelian category is balanced. ■

Corollary 2.1.53 In an abelian category, any morphism $f : M \to N$ factors uniquely up to an isomorphism as an epimorphism followed by a monomorphism. ■

Exercise 2.1.54 Let $\mathcal{B}$ be an abelian category. Show that, if a fully faithful functor $\mathcal{A} \rightarrowtail \mathcal{B}$ has an exact adjoint or coadjoint, then $\mathcal{A}$ also is abelian.

Most statements about abelian categories can be proven by reducing to a category of modules using the following embedding theorem[1]:

Theorem 2.1.55 — Freyd-Mitchell. If $\mathcal{A}$ is a small abelian category, then there exists a ring A and a fully faithful exact functor $\mathcal{A} \rightarrowtail A\text{–Mod}$.

Proof. To do. ■

In other words, a small category is abelian if and only if it is equivalent to an abelian subcategory of a category of modules.

2.1.5 Exactness

Up to the end of this section, we work in a fixed abelian category $\mathcal{A}$.

Definition 2.1.56 A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be *exact in the middle* or *exact in M* or even simply *exact* if $\operatorname{im} f = \ker g$.

Example A sequence $0 \to M' \xrightarrow{f} M$ is exact if and only if f is a monomorphism (and dually, a sequence $M \xrightarrow{f} M'' \to 0$ is exact if and only if f is an epimorphism).

Exercise 2.1.57 Show that the condition for being exact is autodual: $\operatorname{coker} f = \operatorname{im} g$ if and only if $\operatorname{im} f = \ker g$.

Exercise 2.1.58 Show that a sequence

$$0 \longrightarrow M' \xrightarrow{\ i\ } M \xrightarrow{\ p\ } M'' \longrightarrow 0$$

is left exact (resp. right exact, resp. short exact) if and only if it is exact in M', M (resp. M, M'', resp. M', M, M'').

The following very convenient criterion is available in an abelian category:

[1] Feel free to use it if you are too lazy to draw big diagrams.

Exercise 2.1.59 Show that a sequence

$$0 \longrightarrow M' \longrightarrow M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0$$

is short exact if and only if it is left exact and p is an epimorphism (and dual).

On can split, and conversely glue, exact sequences:

Exercise 2.1.60 Show that $M \stackrel{f}{\to} N \stackrel{g}{\to} Q \stackrel{h}{\to} R$ is exact if and only if it splits into right and left exact sequences

$$M \stackrel{f}{\to} N \to P \to 0 \quad \text{and} \quad 0 \to P \to Q \stackrel{h}{\to} R.$$

Exercise 2.1.61 Let $\mathcal{B}$ be another abelian category and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. Show that the following are equivalent:

1. F is exact,
2. if $M' \to M \to M''$ is exact (in the middle), then $F(M') \to F(M) \to F(M'')$ is exact in the middle,
3. F preserves short exact sequences,
4. F is left exact and preserves epimorphisms (and dual).

Proposition 2.1.62 If $f : M \twoheadrightarrow N$ is an epimorphism, then any cartesian diagram

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ \downarrow & \ulcorner & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

is also cocartesian (and dual).

Proof. We know from exercise 2.1.36 that the sequence

$$0 \to M' \to M \oplus N' \to N \to 0$$

is *left* exact. Moreover, the composite morphism $M \oplus N' \twoheadrightarrow M \twoheadrightarrow N$ is an epimorphism. This shows that the sequence is also *right* exact and therefore (exercise 2.1.36 again) that the square is cocartesian. ■

Corollary 2.1.63 Epimorphisms are stable under pull back (and dual).

Proof. Use for example (the dual version of) exercise 1.3.29. ■

It is sometimes hard to provide a formal proof that a given sequence is exact and the next criterion will prove itself quite valuable:

Exercise 2.1.64 Show that a sequence $M' \stackrel{f}{\to} M \stackrel{g}{\to} M''$ is exact in M if and only if $g \circ f = 0$ and, given any $h : L \to M$ such that $g \circ h = 0$, there exists an epimorphism $\pi : L' \twoheadrightarrow L$ such that $h \circ \pi$ factors through f:

$$\begin{array}{ccccc} L' & \overset{\pi}{\dashrightarrow} & L & & \\ \downarrow & & \downarrow{\scriptstyle h} & \searrow^{0} & \\ M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \end{array}$$
$$M' \xrightarrow{0} M''$$

Solution. If the sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact, we can simply compose the cartesian squares

$$\begin{array}{ccccc} L' & \longrightarrow & L & = & L \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow{\scriptstyle h} \\ M' & \twoheadrightarrow & \operatorname{im} f = \ker g & \rightarrowtail & M \end{array}$$

and use the fact that epimorphisms are stable under pullback (corollary 2.1.63). Conversely, if the condition is satisfied, we can consider the case $L = \ker(g)$. Since f factors through $\ker g$, there exists a commutative diagram

$$\begin{array}{ccccc} L' & \twoheadrightarrow & \ker(g) & & \\ \downarrow & \nearrow & \downarrow & \searrow^{0} & \\ M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \end{array}$$

with an epimorphism upstairs. This implies that $M' \twoheadrightarrow \ker g$ is an epimorphism and therefore that $\operatorname{im} f = \ker g$ (by unique factorization). ■

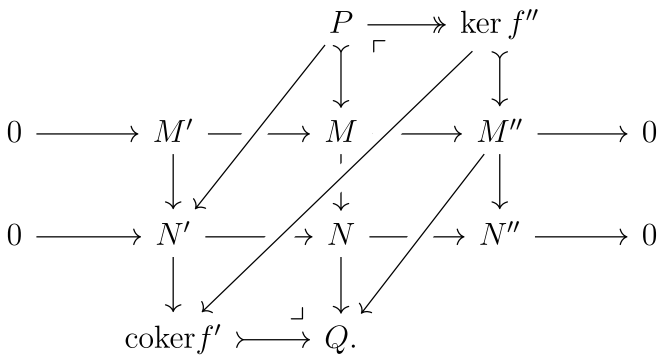
Proposition 2.1.65 — Snake lemma. If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{j} & N & \xrightarrow{q} & N'' & \longrightarrow & 0 \end{array}$$

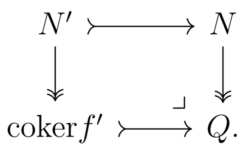
is a commutative diagram with exact rows, then there exists a natural autodual (long) exact sequence

$$0 \to \ker f' \xrightarrow{i} \ker f \xrightarrow{p} \ker f'' \xrightarrow{\delta} \operatorname{coker} f' \xrightarrow{\bar{j}} \operatorname{coker} f \xrightarrow{\bar{q}} \operatorname{coker} f'' \to 0.$$

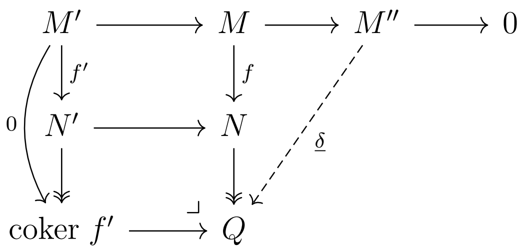
Proof. The construction of δ is shown on the following diagram:



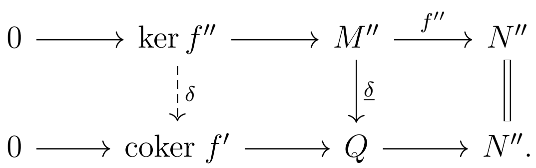
Let us give some details. We first form the cocartesian (which is also cartesian by proposition 2.1.62) diagram



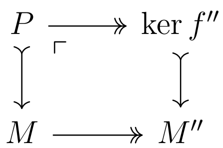
Note that exercise 2.1.38 provides us with short exact sequences $0 \to \operatorname{coker} f' \to Q \to N'' \to 0$ (so that we turned a cokernel into a kernel) and $0 \to \operatorname{im} f' \to N \to Q \to 0$ (that we shall only use in the end). We can consider the following factorization



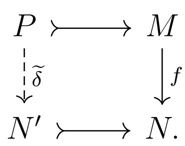
and then the induced morphism



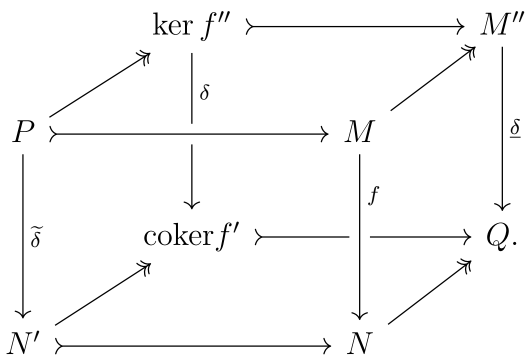
It follows from proposition 2.1.32 that $\ker \delta = \ker \underline{\delta}$ (but this will only be used in the end). Dually, we have a cartesian square and a factorization



and

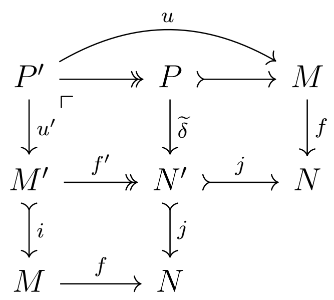


The fact that all horizontal morphisms are monomorphisms and all faces besides the left hand side are commutative in

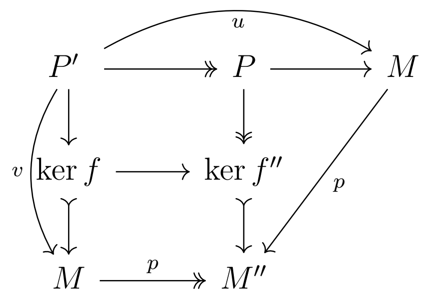


implies that it is indeed a commutative diagram so that our definition is autodual. According to exercise 2.1.32, we are only left with the proof of exactness in $\ker f''$.

As an intermediate step, we first assume that f' is an epimorphism and we have to prove that $\ker f \to \ker f''$ also is an epimorphism. For this purpose, we consider the following commutative diagram with cartesian upleft square:



and set $v := u - i \circ u' : P' \to M$ so that $f \circ v = 0$. It follows that v factors through $\ker f$. Then, since $p \circ v = p \circ u$ (because $p \circ i = 0$), the diagram



is commutative and the middle horizontal morphism must be an epimorphism.

We now turn to the general case. We have to show that that p induces an epimorphism $\ker f \twoheadrightarrow \ker \delta = \ker \underline{\delta}$. It is sufficient to apply the intermediate step (case f' surjective) to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker f & \longrightarrow & \ker \underline{\delta} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \underline{\delta}} & & \\
0 & \longrightarrow & \operatorname{im} f' & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0.
\end{array}$$

■

Exercise 2.1.66 Show that if

$$\begin{array}{ccccccc}
 & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & &
\end{array}$$

is a commutative diagram with exact rows, then there exists a natural autodual (long) exact sequence

$$\ker f' \to \ker f \to \ker f'' \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f''.$$

Exercise 2.1.67 Prove the four-lemma: if

$$\begin{array}{ccccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4
\end{array}$$

is a commutative diagram with exact rows with f_1 an epimorphism and f_2, f_4 monomorphisms, then f_3 is a monomorphism (and dual).

Solution. Split into

$$\begin{array}{ccccccccc}
 & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_2' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2'} & & \\
0 & \longrightarrow & N_1' & \longrightarrow & N_2 & \longrightarrow & N_2' & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_2' & \longrightarrow & M_3 & \longrightarrow & M_4' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2'} & & \downarrow{\scriptstyle f_3} & & \downarrow & & \\
0 & \longrightarrow & N_2' & \longrightarrow & N_3 & \longrightarrow & N_4 & &
\end{array}$$

and apply snake lemma (version of exercise 2.1.66) to both. ■

Exercise 2.1.68 Prove the five-lemma: if

$$\begin{CD}
M_1 @>>> M_2 @>>> M_3 @>>> M_4 @>>> M_5 \\
@VV{f_1}V @V{\simeq}V{f_2}V @VV{f_3}V @V{\simeq}V{f_4}V @VV{f_5}V \\
N_1 @>>> N_2 @>>> N_3 @>>> N_4 @>>> N_5
\end{CD}$$

is a commutative diagram with exact rows with f_1 an epimorphism and f_2, f_4 isomorphisms and f_5 a monomorphism, then f_3 is an isomorphism.

Proposition 2.1.69 Assume that, in the commutative diagram

$$\begin{CD}
0 @>>> M' @>>> M @>>> M'' @>>> 0 \\
@. @| @VV{f}V @VV{f''}V @. \\
0 @>>> N' @>>> N @>>> N'' @>>> 0,
\end{CD}$$

the bottom row is exact. Then, the top row also is exact if and only if the right hand square is cartesian (and dual).

Solution. It follows from exercise 2.1.38, proposition 2.1.63 and exercise 2.1.59 that the condition is sufficient. For the converse, we can first pull back the bottom row and apply the five lemma upstairs:

$$\begin{CD}
0 @>>> M' @>>> M @>>> M'' @>>> 0 \\
@. @| @VVV @| @. \\
0 @>>> N'' @>>> \widetilde{M} @>>> M'' @>>> 0 \\
@. @| @VVV @VV{f''}V @. \\
0 @>>> N' @>>> N @>>> N'' @>>> 0.
\end{CD}$$

■

Definition 2.1.70 An *extension* of M by N is a short exact sequence $0 \to N \to E \to M \to 0$. The extension is said to be *trivial* if the short exact sequence is split.

We shall denote by $\mathrm{Ext}(M, N)$ the set of extensions of M by N up to (iso) morphism

$$\begin{CD}
0 @>>> N @>>> E @>>> M @>>> 0 \\
@. @| @VV{\simeq}V @| @. \\
0 @>>> N @>>> E' @>>> M @>>> 0
\end{CD}$$

(by the five lemma, any morphism is automatically an isomorphism). The *pull-back* of an extension E of M by N along a morphism $f : M' \to M$ is the unique extension (up to isomorphism) making commutative the diagram

$$\begin{CD}
0 @>>> N @>>> f^*E @>>> M' @>>> 0 \\
@. @| @VVV @VV{f}V @. \\
0 @>>> N @>>> E @>>> M @>>> 0.
\end{CD}$$

Uniqueness is a consequence of exercise 2.1.69 that also shows that the right hand square is automatically cartesian. This is clearly functorial. The *push-out* of E along $g : N \to N'$ is defined dually by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & \lrcorner & \downarrow & & \| & & \\
0 & \longrightarrow & N' & \longrightarrow & g_*E & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Exercise 2.1.71 Show that if E is an extension of M by N, $f : M' \to M$ and $g : N \to N'$, then $f^*g_*E = g_*f^*E$ (up to isomorphism).

Definition 2.1.72 The *Baer sum* of two extensions E and E' of M by N is

$$E + E' := \nabla_*\Delta^*(E \oplus E') = \Delta^*\nabla_*(E \oplus E')$$

with $\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : M \to M \oplus M$ and $\nabla = \begin{bmatrix} 1 & 1 \end{bmatrix} : N \oplus N \to N$.

This is pictured as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \oplus N & \longrightarrow & E \oplus E' & \longrightarrow & M \oplus M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla} & \lrcorner & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M \oplus M & \longrightarrow & 0 \\
 & & \| & & \uparrow & \llcorner & {\scriptstyle \Delta}\uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & E + E' & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Exercise 2.1.73 Show that $\mathrm{Ext}(M, N)$ is an abelian group for the Baer sum.

Exercise 2.1.74 Show that (and dual)

1. if $g : N \to N'$ then $g_* : \mathrm{Ext}(M, N) \to \mathrm{Ext}(M, N')$ is a morphism of groups,
2. if E is an extension of M by N then the map

$$\mathrm{Hom}(M', M) \to \mathrm{Ext}(M', N), \quad f \mapsto f^*E$$

is a morphism of groups.

Exercise 2.1.75 Show that if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then there exists a long exact sequence (and dual)

$$\begin{array}{l}
0 \longrightarrow \mathrm{Hom}(M, N') \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N'') \longrightarrow \\
\quad \longrightarrow \mathrm{Ext}(M, N') \longrightarrow \mathrm{Ext}(M, N) \longrightarrow \mathrm{Ext}(M, N'').
\end{array}$$

2.1.6 Grothendieck category

We now introduce Grothendieck axioms:

Definition 2.1.76 A category is:
1. AB3 : Abelian with all colimits (AB3* : dual).
2. AB4 : AB3 and coproducts are exact (AB4* : dual).
3. AB5 : AB4 and filtered colimits are exact (AB5* : dual).
4. AB6 : AB5 and filtered colimits commute with products (AB6* : dual).

Remark This may be extended to
1. AB0 : preadditive with finite coproducts (so that AB0* = AB0 = additive)
2. AB1 : AB0 with cokernels (so that AB1 + AB1* = preabelian)
3. AB2 : AB1 with epimorphisms being regular (so that AB2 + AB2* = abelian)

Examples
1. The category **Ab** satisfies AB6 + AB4*.
2. The category **AbTop** is only preabelian (AB1 + AB1*).
3. The category **AbCHaus** $\simeq$ **Ab**$^{\text{op}}$ (Pontryagin duality) satisfies AB4 + AB6*.
4. If $\mathcal{C}$ is any category, then $\mathbf{Ab}\widehat{\mathcal{C}} \simeq \widehat{\mathcal{C}}(\mathbf{Ab})$ satisfies AB6 + AB4*.
5. If $\mathcal{P}$ is a small preadditive category, then $\mathcal{P}$-**Mod** satisfies AB6 + AB4*.
6. The category of *sheaves* of abelian groups (see below) on a topological space satisfies AB5 + AB3* (but not AB6 nor AB4* in general).
7. There is no category satisfying AB5 + AB5* besides $\{0\}$.

Definition 2.1.77 A Grothendieck category is an AB5 category that has a generator.

Examples
1. A-**Mod** is a Grothendieck category.
2. If $\mathcal{A}$ is a small abelian category, then $\text{Ind}(\mathcal{A})$ is a Grothendieck category.
3. If $\mathcal{C}$ is a small category, then $\mathbf{Ab}(\widehat{\mathcal{C}})$ is a Grothendieck category.
4. If $\mathcal{P}$ is a small preadditive category, then $\mathcal{P}$-**Mod** is a grothendieck category
5. The category of *sheaves* of abelian groups (see below) on a topological space is a Grothendieck category.

Exercise 2.1.78 Show that if $\mathcal{C}$ is an AB5 category, then G is a generator if and only if, for all $M \in \mathcal{C}$, there exists an epimorphism $G^{(I)} := \bigoplus_I G \twoheadrightarrow M$.

Proposition 2.1.79 A Grothendieck category is automatically AB3*.

Proof. To do. ∎

Proposition 2.1.80 A Grothendieck category has enough injectives.

Proof. To do. ∎

2.2 Complex

2.2.1 Definition/Examples

We fix an additive category $\mathcal{P}$.

Definition 2.2.1
1. A *(long) sequence* of $\mathcal{P}$ is a commutative diagram on the linearly ordered set $(\mathbb{Z}, \leq)$:

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d_{n-1}} K^n \xrightarrow{d_n} K^{n+1} \longrightarrow \cdots$$

We call K^n (resp. d_n) the *term* (resp. *differential*) of *degree* n.

2. If $d_n \circ d_{n-1} = 0$ for each $n \in \mathbb{Z}$, then $K^\bullet$ is called a *(cochain) complex*.
3. A complex $K^\bullet$ is said to be *bounded below* (resp. *bounded above*, resp. *bounded*) if $K^n = 0$ for $n << 0$ (resp. for $n >> 0$, resp. both for $n << 0$ and $n >> 0$).

A morphism $f : K^\bullet \to L^\bullet$ is a sequence of morphisms $f_n : K^n \to L^n$ satisfying $f_{n+1} \circ d_n = d_n \circ f_n$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K^{n-1} & \xrightarrow{d_{n-1}} & K^n & \xrightarrow{d_n} & K^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ \cdots & \longrightarrow & L^{n-1} & \xrightarrow{d_{n-1}} & L^n & \xrightarrow{d_n} & L^{n+1} & \longrightarrow & \cdots . \end{array}$$

Proposition 2.2.2 1. Complexes of $\mathcal{P}$ form a (full) additive subcategory $\mathbf{C}(\mathcal{P})$ of $\mathcal{P}^{(\mathbb{Z},\leq)}$,
2. limits and colimits are computed termwise in $\mathbf{C}(\mathcal{P})$,
3. any additive functor $F : \mathcal{P} \to \mathcal{P}'$ provides an additive functor

$$F^\bullet : \mathbf{C}(\mathcal{P}) \to \mathbf{C}(\mathcal{P}').$$

Proof. All similar assertions are known for $\mathcal{P}^{(\mathbb{Z},\leq)}$. Moreover, any limit of colimit of complexes is still a complex since any limit or colimit of commutative diagrams of the form

$$\begin{array}{ccc} K^{n-1} & \longrightarrow & K^n \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K^{n+1} \end{array}$$

has the same form. Everything follows. ∎

Exercise 2.2.3 1. Show that the inclusion $\mathbf{1} = \{0\} \rightarrowtail (\mathbb{Z}, \leq)$ induces a functor $\mathcal{P} \rightarrowtail \mathbf{C}(\mathcal{P})$ which is fully faithful and preserves all limits and colimits. We shall identify $\mathcal{P}$ with its image in $\mathbf{C}(\mathcal{P})$ so that $M^0 = M$ and $M^n = 0$ otherwise.
2. Show that the inclusion $[\mathbf{1}] := \{0 < 1\} \rightarrowtail (\mathbb{Z}, \leq)$ induces a functor

$$\mathbf{Mor}(\mathcal{P}) \rightarrowtail \mathbf{C}(\mathcal{P}), \quad (M \xrightarrow{f} N) \mapsto [M \xrightarrow{f} N]$$

(with M in degree 0 and N in degree 1) which again is fully faithful and preserves all limits and colimits.
3. Analog with left, right or short exact sequences ?

We shall denote by $\mathbf{C}^+(\mathcal{P})$, $\mathbf{C}^-(\mathcal{P})$ and $\mathbf{C}^b(\mathcal{P})$ the categories of complexes that are respectively bounded below, bounded above and bounded. All the coming developments has an equivalent with $+$, $-$ and b.

A *chain complex* is defined exactly as a cochain complex but using diagrams on $(\mathbb{Z}, \geq) = (\mathbb{Z}, \leq)^{\text{op}}$ instead and writing $d_n : K_n \to K_{n-1}$. Equivalently, a chain complex is a cochain complex in $\mathcal{P}^{\mathrm{op}}$ (with morphisms in the reverse direction). Actually, the isomorphism $(\mathbb{Z}, \leq) \simeq (\mathbb{Z}, \geq), n \mapsto -n$ provides an isomorphism $\mathbf{C}(\mathcal{P}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{C}(\mathcal{P})$ between chain complexes and cochain complexes. We shall make that identification (so that $K_n = K^{-n}$ and d_n is turned in d_{-n}) and only consider a chain complex as a change in the notation. We shall let the reader deal by him/herself with the dual situation but will systematically use the corresponding results when needed.

Exercise 2.2.4 Show that if $K_\bullet$ is a (semi) simplicial object of $\mathcal{P}$ and we set $d_n := \sum_{i=0}^{n-1}(-1)^i d_n^i$, then $K_\bullet$ becomes a chain complex. Show that this provides a functor $\widehat{\mathbf{\Delta}}(\mathcal{P}) \to \mathbf{C}(\mathcal{P})$ (called the Dold-Kan correspondance[a]).

[a] When $\mathcal{P}$ is abelian, we obtain an equivalence between simplicial complexes and chain complexes indexed by $\mathbb{N}$.

Examples 1. Recall that there exists a functor

$$\mathbf{Top} \to \widehat{\mathbf{\Delta}}, \quad X \to S_\bullet(X) \quad \text{with} \quad S_n(X) = \mathcal{C}(|\Delta^n|, X).$$

On the other hand, the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$ induces by composition a functor $\widehat{\mathbf{\Delta}} := \widehat{\mathbf{\Delta}}(\mathbf{Set}) \to \widehat{\mathbf{\Delta}}(\mathbf{Ab})$. Moreover, we have the Dold-Kan correspondance $\widehat{\mathbf{\Delta}}(\mathbf{Ab}) \to \mathbf{C}(\mathbf{Ab})$. All in all, we get a functor

$$\mathbf{Top} \to \mathbf{C}(\mathbf{Ab}), \quad X \to C_\bullet(X) \quad \text{with} \quad C_n(X) := \mathbb{Z} \cdot S_n(X).$$

2. Alternatively, using the functor

$$\mathbf{Set}^{\mathrm{op}} \to \mathbf{Ab}, \quad S \mapsto \mathcal{F}(S, M),$$

where M is a fixed abelian group, we obtain

$$\mathbf{Top}^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab}), \quad X \to C^\bullet(X, M)$$

with

$$C^n(X, M) := \mathcal{F}(S_n(X), M) \simeq \mathrm{Hom}_{\mathbf{Ab}}(C_n(X), M).$$

The augmentation map provides a natural morphism of complexes $M \to C^\bullet(X, M)$.

3. If X is a smooth/analytic/complex manifold, then there exists the de Rham complex $\Omega^\bullet(X)$. The fact that this is indeed a complex relies in the end on Schwarz's theorem. There also exists a morphism of complexes $\mathbb{R} \to \Omega^\bullet(X)$ (or $\mathbb{C}$ in the complex case).
4. If A is a commutative R-algebra, then there exists also a de Rham complex $\Omega^\bullet_{A/R}$ and a morphism $R \to \Omega^\bullet_{A/R}$.

There exists an alternative approach when direct sums exist in $\mathcal{P}$. A *differential object*[2] of $\mathcal{P}$ is an object M endowed with an endomorphism d such that $d \circ d = 0$. We can then consider the complex

$$\cdots \longrightarrow M \xrightarrow{d} M \xrightarrow{d} M \longrightarrow \cdots$$

and this is functorial. There exists a functor in the other direction sending a complex $K^\bullet$ to the direct sum $\oplus_{n \in \mathbb{Z}} K^n$ endowed with the endomorphism d induced by the d_n's.

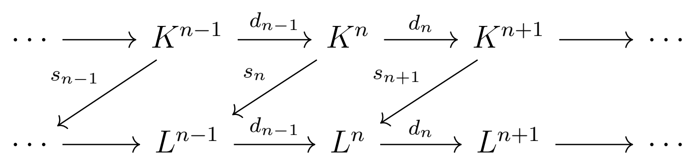
[2] Be careful that other authors may have sensibly different definitions.

Exercise 2.2.5 Show that there exists an adjunction between differential objects and complexes.

2.2.2 Homotopy

Recall that $\mathcal{P}$ denotes an additive category.

Definition 2.2.6 Two morphisms $f, g : K^\bullet \to L^\bullet$ are said to be *homotopic* if there exists a family of morphisms $s_n : K^n \to L^{n-1}$:



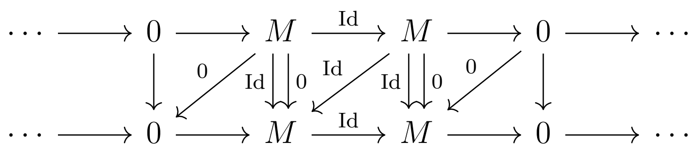
such that, for all n,

$$f_n - g_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n.$$

We shall then write $s : f \sim g$.

Clearly, $f \sim g \Leftrightarrow f - g \sim 0$.

Example In the case $K^\bullet = L^\bullet = [M \overset{\mathrm{Id}}{\to} M]$, $f = \mathrm{Id}$, $g = 0$, there exists a homotopy:



Another example is given by:

Proposition 2.2.7 Show that, if $f, g : X \to Y$ are homotopic continuous maps, then the morphisms $f_*, g_* : C_\bullet(X) \to C_\bullet(Y)$ obtained by functoriality are homotopic.

Proof. The construction is based on the notion of *prism*

$$p_n := \sum_{i=0}^{n} (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_n] \in \mathrm{C}_{n+1}(|\Delta^n| \times [0,1])$$

where $v_i = (e_i, 0), w_i := (e_i, 1) \in |\Delta^n| \times [0,1]$. Then if we are given a homotopy $h : X \times [0,1] \to Y$ between f and $g$and $\sigma \in S_n(X)$, we consider the composite map

$$h_\sigma : |\Delta^n| \times [0,1] \overset{\sigma \times \mathrm{Id}_{[0,1]}}{\longrightarrow} X \times [0,1] \overset{h}{\longrightarrow} Y.$$

and let

$$s_n : C_n(X) \to C_{n+1}(Y), \quad \sigma \mapsto h_{\sigma *}(p_n).$$

It is then a matter of checking that this is indeed a homotopy. ■

Exercise 2.2.8 Show that morphisms that are homotopic to 0 form a subgroup of $\mathrm{Hom}_{\mathbf{C}(\mathcal{P})}(K^\bullet, L^\bullet)$.

Exercise 2.2.9 Show that if $f \sim g : K^\bullet \to L^\bullet$ and $\varphi : L^\bullet \to M^\bullet$ (resp. $\psi : J^\bullet \to K^\bullet$), then $\varphi \circ f \sim \varphi \circ g$ (resp. $f \circ \psi \sim g \circ \psi$).

Solution. This is readily checked (the other assertion being obtained by duality):

$$\begin{aligned}\varphi_n \circ f_n - \varphi_n \circ g_n &= \varphi_n \circ s_{n+1} \circ d_n + \varphi_n \circ d_{n-1} \circ s_n \\ &= \varphi_n \circ s_{n+1} \circ d_n + d_{n-1} \circ \varphi_{n-1} \circ s_n. \end{aligned}$$ ■

Proposition 2.2.10 Homotopy is an equivalence relation compatible with composition on both sides.

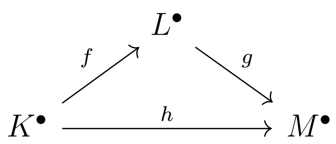
Proof. Clear. ■

Definition 2.2.11 The category $\mathbf{K}(\mathcal{P})$ of *complexes up to homotopy* is the category that has the same objects as $\mathbf{C}(\mathcal{P})$ and

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathbf{C}(\mathcal{P})}(K^\bullet, L^\bullet)/\sim$$

with induced composition. A morphism of complexes that becomes an isomorphism in $\mathbf{K}(\mathcal{P})$ is called a *homotopy equivalence*. A complex that becomes (isomorphic to) 0 in $\mathbf{K}(\mathcal{P})$ is said to be *contractible*.

There exists an obvious functor $\mathbf{C}(\mathcal{P}) \to \mathbf{K}(\mathcal{P})$ sending an object $K^\bullet$ to itself and a morphism f to its equivalence class "f up to homotopy"[3]. In practice, we shall simply say f but we could as well write $[f]$ to remove any ambiguity. For example, we shall say that a diagram



(where f, g, h are genuine morphisms of complexes) commutes up to homotopy is there exists a homotopy $s : g \circ f \sim h$. There also exists a notion of homotopy for *chain* complexes but, again, one shall simply see it as a change in the notations because $\mathbf{K}(\mathcal{P}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{K}(\mathcal{P})$.

Exercise 2.2.12
1. Show that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a homotopy equivalence if and only if there exists a morphism $g : L^\bullet \to K^\bullet$ such that $g \circ f \sim \mathrm{Id}$ and $f \circ g \sim \mathrm{Id}$.
2. Show that a complex $K^\bullet$ is contractible if and only if $\mathrm{Id}_{K^\bullet} \sim 0_{K^\bullet}$.

Exercise 2.2.13 Show that a complex $[M \overset{\mathrm{Id}}{\to} M]$ is always contractible but the

[3] Exactly as we talk about an angle "θ modulo 2π".

complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$

is not (although the sequence is exact).

Solution. For the complex $[M \xrightarrow{\mathrm{Id}} M]$, we have by definition $d_0 = \mathrm{Id}_M$ and $d_n = 0$ otherwise. Moreover, if we denote by f the identity of this complex, then we have $f_0 = f_1 = \mathrm{Id}_M$ and $f_n = 0$ otherwise. Now, we set $s_1 := \mathrm{Id}_M$ and $s_n = 0$ otherwise. Since $f_0 = s_1 \circ d_0$ and $f_1 = d_0 \circ s_1$, we have

$$\forall n \in \mathbb{Z}, \quad f_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n.$$

It means that there exists a homotopy $\mathrm{Id}_M \simeq 0_M$ showing that M is trivial in the homotopy category or, equivalently, contractible.

Now, if the complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$ were contractible, there would then exist a homotopy between the identily and the zero map. Since the only morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the zero map, there would exist a morphism $s : \mathbb{Z} \to \mathbb{Z}$ such that $\mathrm{Id}_{\mathbb{Z}} = d \circ s$ where d is multiplication by 2. This cannot happen. ■

Proposition 2.2.14 $\mathbf{K}(\mathcal{P})$ is an additive category and $\mathbf{C}(\mathcal{P}) \to \mathbf{K}(\mathcal{P})$ is an additive functor. The composite functor $\mathcal{P} \to \mathbf{C}(\mathcal{P}) \to \mathbf{K}(\mathcal{P})$ is fully faithful.

Proof. For the first assertion, use the last criterion of proposition 2.1.11. For the second one, note that for $f : M \to N$, we have $f \sim 0 \Leftrightarrow f = 0$. ■

Exercise 2.2.15 Show that any additive functor $F : \mathcal{P} \to \mathcal{P}'$ provides a functor

$$F^\bullet : \mathbf{K}(\mathcal{P}) \to \mathbf{K}(\mathcal{P}').$$

On also defines $\mathbf{K}^+(\mathcal{P})$, $\mathbf{K}^-(\mathcal{P})$ and $\mathbf{K}^{\mathrm{b}}(\mathcal{P})$ as above (may also be seen as full subcategories of $\mathbf{K}(\mathcal{P})$).

2.2.3 Distinguished triangle: definition

We still assume that $\mathcal{P}$ is an additive category. The notions of a kernel or cokernel are not relevant in $\mathbf{K}(\mathcal{P})$ and have to be replaced by that of a *mapping cone*. The notion of an exact sequence will then be replaced by that of a *distinguished* triangle.

Definition 2.2.16 The *kth shift* $K^\bullet[k]$ of a complex $K^\bullet$ is defined by $K^\bullet[k]^n := K^{n+k}$ endowed with $(-1)^k d_{n+k}$.

Actually, there exists a functor $K^\bullet \mapsto K^\bullet[k]$ (with $f[k]_n := f_{n+k}$) which is an automorphism of the category $\mathbf{C}(\mathcal{P})$. It induces an automorphism of $\mathbf{K}(\mathcal{P})$.

Be careful that, when $M \in \mathcal{P}$, $M[k]$ means M in degree $-k$ (and 0 elsewhere).

Exercise 2.2.17 Show that, for $M, N \in \mathcal{P}$, we have

$$\mathrm{Hom}_{\mathbf{C}(\mathcal{P})}(M[k], N[\ell]) = \mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(M[k], N[\ell]) = \begin{cases} \mathrm{Hom}_{\mathcal{P}}(M, N) \text{ if } k = \ell \\ 0 \text{ otherwise.} \end{cases}$$

We now introduce the following technical but fundamental notion:

Definition 2.2.18 The *mapping cone* of a morphism of complexes $f : K^\bullet \to L^\bullet$ is the complex $M(f)^\bullet$ with

$$\mathrm{M}(f)^n := K^{n+1} \oplus L^n \quad \text{and} \quad d_n := \begin{bmatrix} -d_{n+1} & 0 \\ f_{n+1} & d_n \end{bmatrix}.$$

Exercise 2.2.19 Show that the mapping cone is indeed a complex.

Examples 1. In the case $f = 0 : K^\bullet \to L^\bullet$, we get $\mathrm{M}(0)^\bullet = K^\bullet[1] \oplus L^\bullet$.
2. If $f : M \to N$ is a morphism in $\mathcal{P}$, then

$$\mathrm{M}(f) = [M \xrightarrow{f} N][1].$$

Definition 2.2.20 A sequence $0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$ is *termwise split* if $0 \to K^n \xrightarrow{f_n} L^n \xrightarrow{g_n} M^n \to 0$ a split exact sequence for all $n \in \mathbb{Z}$.

Note that $L^\bullet$ is then an extension of $M^\bullet$ by $K^\bullet$ that will be said to be *termwise split*.

Lemma 2.2.21 If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, then $\mathrm{M}(f)$ is a termwise split extension of $K^\bullet[1]$ by $L^\bullet$.

Proof. Actually, there exists a sequence of complexes

$$0 \to L^\bullet \xrightarrow{g} \mathrm{M}(f)^\bullet \xrightarrow{h} K^\bullet[1] \to 0$$

with $g_n := \begin{bmatrix} 0 \\ 1 \end{bmatrix} : L^n \to K^{n+1} \oplus L^n$ and $h_n = \begin{bmatrix} 1 & 0 \end{bmatrix} : K^{n+1} \oplus L^n \to K^{n+1}$. This is proved as follows:

$$g_{n+1} \circ d_n = \begin{bmatrix} 0 \\ d_n \end{bmatrix} = \begin{bmatrix} -d_{n+1} & 0 \\ f_{n+1} & d_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = d_n \circ g_n$$

and

$$h_{n+1} \circ d_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -d_{n+1} & 0 \\ f_{n+1} & d_n \end{bmatrix} = \begin{bmatrix} -d_{n+1} & 0 \end{bmatrix} = d_n \circ h_n$$

(where the last d_n is meant as the differential of $K^\bullet[1]$ which is $-d_{n+1}$ – if we start from of $K^\bullet$). ∎

Examples 1. In the case $f = 0$, we have a trivial extension

$$0 \to L^\bullet \to K^\bullet[1] \oplus L^\bullet \to K^\bullet[1] \to 0$$

of complexes.
2. If $f : M \to N$ is a morphism in $\mathcal{P}$, then the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & = & N & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & M & = & M & \longrightarrow & 0 \end{array}$$

of vertical complexes is exact.

Exercise 2.2.22 Show that, conversely, if $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ is a termwise split extension, then there exists a morphism $h : M^\bullet[-1] \to K^\bullet$ and an isomorphism $L^\bullet \simeq \mathrm{M}(h)$ (of extensions).

From now on, we need to work up to homotopy.

Definition 2.2.23 A *triangle* in $\mathbf{K}(\mathcal{P})$ is a diagram

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1].$$

A *morphism of triangles* is a commutative diagram (in $\mathbf{K}(\mathcal{P})$)

$$\begin{array}{ccccccc}
K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1].
\end{array}$$

The triangle is said to be *distinguished* if it is isomorphic (in $\mathbf{K}(\mathcal{P})$) to

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} \mathrm{M}(f)^\bullet \xrightarrow{h} K^\bullet[1].$$

with $g_n := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $h_n = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

In fancy terms, the category of triangles is the kernel of the functors $\mathbf{K}(\mathcal{P})^{[3]} \to \mathbf{K}(\mathcal{P})$ that send $K^\bullet \to L^\bullet \to M^\bullet \to N^\bullet$ to $K^\bullet[1]$ and $N^\bullet$ respectively. The name "triangle" comes from the following representation:

$$\begin{array}{ccc}
L^\bullet & \longrightarrow & M^\bullet \\
 & \nwarrow \quad \swarrow_{+1} & \\
 & K^\bullet. &
\end{array}$$

This notion may be seen as an analog, in the category of complexes up to homotopy, of the notion of a short exact sequence in an abelian category.

Examples

1. The triangle

$$K^\bullet \xrightarrow{0} L^\bullet \to L^\bullet \oplus K^\bullet[1] \to K^\bullet[1]$$

is distinguished.

2. The triangle

$$M \xrightarrow{f} N \to [M \xrightarrow{f} N][1] \to M[1]$$

is distinguished.

3. There is *no* distinguished triangle $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}[1]$.

Proposition 2.2.24 If F is an additive functor and $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ is distinguished, then $F(K^\bullet) \to F(L^\bullet) \to F(M^\bullet) \to F(K^\bullet)[1]$ is also distinguished.

Proof. Follows from $F(\mathrm{M}(f)) = \mathrm{M}(F(f))$. ■

2.2.4 Distinguished triangle: properties

We still assume that $\mathcal{P}$ is an additive category.

The next fundamental result[4] shows that distinguished triangles are stable under shifting:

Theorem 2.2.25 A triangle

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \xrightarrow{h} K^\bullet[1]$$

is distinguished if and only if the triangle

$$L^\bullet \xrightarrow{g} M^\bullet \xrightarrow{h} K^\bullet[1] \xrightarrow{-f[1]} L^\bullet[1]$$

is distinguished.

Proof. For the reverse implication, it is equivalent to show that $K^\bullet[1] \to L^\bullet[1] \to M^\bullet[1] \to K^\bullet[2]$ is distinguished and we may apply twice the direct implication to the triangle $L^\bullet \to M^\bullet \to K^\bullet[1] \to L^\bullet[1]$.

To prove the direct implication, we may assume that $M = \mathrm{M}(f)$ with g, h as usual and it is sufficient to show that there exists a homotopy equivalence $\phi : K^\bullet[1] \to \mathrm{M}(g)^\bullet$ making the diagram

$$\begin{array}{ccccccc} L^\bullet & \xrightarrow{g} & \mathrm{M}(f)^\bullet & \xrightarrow{h} & K^\bullet[1] & \xrightarrow{-f[1]} & L^\bullet[1] \\ \| & & \| & & \downarrow\phi & & \| \\ L^\bullet & \xrightarrow{g} & \mathrm{M}(f)^\bullet & \xrightarrow{g'} & \mathrm{M}(g)^\bullet & \xrightarrow{h'} & L^\bullet[1] \end{array}$$

commutative up to homotopy with

$$g_n := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, h_n = \begin{bmatrix} 1 & 0 \end{bmatrix}, g'_n := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, h'_n = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

If we set

$$\phi_n := \begin{bmatrix} -f_{n+1} \\ 1 \\ 0 \end{bmatrix} : K^\bullet[1]^n := K^{n+1} \to \mathrm{M}(g)^n := L^{n+1} \oplus K^{n+1} \oplus L^n,$$

then

$$\phi_{n+1} \circ d_n = \begin{bmatrix} f_{n+2} \circ d_{n+1} \\ -d_{n+1} \\ 0 \end{bmatrix}$$

[4] In fancy language, it means that $\mathbf{K}(\mathcal{P})$ is a *triangulated* category.

and

$$d_n \circ \phi_n = \begin{bmatrix} -d_{n+1} & 0 & 0 \\ 0 & -d_{n+1} & 0 \\ 1 & f_{n+1} & d_n \end{bmatrix} \begin{bmatrix} -f_{n+1} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{n+1} \circ f_{n+1} \\ -d_{n+1} \\ 0 \end{bmatrix},$$

showing that ϕ is a morphism of complexes.

To define a morphism in the other direction, we set

$$\psi_n := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} : \mathrm{M}(g)^n := L^{n+1} \oplus K^{n+1} \oplus L^n \to K[1]^n := K^{n+1}$$

and we check:

$$\begin{aligned} \psi_{n+1} \circ d_n &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -d_{n+1} & 0 & 0 \\ 0 & -d_{n+1} & 0 \\ 1 & f_{n+1} & d_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & -d_{n+1} & 0 \end{bmatrix} \\ &= d_n \circ \psi_n. \end{aligned}$$

Now, we have

$$\psi_n \circ \phi_n = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -f_{n+1} \\ 1 \\ 0 \end{bmatrix} = \mathrm{Id}$$

so that $\psi \circ \varphi = \mathrm{Id}$. In the other direction, we build a homotopy $s : \mathrm{Id} \sim \varphi \circ \psi$ by setting:

$$s_n := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the one hand,

$$\begin{aligned} \mathrm{Id} - \phi_n \circ \psi_n &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -f_{n+1} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & f_{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and on the other hand:

$$\begin{aligned} s_{n+1} \circ d_n + d_{n-1} \circ s_n &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -d_{n+1} & 0 & 0 \\ 0 & -d_{n+1} & 0 \\ 1 & f_{n+1} & d_{n+1} \end{bmatrix} \\ &+ \begin{bmatrix} -d_n & 0 & 0 \\ 0 & -d_n & 0 \\ 1 & f_n & d_n \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & f_{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus we see that ϕ and ψ are inverse homotopy equivalences but it still remains to check that the diagram is commutative (up to homotopy). This follows from equalities

$$h'_n \circ \varphi_n = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -f_{n+1} \\ 1 \\ 0 \end{bmatrix} = -f_{n+1}$$

showing that $h' \circ \varphi = -f[1]$ and

$$\psi_n \circ g'_n = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = g_n$$

showing that $\psi \circ g' = g$. ■

Exercise 2.2.26 Show that

1. $K^\bullet \overset{\mathrm{Id}}{\to} K^\bullet \to 0 \to K^\bullet[1]$ and
2. $[M \overset{f}{\to} N] \to M \overset{f}{\to} N \to [M \overset{f}{\to} N][1]$

are always distinguished.

Corollary 2.2.27 If $K^\bullet \overset{f}{\to} L^\bullet \overset{g}{\to} M^\bullet \to K^\bullet[1]$ is distinguished, then $g \circ f$ is homotopic to zero.

Proof. Any distinguished triangle is isomorphic to *some*

$$L^\bullet \overset{g}{\to} \mathrm{M}(f) \overset{h}{\to} K^\bullet[1] \overset{f[1]}{\to} L^\bullet[1]$$

where $h \circ g = 0$. ■

Exercise 2.2.28 Show that

1. if $0 \to K^\bullet \overset{f}{\to} L^\bullet \overset{g}{\to} M^\bullet \to 0$ is a termwise split extension, then there exists a distinguished triangle $K^\bullet \overset{f}{\to} L^\bullet \overset{g}{\to} M^\bullet \to K^\bullet[1]$,
2. conversely, any distinguished triangle is isomorphic to a triangle coming from a termwise split extension.

The next statement shows that the mapping cone is "almost" functorial:

Proposition 2.2.29

1. Any morphism $f : K^\bullet \to L^\bullet$ can be extended to a (not unique) distinguished triangle $K^\bullet \overset{f}{\to} L^\bullet \to M^\bullet \to K^\bullet[1]$.
2. Any diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{f} & L^\bullet \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ K'^\bullet & \xrightarrow{f'} & L'^\bullet \end{array}$$

that is commutative up to homotopy can then be extended to a (not unique) morphism

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1] \end{array}$$

of distinguished triangles.

Proof. 1. Use the mapping cone $\mathrm{M}(f)^\bullet$.
2. We can write $v_n \circ f_n - f'_n \circ u_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n$ and then set

$$w_n := \begin{bmatrix} u_{n+1} & 0 \\ s_{n+1} & v_n \end{bmatrix} = \mathrm{M}(f)^n \to \mathrm{M}(f')^n.$$

We first need to check that it defines a morphism of complexes:

$$\begin{aligned} w_{n+1} \circ d_n &= \begin{bmatrix} u_{n+2} & 0 \\ s_{n+2} & v_{n+1} \end{bmatrix} \begin{bmatrix} -d_{n+1} & 0 \\ f_{n+1} & d_n \end{bmatrix} \\ &= \begin{bmatrix} -u_{n+2} \circ d_{n+1} & 0 \\ -s_{n+2} \circ d_{n+1} + v_{n+1} \circ f_{n+1} & v_{n+1} \circ d_n \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} d_n \circ w_n &= \begin{bmatrix} -d_{n+1} & 0 \\ f'_{n+1} & d_n \end{bmatrix} \begin{bmatrix} u_{n+1} & 0 \\ s_{n+1} & v_n \end{bmatrix} \\ &= \begin{bmatrix} -d_{n+1} \circ u_{n+1} & 0 \\ f'_{n+1} \circ u_{n+1} + d_n \circ s_{n+1} & d_n \circ v_n \end{bmatrix}. \end{aligned}$$

It is also necessary to check that the diagram is commutative:

$$w_n \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \circ v_n$$

and

$$u_{n+1} \circ \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} u_{n+1} & 0 = \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \circ w_n.$$

■

Lemma 2.2.30 If we are given a morphism of distinguished triangles

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \xrightarrow{h} & K^\bullet[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle 0} \\ K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \xrightarrow{h} & K^\bullet[1], \end{array}$$

then $w^2 = 0$ up to homotopy.

Proof. We may assume that $M^\bullet = \mathrm{M}(f)^\bullet$ with g, h as usual and write

$$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}.$$

We then have $\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = w \circ g \sim 0$ and $\begin{bmatrix} w_{21} & w_{22} \end{bmatrix} = h \circ w \sim 0$. Therefore,

$$w \sim w' := \begin{bmatrix} 0 & w_{12} \\ 0 & w_{22} \end{bmatrix} \quad \text{and} \quad w \sim w'' := \begin{bmatrix} w_{11} & w_{12} \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$w^2 \sim w' \circ w'' = \begin{bmatrix} 0 & w_{12} \\ 0 & w_{22} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ 0 & 0 \end{bmatrix} = 0.$$ ■

Proposition 2.2.31 Let

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ K'^\bullet & \xrightarrow{f'} & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1] \end{array}$$

be a morphism of distinguished triangles. If two of the morphisms u, v and w are homotopy equivalences, then so is the third (and the triangles are isomorphic).

Proof. Using proposition 2.2.25, we may assume that u and v are homotopy equivalences and we need to show that the same holds for w. If we denote by u' and v' homotopy inverses for u and v respectively, then, as it would be the case in any category, $f \circ u' \sim v' \circ f$. Using proposition 2.2.29, we can then build a morphism of triangles in the other direction:

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ {\scriptstyle u'}\uparrow & & {\scriptstyle v'}\uparrow & & {\scriptstyle w'}\uparrow & & {\scriptstyle u'[1]}\uparrow \\ K'^\bullet & \xrightarrow{f'} & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1]. \end{array}$$

As this is again the case in any category, w and w' are homotopy equivalences if and only if both $w \circ w'$ and $w' \circ w$ are homotopy equivalences. We are therefore reduced to the case $u = \mathrm{Id}$ and $v = \mathrm{Id}$. Then, we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle w-\mathrm{Id}} & & \downarrow{\scriptstyle 0} \\ K^\bullet & \xrightarrow{f} & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1]. \end{array}$$

If follows from lemma 2.2.30 that $(w - \mathrm{Id})^2 \sim 0$. Thus we see that $2\mathrm{Id} - w$ is a homotopy inverse for w. ■

As a consequence, the mapping cone is "almost" unique up to a homotopy equivalence.

Corollary 2.2.32 In a distinguished triangle

$$K^\bullet \xrightarrow{f} L^\bullet \to M^\bullet \to K^\bullet[1],$$

f is a homotopy equivalence if and only if $M^\bullet$ is contractible.

Proof. It is sufficient to contemplate the following diagram:

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow f & & \| & & \downarrow & & \downarrow f[1] \\ L^\bullet & = & L^\bullet & \longrightarrow & 0 & \longrightarrow & L^\bullet[1]. \end{array}$$

■

2.2.5 Cohomology

Let $\mathcal{A}$ be an *abelian* category.

Exercise 2.2.33 Show that $\mathbf{C}(\mathcal{A})$ also is abelian.

Definition 2.2.34 The n-th *cohomology* object of a cochain complex $K^\bullet$ of $\mathcal{A}$ is

$$\mathrm{H}^n(K^\bullet) := \operatorname{coker}(\operatorname{im} d_{n-1} \rightarrowtail \ker d_n).$$

It may also be convenient to consider the $\mathrm{Z}^n(K^\bullet) := \ker d_n$ (n-th cycles) and $\mathrm{B}^n(K^\bullet) := \operatorname{im} d_{n-1}$ (nth boundary). Then, there exists short exact sequences

$$0 \to \mathrm{B}^n(K^\bullet) \to \mathrm{Z}^n(K^\bullet) \to \mathrm{H}^n(K^\bullet) \to 0 \tag{2.6}$$

and

$$0 \to \mathrm{Z}^n(K^\bullet) \to K^n \to \mathrm{B}^{n+1}(K^\bullet) \to 0. \tag{2.7}$$

One defines in the same way the homology of a chain complex.

Examples 1. We have $\mathrm{H}^n(K^\bullet[k]) = \mathrm{H}^{n+k}(K^\bullet)$.
2. If $M \in \mathcal{A}$, then $\mathrm{H}^0(M) = M$ and $\mathrm{H}^n(M) = 0$ otherwise.
3. We have $\mathrm{H}^0([M \xrightarrow{f} N]) = \ker f$ and $\mathrm{H}^1([M \xrightarrow{f} N]) = \operatorname{coker} f$ (and 0 otherwise).
4. A short exact sequence $0 \to M' \to M \to M'' \to 0$ has 0 cohomology everywhere.
5. If X is a topological space, we set

$$\mathrm{H}_n^{\mathrm{sing}}(X) := \mathrm{H}_n(\mathrm{C}_\bullet(X)) \quad \text{and} \quad \mathrm{H}^n_{\mathrm{sing}}(X, M) := \mathrm{H}^n(\mathrm{C}^\bullet(X, M)).$$

6. If X is a smooth manifold, we set $\mathrm{H}^n_{\mathrm{dR}}(X) := \mathrm{H}^n(\Omega^\bullet(X))$.
7. If A is a commutative R algebra, one can define $H^n_{\mathrm{dR}}(A/R) := H^n(\Omega^\bullet_{A/R})$ but this is only used when A is *smooth* over R.

Note that one can also define the (co-)homology of a differential object (M, d) as $\mathrm{H}(M) := \operatorname{coker}(\operatorname{im} d \rightarrowtail \ker d)$. We shall then have $\mathrm{H}^n(\underline{M}) = \mathrm{H}(M)$ for all $n \in \mathbb{Z}$ and conversely, $\mathrm{H}\left(\bigoplus_{n\in\mathbb{Z}} K^n\right) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(K^\bullet)$ when $K^\bullet$ is a complex (and we have infinite direct sums).

Exercise 2.2.35 Show that cohomology provides us with an additive functor

$$\mathrm{H}^n : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$$

which is *not* left or right exact in general.

Hint. For the counter-example, consider the short exact sequence of complexes of abelian groups

$$0 \to [\mathbb{Z} \xrightarrow{2} \mathbb{Z}] \xrightarrow{2} [\mathbb{Z} \xrightarrow{2} \mathbb{Z}] \to [\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2] \to 0.$$ ■

The next result applies for example when Q is a field of characteristic zero:

Exercise 2.2.36 Show that if Q is a divisible group, then

$$\mathrm{H}^n_{\mathrm{sing}}(X, Q) \simeq \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{H}^{\mathrm{sing}}_n(X), Q).$$

We shall need later the following elementary fact:

Lemma 2.2.37 If $M \in \mathcal{A}$ and $L^\bullet \in \mathbf{C}(\mathcal{A})$, then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, L^\bullet) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{A}}(M, L^\bullet)).$$

Proof. A quick inspection shows that

$$\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(M, L^\bullet) = \mathrm{Z}^0(\mathrm{Hom}_{\mathcal{A}}(M, L^\bullet))$$

and $f \in \mathrm{B}^0(\mathrm{Hom}(M, L^\bullet))$ if and only if the corresponding map of complexes is homotopic to zero. ■

The next formula provides a symmetric formula for cohomology (both as a kernel and as a cokernel).

Lemma 2.2.38 If $K^\bullet$ is a complex, then there exists for all $n \in \mathbb{Z}$ an exact sequence

$$0 \to \mathrm{H}^n(K^\bullet) \to \mathrm{coker}\, d_{n-1} \to \mathrm{ker} d_{n+1} \to \mathrm{H}^{n+1}(K^\bullet) \to 0.$$

Proof. We first apply snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}\ d_{n-1} & = & \mathrm{im}\ d_{n-1} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_n & \longrightarrow & K^n & \longrightarrow & \mathrm{im}\ d_n & \longrightarrow & 0 \end{array}$$

and get an exact sequence $0 \to \mathrm{H}^n(K^\bullet) \to \mathrm{coker}\, d_{n-1} \to \mathrm{im}\ d_n \to 0$. We can then glue with the exact sequence $0 \to \mathrm{im}\ d_n \to \ker d_{n+1} \to \mathrm{H}^{n+1}(K^\bullet) \to 0$. ■

We can now state the snake theorem:

Theorem 2.2.39 Any short exact sequence of complexes $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$

in $\mathcal{A}$ gives rise to a functorial long exact sequence

$$\cdots \to \mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(L^\bullet) \to \mathrm{H}^n(M^\bullet) \to \mathrm{H}^{n+1}(K^\bullet) \to \cdots$$

Proof. Apply snake lemma (version of exercise 2.1.66) to

$$\begin{array}{ccccccc} & & \mathrm{coker} d_{K,n-1} & \longrightarrow & \mathrm{coker} d_{L,n-1} & \longrightarrow & \mathrm{coker} d_{M,n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_{K,n+1} & \longrightarrow & \ker d_{L,n+1} & \longrightarrow & \ker d_{M,n+1} & & \end{array}$$

and use exercise 2.2.38. ■

Definition 2.2.40 If $\mathcal{P}$ is an additive category, then an additive functor $H : \mathbf{K}(\mathcal{P}) \to \mathcal{A}$ is said to be *cohomological* if, whenever $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ is distinguished, then the sequence $H(K^\bullet) \to H(L^\bullet) \to H(M^\bullet)$ is exact (in the middle).

Exercise 2.2.41 Show that, if $H : \mathbf{K}(\mathcal{P}) \to \mathcal{A}$ is a *cohomological* functor and $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ a distinguished triangle, then there exists a natural long exact sequence of complexes

$$\cdots \to H(K^\bullet[n]) \to H(L^\bullet[n]) \to H(M^\bullet[n]) \to H(K^\bullet[n+1]) \to \cdots .$$

Before going any further, we give the following standard example:

Proposition 2.2.42 If $\mathcal{P}$ is an additive category then, for fixed $K^\bullet$, the functor

$$\mathbf{K}(\mathcal{P}) \to \mathbf{A}\mathrm{b}, \quad L^\bullet \mapsto \mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(K^\bullet, L^\bullet)$$

is cohomological (and dual).

Proof. We have to show that if $L'^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} L''^\bullet \to L'^\bullet[1]$ is a distinguished triangle, then the sequence

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(K^\bullet, L'^\bullet) \xrightarrow{f_*} \mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(K^\bullet, L^\bullet) \xrightarrow{g_*} \mathrm{Hom}_{\mathbf{K}(\mathcal{P})}(K^\bullet, L''^\bullet)$$

is exact in the middle. Since $g \circ f$ is homotopic to zero, it is sufficient to notice that, if we are given a morphism $u : K^\bullet \to L^\bullet$ such that the composite $g \circ u : K^\bullet \to L^\bullet \to L''^\bullet$ is homotopic to zero, then there exists a morphism of triangles

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & 0 & \longrightarrow & K^\bullet[1] & = & K^\bullet[1] \\ \downarrow u & & \downarrow & & \downarrow u'[1] & & \downarrow u[1] \\ L^\bullet & \xrightarrow{g} & L''^\bullet & \longrightarrow & L'^\bullet[1] & \xrightarrow{f[1]} & L'^\bullet[1] \end{array}$$

providing u' such that $f \circ u' = u$. ■

Lemma 2.2.43 For all $n \in \mathbb{N}$, the functor H^n factors (uniquely) through $\mathbf{K}(\mathcal{A})$.

Proof. We have to show that $\mathrm{H}^n(f) = 0$ when $f : K^\bullet \to L^\bullet$ is homotopic to zero. But we can then write $f_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n$. Since $s_{n+1} \circ d_n$ is zero on $\mathrm{Z}^n(K^\bullet)$ and the image of $d_{n-1} \circ s_n$ is contained in $\mathrm{B}^n(L^\bullet)$, the image of the restriction of f_n to $\mathrm{Z}^n(K^\bullet)$ is contained in $\mathrm{B}^n(L^\bullet)$. ■

Proposition 2.2.44 For all $n \in \mathbb{Z}$, the functor $\mathrm{H}^n : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological.

Proof. Up to isomorphism, any distinguished triangle comes from a termwise split extension and we can apply snake theorem 2.2.39. ■

Corollary 2.2.45 Any distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ gives rise to a functorial long exact sequence

$$\cdots \to \mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(L^\bullet) \to \mathrm{H}^n(M^\bullet) \to \mathrm{H}^{n+1}(K^\bullet) \to \cdots . \quad ■$$

Definition 2.2.46 A *quasi-isomorphism* of complexes $f : K^\bullet \to L^\bullet$ is a morphism such that $\mathrm{H}^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. A complex $K^\bullet$ is said to be *acyclic* if $\mathrm{H}^n(K^\bullet) = 0$ for all $n \in \mathbb{Z}$.

Being quasi-isomorphic is a homotopy invariant property since $\mathrm{H}^n(f) = \mathrm{H}^n(g)$ when $f \sim g$ from lemma 2.2.43. In particular, a homotopy equivalence is automatically a quasi-isomorphism and a contractile complex is automatically acyclic. Note also that a complex $K^\bullet$ is acyclic if and only if $K^\bullet$ is quasi-isomorphic to 0 if and only if 0 is quasi-isomorphic to $K^\bullet$. Be careful however that being quasi-isomorphic is *not* a symmetric relation and therefore not an equivalence relation either. We also have:

Proposition 2.2.47 If $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ is a short exact sequence, then $K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if $M^\bullet$ is acyclic (and dual).

Proof. This is an immediate corollary of theorem 2.2.39. ■

To give an important example, when E is a set (or a family) of open subsets of a topological space X, we shall denote by

$$S_n(E) := \bigcup_{U \in E} S_n(U) \subset S_n(X)$$

the set of singular n-chains contained in some $U \in E$. We may then consider the complex $C_\bullet(E) \subset C_\bullet(X)$ with $C_n(E) := \mathbb{Z} \cdot S_n(E)$. Then, the *theorem of small simplices* is as follows:

Proposition 2.2.48 The inclusion morphism

$$C_\bullet(E) \rightarrowtail C_\bullet(X)$$

is a quasi-isomorphism when E is a covering of X.

Proof. To do. ■

Exercise 2.2.49 Show that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$ if and only if the morphism $M' \to [M \to M'']$ (resp. $[M' \to M][1] \to M''$) is a quasi-isomorphism.

Exercise 2.2.50 Assume $K^\bullet \xrightarrow{f} L^\bullet \to M^\bullet \to K^\bullet[1]$ is a distinguished triangle. Show that f is a quasi-isomorphism if and only if $M^\bullet$ is acyclic (and idem with a short exact sequence).

Exercise 2.2.51 Show that, if

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ K'^\bullet & \longrightarrow & L'^\bullet & \longrightarrow & M'^\bullet & \longrightarrow & K'^\bullet[1] \end{array}$$

is a morphism of distinguished triangles and two among u, v and w are quasi-isomorphisms, then so is the third (and idem with a short exact sequences).

Solution. In both cases, there exists a morphism of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{H}^n(K^\bullet) & \longrightarrow & \mathrm{H}^n(L^\bullet) & \longrightarrow & \mathrm{H}^n(M^\bullet) & \longrightarrow & \mathrm{H}^{n+1}(K^\bullet) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}^n(K'^\bullet) & \longrightarrow & \mathrm{H}^n(L'^\bullet) & \longrightarrow & \mathrm{H}^n(M'^\bullet) & \longrightarrow & \mathrm{H}^{n+1}(K'^\bullet) & \longrightarrow & \cdots \end{array}$$

Any vertical morphism which is not an isomorphism by hypothesis is surrounded by two isomorphisms. It then follows from the five-lemma that this is actually also an isomorphism. ■

2.2.6 Injective/projective resolutions

We still let $\mathcal{A}$ be an abelian category.

The notion of a projective/injective object was studied in section 1.5.2 but there are many other ways to see it in our context (we shall focus here on the injective case – the other one being obtained by duality):

Proposition 2.2.52 For $I \in \mathcal{A}$, the following are equivalent (and dual):

1. I is injective,
2. any monomorphism $I \rightarrowtail M$ splits,
3. $\forall M \in \mathcal{A}, \mathrm{Ext}(M, I) = 0$,
4. h_I is exact.

Proof.
- $(1) \Leftrightarrow (2)$: Follows by duality from propositions 2.1.63 and 1.5.13.
- $(2) \Leftrightarrow (3)$: Follows from lemma 2.1.48.

- (1) $\Leftrightarrow$ (4) : Since h_I is always left exact, it is exact if and only if it preserves epimorphisms (turns mono into epi) which is the very definition of an injective object. ■

Exercise 2.2.53 Prove the following :

1. if $M = M' \oplus M''$, then M is injective if and only if both M' and M'' are injective (and dual).
2. if $0 \to M' \to M \to M'' \to 0$ is an exact sequence with M' injective, then M is injective if and only if M'' is (and dual).

Solution.

1. There exists a natural isomorphism $h_M(N) = h_{M'}(N) \oplus h_{M''}(N)$ and it formally follows (use exercise 2.1.29) that h_M is exact if and only if both $h_{M'}$ and $h_{M''}$ are.
2. We have $M \simeq M' \oplus M''$. ■

Exercise 2.2.54 Assume $\mathcal{B}$ is another abelian category and $F : \mathcal{A} \to \mathcal{B}$ is adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$. Show that (and dual)

1. if F is exact then G preserves injectives,
2. if F is faithful exact and $\mathcal{B}$ has enough injectives, then $\mathcal{A}$ too has enough injectives.

Solution. We have an adjunction $\mathrm{Hom}(FM, I) \simeq \mathrm{Hom}(M, GI)$ showing that $h_{GI} = h_I \circ F$ is exact when I is injective and F is exact. It means that GI will be injective too.

If $M \in \mathcal{A}$, then there exists a monomorphism $FM \rightarrowtail I$ with I injective. Since G has an adjoint, it preserves all limits and we have a monomorphism $GFM \rightarrowtail GI$. If F is faithful, then the unit also is a monomorphism $M \rightarrowtail GFM$. By composition, we obtain a monomorphism $M \rightarrowtail GI$. To conclude, it is sufficient to recall what we just proved: GI is injective since F is exact.

By duality, if G is exact then F preserves projectives and if G is also faithful and $\mathcal{A}$ has enough projectives, then $\mathcal{B}$ too has enough projectives. ■

Proposition 2.2.55 If A is any ring, then the category of A-modules has enough projectives and injectives.

Proof. The forgetful functor $A\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is faithful (exact) with adjoint $M \mapsto A \otimes_{\mathbb{Z}} M$ and coadjoint $M \mapsto \mathrm{Hom}_{\mathbb{Z}}(A, M)$. We can apply the second assertion of exercise 2.2.54 thanks to proposition 1.5.17. ■

We denote by $\mathcal{I}$ (resp. $\mathcal{P}$) the full subcategory of injective (resp. projective) objects of $\mathcal{A}$.

Lemma 2.2.56 If $K^\bullet$ is acyclic and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$ then any morphism $f : K^\bullet \to I^\bullet$ is homotopic to zero (and dual).

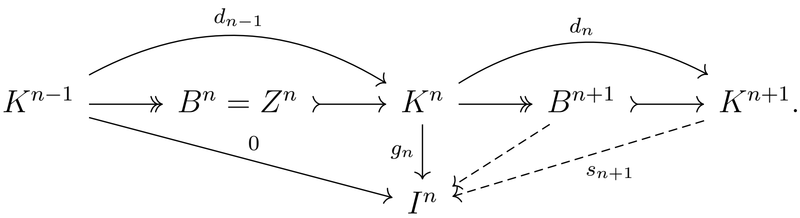
Proof. [5] Since $I^\bullet$ is bounded below, we may assume that we have built the maps s_k for $k \leq n$. In particular, we have $f_{n-1} = s_n \circ d_{n-1} + d_{n-2} \circ s_{n-1}$. Thus, if we set $g_n := f_n - d_{n-1} \circ s_n$, we have

$$\begin{aligned} g_n \circ d_{n-1} &= f_n \circ d_{n-1} - d_{n-1} \circ s_n \circ d_{n-1} \\ &= d_{n-1} \circ f_{n-1} - d_{n-1} \circ s_n \circ d_{n-1} \\ &= d_{n-1} \circ (f_{n-1} - s_n \circ d_{n-1}) \\ &= d_{n-1} \circ d_{n-2} \circ s_{n-1} \\ &= 0. \end{aligned}$$

In other words g_n is zero on $B^n(K^\bullet) = Z^n(K^\bullet)$. It means that g_n factors through $B^{n+1}(K^\bullet)$. Since I^n is injective, this last map can then be extended to $s_{n+1} : K^{n+1} \to I^n$ and we have $g_n = s_{n+1} \circ d_n$ so that

$$f_n = s_{n+1} \circ d_n + d_{n-1} \circ s_n. \qquad \blacksquare$$

The construction may be pictured as follows:



Proposition 2.2.57 If $K^\bullet \to L^\bullet$ is a quasi-isomorphism and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$, then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(L^\bullet, I^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) \quad \text{(and dual)}.$$

Proof. There exists a distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$. Applying the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(-, I^\bullet)$ provides an exact sequence

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M^\bullet[-1], I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M^\bullet, I^\bullet).$$

Exercise 2.2.50 tells us that $M^\bullet$ is acyclic and we can apply lemma 2.2.56. ■

Corollary 2.2.58 A complex $I^\bullet \in \mathbf{C}^+(\mathcal{I})$ is acyclic if and only if it is contractible (and dual).

Proof. Apply directly lemma 2.2.56 to the identity. ■

Proposition 2.2.59 A morphism $u : I^\bullet \to J^\bullet$ in $\mathbf{C}^+(\mathcal{I})$ is a quasi-isomorphism if and only if it is a homotopy equivalence (and dual).

Proof. Only the direct implication in the first assertion really requires an argument. But we may then consider the cone of u in $K^+(\mathcal{I})$. Thanks to exercise 2.2.50 and corollary 2.2.32, our assertion follows from the previous corollary. ■

[5] At the time of this writing, the proof in [Sta19, Tag 013R] is not correct as written.

Corollary 2.2.60 If $u : I^\bullet \to J^\bullet$ in $\mathbf{C}^+(\mathcal{I})$ is a quasi-isomorphism and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then $F(u) : F(I^\bullet) \to F(J^\bullet)$ is a quasi-isomorphism (and dual). ■

Exercise 2.2.61 Show that if E is an open covering of a topological space X, M is an abelian group, then there exists a complex $C^\bullet(E, M)$ with $C^n(E, M) := \mathcal{F}(S_n(E), M)$ and a quasi-isomorphism

$$C^\bullet(X, M) \twoheadrightarrow C^\bullet(E, M).$$

Solution. Since both complexes $C_\bullet(X)$ and $C_\bullet(E)$ have projective (actually free) terms and are quasi-isomorphic thanks to the *theorem of small simplices*, there exists a quasi-isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(C_\bullet(X), M) \to \mathrm{Hom}_{\mathbf{Ab}}(C_\bullet(E), M).$$

This is exactly what we want. ■

Definition 2.2.62 If $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism, then $L^\bullet$ is called a *right resolution*[a] of $K^\bullet$. It is called an *injective resolution* if each $L^\bullet \in \mathbf{C}(\mathcal{I})$.

[a] Or *replacement*.

A *left* (resp. a *projective*) *resolution* is a right (resp. an injective) resolution in $\mathcal{A}^{\mathrm{op}}$.

Examples
1. A sequence $0 \to M \to K^0 \to K^1 \to \cdots$ is exact if and only if $K^\bullet$ is a right resolution of M with $K^n = 0$ for $n < 0$.
2. Any abelian group M has a free (projective) resolution of length 2: there exists a short exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with L_0 and L_1 free.
3. $[\mathbb{Z} \overset{n}{\to} \mathbb{Z}][1]$ is a projective (right) resolution of $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ (or $\mathbb{R} \to \mathbb{S}, t \mapsto e^{2i\pi t}$ if you prefer) is a (left) injective resolution of $\mathbb{Z}$.

Exercise 2.2.63 Show that if X is a contractible topological space (resp. and M is an abelian group), then $C_\bullet(X)$ (resp. $C^\bullet(X, M)$) is a left (resp. right) resolution of $\mathbb{Z}$ (resp. M).

Hint. Use exercise 2.2.7. ■

Exercise 2.2.64 Show that, if $\mathbb{R}^n$ is seen as a smooth manifold, then $\Omega^\bullet(\mathbb{R}^n)$ is a right resolution of $\mathbb{R}$.

Solution. We already know that $\mathrm{d}f = 0$ implies f constant. Now, we assume that

$$\omega := \sum_{1 \le i_1 < \ldots i_p \le k} f_{i_1,\ldots,i_p} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p} \in \Omega^p(\mathbb{R}^n)$$

is a cycle and we show by induction on k that it is a boundary. For $i_p = k$, we write

$$F_{i_1,\ldots,i_{p-1}}(x_1, \ldots, x_n) := \int_0^{x_p} f_{i_1,\ldots,i_p}(x_0, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)\mathrm{d}t,$$

we define

$$\eta := \sum_{1 \leq i_1 < \ldots i_p = k} F_{i_1,\ldots,i_{p-1}} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_{p-1}} \in \Omega^{k-1}(\mathbb{R}^n).$$

and we apply our induction hypothesis to $\omega - \mathrm{d}\eta$. Note that $\frac{\partial}{\partial_i} f_{i_1,\ldots,i_p}(x_1,\ldots,x_n) = 0$ for $i \neq i_1,\ldots,i_p$ so that $f_{i_1,\ldots,i_p}$ only depends on x_i for $i \leq k$ (and the same therefore holds for ω and η). ■

Proposition 2.2.65 If $K^\bullet \to I^\bullet$ and $L^\bullet \to J^\bullet$ are two injective resolutions in $\mathbf{C}^+(\mathcal{A})$, then any morphism $f : K^\bullet \to L^\bullet$ extends to $I^\bullet \to J^\bullet$ up to homotopy (and dual).

Proof. We may first extend f to $K^\bullet \to L^\bullet \to J^\bullet$ and then apply proposition 2.2.57:

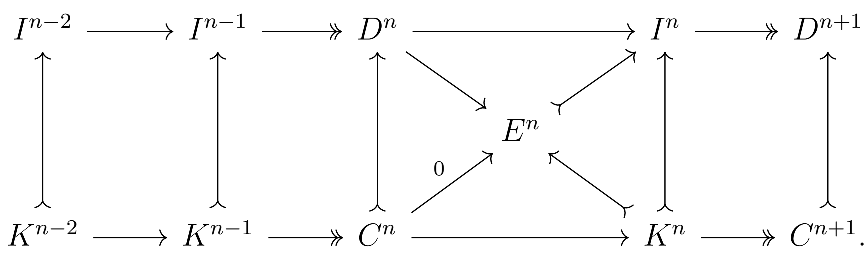
$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(I^\bullet, J^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, J^\bullet).$$ ■

Proposition 2.2.66 If $\mathcal{A}$ has enough injectives and $K^\bullet \in \mathbf{C}^+(\mathcal{A})$, then there exists an injective resolution $K^\bullet \rightarrowtail I^\bullet$ which is a monomorphism (and dual).

Proof. After a shift, we may assume that $K^n = 0$ for $n < 0$ and, by induction, that there exists a monomorphism of complexes with right exact lines

$$\begin{array}{ccccccccccccc} I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots & \longrightarrow & I^{n-2} & \longrightarrow & I^{n-1} & \longrightarrow & D^n & \longrightarrow & 0 \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\ K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots & \longrightarrow & K^{n-2} & \longrightarrow & K^{n-1} & \longrightarrow & C^n & \longrightarrow & 0 \end{array}$$

such that $\mathrm{H}^m(K^\bullet) \simeq \mathrm{H}^m(I^\bullet)$ for $m < n-1$ (the monomorphism on the right is part of the induction hypothesis). Then, there exists an injection $E^n := K^n \times_{C^n} D^n \rightarrowtail I^n$ into an injective. We obtain monomorphisms $K^n \rightarrowtail I^n$ and $C^{n+1} \rightarrowtail D^{n+1}$ on the cokernels and a morphism $I^{n-1} \to I^n$ such that $\mathrm{H}^{n-1}(K^\bullet) \simeq \mathrm{H}^{n-1}(I^\bullet)$. This may be shown by playing around with the following diagram:



Details are left to the reader. ■

2.3 Deriving

2.3.1 Derived category

Let $\mathcal{A}$ be an abelian category.

Recall that we introduced in section 1.5.3 the notion of a localization (as well as calculus of fractions).

Proposition 2.3.1 The category $\mathbf{K}(\mathcal{A})$ admits both left and right calculus of fractions with respect to quasi-isomorphisms.

Proof. An identity is a quasi-isomorphism and quasi-isomorphisms are clearly stable under composition. Assume now that we are given a morphism $K^\bullet \to L^\bullet$ and a quasi-isomorphism $L'^\bullet \to L^\bullet$. Then, there exists a morphism of distinguished triangles (build $N^\bullet$ first, then $K'^\bullet$)

$$\begin{array}{ccccccc}
K'^\bullet & \dashrightarrow & K^\bullet & \longrightarrow & N^\bullet & \dashrightarrow & K'^\bullet[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
L'^\bullet & \longrightarrow & L^\bullet & \longrightarrow & N^\bullet & \longrightarrow & L'^\bullet[1].
\end{array}$$

Since $L'^\bullet \to L^\bullet$ is a quasi-isomorphism, $N^\bullet$ is acyclic and therefore $K'^\bullet \to K^\bullet$ is also a quasi-isomorphism. The last condition is proved in the same way with a diagram

$$\begin{array}{ccccccc}
K'^\bullet & & & & & & \\
\downarrow & & & & & & \\
K^\bullet & = & K^\bullet & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow u & & \downarrow & & \downarrow \\
N^\bullet & \dashrightarrow & L^\bullet & \longrightarrow & L'^\bullet & \dashrightarrow & N^\bullet[1] \\
\downarrow & & & & & & \\
K'^\bullet[1]. & & & & & &
\end{array}$$

The left property is obtained by duality. ∎

Definition 2.3.2 The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is the localization of $\mathbf{K}(\mathcal{A})$ at quasi-isomorphisms.

Thus, $\mathbf{D}(\mathcal{A})$ has the same objects as $\mathbf{K}(\mathcal{A})$ (or equivalently $\mathbf{C}(\mathcal{A})$) and a morphism $K^\bullet$ to $L^\bullet$ is (up to equivalence) a diagram

$$\begin{array}{ccc}
K'^\bullet & & \\
\downarrow & \searrow & \\
K^\bullet & & L^\bullet
\end{array}$$

where the vertical map is a quasi-isomorphism. A morphism of complexes $K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if it becomes an isomorphism in $\mathbf{D}(\mathcal{A})$.

Exercise 2.3.3 Show that $\mathbf{D}(\mathcal{A})$ is also the localization of $\mathbf{C}(\mathcal{A})$ at quasi-isomorphisms (even it this last category does not admits right or left calculus of fractions).

One can also define $\mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A})$ along the same lines. Moreover, there exists an embedding $\mathbf{D}^+(\mathcal{A}) \rightarrowtail \mathbf{D}(\mathcal{A})$ whose essential image is made of complexes such that $\mathrm{H}^n(K^\bullet) = 0$ for $n << 0$ (and analogous statements with $-$ and b).

One defines a *triangle*, a *morphism of triangles* or a *distinguished triangle* in $\mathbf{D}(\mathcal{A})$ exactly as in definition 2.2.23. In particular, a triangle in $\mathbf{D}(\mathcal{A})$ is distinguished if and only if it is isomorphic in $\mathbf{D}(\mathcal{A})$ to a distinguished triangle of $\mathbf{K}(\mathcal{A})$.

Proposition 2.3.4 If $0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$ is an exact sequence of complexes, then there exists a morphism $M^\bullet \to K^\bullet[1]$ in $\mathbf{D}(\mathcal{A})$ making $K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to K^\bullet[1]$ distinguished.

Proof. The maps

$$u_n := \begin{bmatrix} 0 & g_n \end{bmatrix} : \mathrm{M}(f)^n = K^{n+1} \oplus L^n \to M^n$$

define a morphism of complexes since

$$u_{n+1} \circ d_n = \begin{bmatrix} 0 & g_{n+1} \end{bmatrix} \begin{bmatrix} -d_{n+1} & 0 \\ f_{n+1} & d_n \end{bmatrix} = \begin{bmatrix} g_{n+1} \circ f_{n+1} & g_{n+1} \circ d_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & d_n \circ g_n \end{bmatrix} = d_n \circ \begin{bmatrix} 0 & g_n \end{bmatrix} = d_n \circ u_n.$$

It is then sufficient to show that $u : \mathrm{M}(f)^\bullet \to M^\bullet$ is a quasi-isomorphism. There exists short exact sequences

$$0 \longrightarrow K^{n+1} \oplus K^n \xrightarrow{v_n} K^{n+1} \oplus L^n \xrightarrow{u_n} M^n \longrightarrow 0$$

with

$$v_n := \begin{bmatrix} 1 & 0 \\ 0 & f_n \end{bmatrix} : \mathrm{M}(\mathrm{Id}_{K^\bullet})^n \to M^\bullet(f)^n$$

providing an exact sequence

$$0 \to \mathrm{M}(\mathrm{Id}_{K^\bullet}) \xrightarrow{v} M^\bullet(f) \xrightarrow{u} M^\bullet \to 0$$

of complexes. But $\mathrm{M}(\mathrm{Id}_{K^\bullet})$ is homotopically equivalent to 0 (see exercise 2.2.26) and in particular acyclic. It then follows from corollary 2.2.47 that u is a quasi-isomorphism. ■

Proposition 2.3.5 If $K^\bullet \in \mathbf{C}(\mathcal{A})$ and $I^\bullet \in \mathbf{C}^+(\mathcal{I})$, then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(K^\bullet, I^\bullet) \quad \text{(and dual).}$$

Proof. Formal consequence of proposition 2.2.57. ■

Theorem 2.3.6 If $\mathcal{A}$ has enough injectives, then there exists an equivalence of categories

$$\mathbf{K}^+(\mathcal{I}) \simeq \mathbf{D}^+(\mathcal{A}) \quad \text{(and dual)}.$$

Proof. Full faithfulness follows from proposition 2.3.5 and essential surjectivity from proposition 2.2.66. ■

Definition 2.3.7 The *n-th extension group* of $L^\bullet$ by $K^\bullet$ is

$$\mathrm{Ext}^n(K^\bullet, L^\bullet) := \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(K^\bullet, L^\bullet[n]).$$

Proposition 2.3.8 If $\mathcal{A}$ has enough injectives, $K^\bullet \in \mathbf{C}(\mathcal{A})$ and

$$0 \to L'^\bullet \to L^\bullet \to L''^\bullet \to 0$$

is a short exact sequence in $\mathbf{C}^+(\mathcal{A})$, then there exists a long exact sequence

$$\cdots \to \mathrm{Ext}^n(K^\bullet, L'^\bullet) \to \mathrm{Ext}^n(K^\bullet, L^\bullet) \to \mathrm{Ext}^n(K^\bullet, L''^\bullet) \to \mathrm{Ext}^{n+1}(K^\bullet, L^\bullet) \to \cdots$$

(and dual).

Proof. It follows from proposition 2.3.5 that if $L^\bullet \to I^\bullet$ is an injective resolution, then

$$\mathrm{Ext}^n(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K^\bullet, I^\bullet[n]).$$

We can then apply exercise 2.2.41 according to proposition 2.2.42. ■

We shall mostly consider the case when $M, N \in \mathcal{A}$ are seen as complexes concentrated in degree 0.

Exercise 2.3.9 Show that $\mathrm{Ext}^n(M,N) = 0$ for $n < 0$ and $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$ if $M, N \in \mathcal{A}$.

Proposition 2.3.10 If I is injective, then $\mathrm{Ext}^n(M, I) = 0$ for all $M \in \mathcal{A}$ and $n \neq 0$ (and dual).

Proof. If I is injective, then for $n \neq 0$, we have thanks to exercise 2.2.37,

$$\mathrm{Ext}^n(M,I) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, I[n]) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, I[n]) = \mathrm{H}^0(\mathrm{Hom}(M, I[n])) = 0. ■$$

Exercise 2.3.11 Show that $\mathrm{Ext}^1(M,N) \simeq \mathrm{Ext}(M,N)$ when $\mathcal{A}$ has enough injectives or projectives (although this condition is not necessary).

Solution. By duality, we may assume that there are enough injectives. Then, there exists a short exact sequence

$$0 \to N \to I \xrightarrow{\pi} C \to 0$$

with I injective. We know that $\mathrm{Ext}^1(M,I) = \mathrm{Ext}(M,I) = 0$. There exists therefore two right exact sequences

$$\mathrm{Hom}(M,I) \overset{\pi*}{\to} \mathrm{Hom}(M,C) \to \mathrm{Ext}(M,N) \to 0$$

and

$$\mathrm{Hom}(M,I) \overset{\pi*}{\to} \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,N) \to 0$$

implying the expected identification.

There also exists an unconditional proof by showing directly that the map

$$\mathrm{Ext}(M,N) \to \mathrm{Ext}^1(M,N), \quad E \mapsto \left(\begin{array}{ccc} [N \to E][1] & & \\ {\scriptstyle q-is}\downarrow & \searrow & \\ M & & N[1] \end{array} \right)$$

is an isomorphism of groups. Details are left to the reader but surjectivity is proved as follows: a quasi-isomorphism $K^\bullet \to M$ provides an extension $0 \to B^0 \to Z^0 \to M \to 0$, a morphism $K^\bullet \to N[1]$ provides a morphism $\phi : B^0 \to N$ and we can then set $E = \varphi_* Z^0$.

It is also possible to mix both approaches by sending E to the image of Id_M under the connecting map $\mathrm{Hom}(M,M) \to \mathrm{Ext}^1(M,N)$ coming from E. ■

Exercise 2.3.12 Assume $\mathcal{A}$ has enough injectives. Show that $I \in \mathcal{A}$ is injective if and only if $\mathrm{Ext}^n(M,I) = 0$ for all $M \in \mathcal{A}$ and $n \neq 0$ (and dual).

Exercise 2.3.13 Show that if M, N are two abelian groups, then $\mathrm{Ext}^n(M,N) = 0$ for $n \neq 0, 1$.

Exercise 2.3.14 Show that if M is an abelian group, then

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, M) \simeq M/nM.$$

Solution. We can apply the dual to proposition 2.3.8 to the the short exact sequence $0 \to \mathbb{Z} \overset{n}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ and contemplate the following commutative diagram with exact rows (use the five lemma for example)

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathbb{Z},M) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z},M) & \longrightarrow & \mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},M) & \longrightarrow & \mathrm{Ext}(\mathbb{Z},M) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow & & \| \\ M & \xrightarrow{n} & M & \longrightarrow & M/nM & \longrightarrow & 0. \end{array}$$

■

Exercise 2.3.15 Show that, in the category of abelian groups,

$$\mathrm{Ext}^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0, 1 \\ 0 \text{ otherwise} \end{cases}$$

with $d = m \wedge n$ (for $m, n > 0$).

Solution. Follows from exercise 2.3.14 and the (long) exact sequence

$$0 \to \mathbb{Z}/d\mathbb{Z} \stackrel{m'}{\to} \mathbb{Z}/m\mathbb{Z} \stackrel{n}{\to} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

where $m = dm'$. ■

2.3.2 Derived functor

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between two abelian categories with $\mathcal{A}$ having enough injectives (and dual). From now on, most of the time, we will simply write FM instead of $F(M)$ in order to lighten the notations.

Definition 2.3.16 The *right derived functor*[a] of F is the unique (up to isomorphism) functor $\mathrm{R}F$ making commutative the following diagram:

$$\begin{array}{ccc} \mathbf{K}^+(\mathcal{A}) & \xrightarrow{F} & \mathbf{K}^+(\mathcal{B}) \\ \uparrow & & \downarrow \\ \mathbf{K}^+(\mathcal{I}) & & \\ \downarrow \simeq & & \\ \mathbf{D}^+(\mathcal{A}) & \xrightarrow{\mathrm{R}F} & \mathbf{D}^+(\mathcal{B}). \end{array}$$

[a]This definition is usually only applied to left exact functors.

In practice, if $K^\bullet \to I^\bullet$ is an injective resolution, then $\mathrm{R}FK^\bullet \simeq F^\bullet I^\bullet$ and this can be made functorial. We will then set

$$\forall n \in \mathbb{N}, \quad \mathrm{R}^n FK^\bullet := \mathrm{H}^n(\mathrm{R}FK^\bullet).$$

As a particular case, if $M \in \mathcal{A}$, we may consider $\mathrm{R}FM \in \mathbf{D}^+(\mathcal{B})$ and for all $n \in \mathbb{Z}$, $\mathrm{R}^n FM \in \mathcal{B}$ (with $\mathrm{R}^n FM = 0$ for $n < 0$). One defines dually $\mathrm{L}F := \mathrm{R}F^{\mathrm{op}}$ and $\mathrm{L}_n FK_\bullet := \mathrm{H}_n(\mathrm{L}FK_\bullet)$ when $\mathcal{A}$ has enough projectives. We shall apply right (resp. left) derivation to left (resp. right) exact functors F because, according to exercise 2.3.20 below, we will then have $\mathrm{R}^0 F = F$ (resp. $\mathrm{L}_0 F = F$) on objects.

Examples

1. For fixed $M \in \mathcal{A}$, we can consider the left exact functor

$$h^M : \mathcal{A} \to \mathbf{A}\mathrm{b}, \quad N \to \mathrm{Hom}(M, N).$$

   We shall show in proposition 2.3.21 below that $\mathrm{R}^n h^M(N) = \mathrm{Ext}^n(M, N)$ for all $n \in \mathbb{N}$.

2. In the case $\mathcal{A}$ has enough projectives, we can fix $N \in \mathcal{A}$ and consider the left exact (contravariant) functor

$$h_N : \mathcal{A}^{\mathrm{op}} \to \mathbf{A}\mathrm{b}, \quad M \to \mathrm{Hom}(M, N).$$

   Then, by duality, $\mathrm{R}^n h_N(M) = \mathrm{Ext}^n(M, N)$ again and there is therefore no ambiguity in writing $\mathrm{RHom}(M, N)$.

3. If A is a ring, then for fixed $M \in \mathbf{Mod}-A$, we can consider the right exact functor
$$A\text{-}\mathbf{Mod} \to \mathbf{Ab}, \quad N \to M \otimes_A N$$
and its left derived functor
$$\mathbf{D}^-(A\text{-}\mathbf{Mod}) \to \mathbf{D}^-(\mathbf{Ab}), \quad N \to M \otimes_A^{\mathrm{L}} N.$$
One then sets $\mathrm{Tor}_n^A(M,N) := \mathrm{H}_n(M \otimes_A^{\mathrm{L}} N)$. In the case $A = \mathbb{Z}$, we remove A from the notations.
4. (see exercise 2.1.41) When G is a monoid and k a commutative ring, we may consider the left exact functor of G-invariants
$$\mathbf{Rep}_k(G) \to k\text{-}\mathbf{Mod}, \quad M \mapsto M^G := \{s \in M, \forall g \in G, gs = s\}.$$
One then denote by $\mathrm{H}^n(G,M)$ its cohomology modules (group cohomology).
5. (see exercise 2.1.42) We may consider the (left exact) functor that associates to a module with derivation M on the ring $\mathcal{O} := \mathcal{O}(U)$ of holomorphic functions on an open subset U of $\mathbb{C}$ its solution set
$$\mathrm{Sol}(M) := \{s \in M, \partial_M(s) = 0\}$$
and its derived functor RSol.

Lemma 2.3.17 If
$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$$
is a distinguished triangle in $\mathbf{D}^+(\mathcal{A})$, then
$$\mathrm{R}FK^\bullet \to \mathrm{R}FL^\bullet \to \mathrm{R}FM^\bullet \to \mathrm{R}FK^\bullet[1]$$
also is distinguished (and dual).

Proof. Apply (the bounded below version of) proposition 2.2.24. ■

Theorem 2.3.18 Any short exact sequence
$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$
in $\mathbf{C}^+(\mathcal{A})$ gives rise to a long exact sequence
$$\cdots \to \mathrm{R}^n FK^\bullet \to \mathrm{R}^n FL^\bullet \to \mathrm{R}^n FM^\bullet \to \mathrm{R}^{n+1} FK^\bullet \to \cdots$$
in $\mathcal{B}$ (and dual).

Proof. We know from propositon 2.3.4 that there exists a distinguished triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ in $\mathbf{D}(\mathcal{A})$. We apply lemma 2.3.17 and corollary 2.2.45. ■

Exercise 2.3.19 Show that

$$\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/d\mathbb{Z} \text{ if } k = 0, 1 \\ 0 \text{ otherwise} \end{cases}$$

with $d = m \wedge n$ (for $m, n \geq 1$).

Solution. Recall first that $\mathrm{Tor}_k(\mathbb{Z}, M) = 0$ for $k \neq 0$ and $\mathrm{Tor}_0(\mathbb{Z}, M) = \mathbb{Z} \otimes M \simeq M$. Then the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

provides an exact sequence

$$0 \to \mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \to \mathrm{Tor}_0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to 0$$

and $\mathrm{Tor}_k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = 0$ for $k \neq 0, 1$. It is therefore sufficient to recall the exact sequence

$$0 \to \mathbb{Z}/d\mathbb{Z} \xrightarrow{n'} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \to 0$$

where $n = dn'$. Alternatively, one can compute directly the homology of the complex

$$\mathbb{Z}/n\mathbb{Z} \otimes^L \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \otimes [\mathbb{Z} \xrightarrow{m} \mathbb{Z}][1] \simeq [\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}][1]. \blacksquare$$

The next result shows that right derived functors are only interesting when F is left exact (and dual):

Proposition 2.3.20 The functor F is left exact (resp. exact) if and only if $\forall M \in \mathcal{A}, \mathrm{R}^0 FM \simeq FM$ (resp. $\mathrm{R}FM \simeq \mathrm{F}M$) (and dual).

Solution. Let $0 \to M \to I^0 \to I^1 \to \cdots$ be an injective resolution. Assume first that F is left exact. Then, since $M \simeq \mathrm{H}^0(I^\bullet) \simeq \ker(I^0 \to I^1)$, we will have $FM \simeq \ker(FI^0 \to FI^1) \simeq \mathrm{H}^0(FI^\bullet) = \mathrm{R}^0 FM$. If F is exact, then $FM \simeq FI^\bullet \simeq \mathrm{R}FM$. The converse follows from theorem 2.3.18. ■

Proposition 2.3.21 If $M, N \in \mathcal{A}$, then (when $\mathcal{A}$ has enough projectives for the second term to exist)

$$\mathrm{R}^n h^M(N) = \mathrm{R}^n h_N(M) = \mathrm{Ext}^n(M, N).$$

Proof. We choose an injective resolution $N \to I^\bullet \in \mathbf{C}^+(\mathcal{I})$ and then use proposition 2.3.5 and exercise 2.2.37:

$$\begin{aligned} \mathrm{R}^n h^M(N) &= \mathrm{H}^n(\mathrm{R}h^M(N)) = \mathrm{H}^n(h^M(I)) = \mathrm{H}^0(h^M(I^\bullet))[n]) = \mathrm{H}^0(h^M(I^\bullet[n])) \\ &= \mathrm{H}^0(\mathrm{Hom}_{\mathcal{A}}(M, I^\bullet[n]) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, I^\bullet[n]) \\ &= \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, I^\bullet[n]) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, N[n]) \\ &= \mathrm{Ext}^n(M, N). \end{aligned}$$

■

Exercise 2.3.22 Show that, if G is a group and $k = \mathbb{Z}$, then $\mathrm{H}^1(G, M) \simeq \mathrm{Z}(G, M)/\mathrm{B}(G, M)$ where

$$\mathrm{Z}(G, M) := \{f : G \to M / \forall g, h \in G, f(gh) = f(g) + gf(h)\}$$

is the set of *crossed homomorphisms* and

$$\mathrm{B}(G, M) := \{f : G \to M, g \mapsto gm - m : m \in M\}$$

is the set of *principal homomorphisms.*

Hint. We have $M^G \simeq \mathrm{Hom}_{\mathbb{Z}\cdot G}(\mathbb{Z}, M)$ and therefore, $\mathrm{H}^n(G, M) \simeq \mathrm{Ext}^n_{\mathbb{Z}\cdot G}(\mathbb{Z}, M)$. We consider now the simplicial set $G_\bullet$ from exercise 1.3.14 with $G_n := G^{n+1}$, apply the free functor to get a simplicial abelian group $\mathbb{Z} \cdot G_\bullet$ which by Dold-Kan construction may be seen as a complex of abelian groups. One can check that formulas $g(g_0, \ldots, g_n) := (gg_0, \ldots, gg_n)$ turn $\mathbb{Z} \cdot G^{n+1}$ is a free $\mathbb{Z} \cdot G$ module and that the differentials are $\mathbb{Z} \cdot G$-linear. Actually, this provides a free resolution $\mathbb{Z} \cdot G_\bullet \to \mathbb{Z}$. It follows that $\mathrm{H}^n(G, M)$ is the cohomology of the complex $\mathrm{Hom}_{\mathbb{Z}\cdot G}(\mathbb{Z} \cdot G_\bullet, M)$. Now, we have $\mathrm{Hom}_{\mathbb{Z}\cdot G}(\mathbb{Z} \cdot G^{n+1}, M) \subset \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \cdot G^{n+1}, M) \simeq \mathcal{F}(G^{n+1}, M)$ and it is a matter of writing the conditions down explicitly. ■

Exercise 2.3.23 Assume G is a group and $k = \mathbb{Z}$. Compute $\mathrm{H}^1(G, M)$ when

1. the action of G on M is trivial,
2. $M = \mathbb{Z}$ with the non-trivial action of $\mu_2 := \{1, -1\}$.

Solution. If the action of G on M is trivial, then

$$\mathrm{Z}(G, M) = \{f : G \to M / \forall g, h \in G, f(gh) = f(g) + f(h)\} = \mathrm{Hom}_{\mathrm{Grp}}(G, M)$$

$$\text{and} \quad \mathrm{B}(G, M) := \{f : G \to M, g \mapsto 0\} = 0$$

so that

$$\mathrm{H}^1(G, M) \simeq Z(G, M)/B(G, M) = \mathrm{Hom}_{\mathbf{Grp}}(G, M).$$

If μ_2 acts non-trivially on $\mathbb{Z}$, then a crossed morphism is a map $f : \mu_2 \to \mathbb{Z}$ satisfying

$$f(1) = f(1) + f(1) \qquad f(-1) = f(1) + f(-1),$$
$$f(-1) = f(-1) - f(1), \qquad f(1) = f(-1) - f(-1).$$

This blows down to $f(1) = 0$ and there exists therefore an isomorphism

$$\mathrm{Z}(\mu_2, \mathbb{Z}) \simeq \mathbb{Z}, \quad f \mapsto f(-1).$$

A crossed homomorphism is principal if and only if there exists $n \in \mathbb{Z}$ such that $f(-1) = -2n$ so that

$$\mathrm{B}(\mu_2, \mathbb{Z}) \simeq 2\mathbb{Z}, \quad f \mapsto f(-1)$$

and finally $\mathrm{H}^1(\mu_2, \mathbb{Z}) \simeq \mathrm{Z}(\mu_2, \mathbb{Z})/\mathrm{B}(\mu_2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. ■

Exercise 2.3.24 Show that

$$\mathrm{R}^n\mathrm{Sol}(M) \simeq \begin{cases} \ker \partial_M \text{ if } n = 0 \\ \operatorname{coker} \partial_M \text{ if } n = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Solution. Recall our category is isomorphic to the category of D-modules and one easiliy sees that $\mathrm{Sol}(M) \simeq \mathrm{Hom}_D(\mathcal{O}, M)$. There exists an exact sequence $0 \to D \xrightarrow{\partial} D \to \mathcal{O} \to 0$ providing a projective resolution $[D \xrightarrow{\partial} D] \to \mathcal{O}$. It follows that

$$\begin{aligned} \mathrm{RSol}(M) &\simeq \mathrm{Hom}_D([D \xrightarrow{\partial} D], M) \\ &= [\mathrm{Hom}_D(D, M) \xrightarrow{\partial^*} \mathrm{Hom}_D(D, M)] \\ &= [M \xrightarrow{\partial_M} M]. \end{aligned}$$

■

Exercise 2.3.25 Compute $\mathrm{R}^n\mathrm{Sol}(M)$ when $M = \mathcal{O}$ with $\partial(f) = f'$
1. if U is simply connected,
2. if U is an annulus.

Definition 2.3.26 An object $M \in \mathcal{A}$ is said to be (right) *F-acyclic* if $FM = \mathrm{R}FM$ (i.e. $FM = \mathrm{R}^0FM$ and $\mathrm{R}^nFM = 0$ for $n \neq 0$).

Examples
1. Any object is acyclic if F is exact.
2. An abelian group is acyclic for h_M if and only if it is free.
3. An abelian group is acyclic for $\otimes$ if and only if it is torsion-ree.
4. An A-module is said to be *flat* if it is $\otimes$-acyclic.

Proposition 2.3.27 — Leray acyclicity. If $K^\bullet \in \mathbf{C}^+(\mathcal{A})$ and each K^n is F-acyclic, then $\mathrm{R}FK^\bullet = FK^\bullet$.

Proof. We may assume that $K^n = 0$ for $n < 0$.

Assume first that $K^\bullet$ is bounded and denote by $K'^\bullet := K^1 \to K^2 \to \cdots$. Then, there exists a termwise split exact sequence $0 \to K'^\bullet \to K^\bullet \to K^0 \to 0$. It follows from exercise 2.2.28 and proposition 2.2.24 that there exists a morphism of distinguished triangles

$$\begin{array}{ccccccc} FK'^\bullet & \longrightarrow & FK^\bullet & \longrightarrow & FK^0 & \longrightarrow & FK'^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow{\simeq} & & \downarrow \\ \mathrm{R}FK'^\bullet & \longrightarrow & \mathrm{R}FK^\bullet & \longrightarrow & \mathrm{R}FK^0 & \longrightarrow & \mathrm{R}FK'^\bullet[1] \end{array}$$

in $\mathbf{D}^+(\mathcal{A})$. By induction, the second vertical arrow will also be an isomorphism.

In général, if we set $K'^\bullet = K^{n+2} \to K^{n+3} \to \cdots$ and $K''^\bullet = K^0 \to \cdots \to K^{n+1}$ (bounded), we get a termwise split exact sequence $0 \to K'^\bullet \to K^\bullet \to K''^\bullet \to 0$. It follows that there exists a morphism of exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^nFK^\bullet & \xrightarrow{\simeq} & \mathrm{H}^nFK''^\bullet & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\simeq} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{R}^nFK^\bullet & \xrightarrow{\simeq} & \mathrm{R}^nFK''^\bullet & \longrightarrow & 0 & \longrightarrow & \cdots . \end{array}$$

■

As a consequence, if $M \to K^\bullet$ is a right resolution with each K^n is F-acyclic, then $\mathrm{R}FM = FK^\bullet$ and therefore

$$\forall n \in \mathbb{N}, \quad \mathrm{R}^n FM = \mathrm{H}^n(FK^\bullet).$$

Exercise 2.3.28 Show that an object is injective if and only if it is F-acyclic for all functors F if and only if this is the case when $F = \mathrm{Hom}(M, -)$ for all $M \in \mathcal{A}$ (and dual).

Corollary 2.3.29 Assume $\mathcal{B}$ also has enough injectives, $\mathcal{C}$ is another abelian category and $G : \mathcal{B} \to \mathcal{C}$ an additive functor. Assume that FI is G-acyclic whenever I is injective. Then $\mathrm{R}(G \circ F) = \mathrm{R}G \circ \mathrm{R}F$.

Proof. Follows from Leray acyclicity. ∎

Exercise 2.3.30 Show that the condition is automatic when F has an exact adjoint.

Exercise 2.3.31 Show that if F is left exact, then $\mathrm{R}F$ is the (left) Kan extension of the composite map $\mathbf{K}^+(\mathcal{A}) \xrightarrow{F} \mathbf{K}^+(\mathcal{B}) \to \mathbf{D}^+(\mathcal{B})$ along $\mathbf{K}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{A})$ (and dual).

2.3.3 Spectral sequence

We let $\mathcal{A}$ be an abelian category.

It will be convenient here to assume that countable direct sums exist in $\mathcal{A}$ and use the subobject and quotient terminology. A *subobject* (resp. *quotient*) N of M is a isomorphism class of monomorphisms $N \rightarrowtail M$ (resp. epimorphisms $M \twoheadrightarrow N$). Kernel and cokernel induce a bijections between subobject of quotients. We shall write $N \subset M$ when N is a subobject of M and denote by M/N the corresponding quotient. Limits and colimits induce operations on subobjects called intersection and union.

Definition 2.3.32 A *(decreasing) filtration* on an object $M \in \mathcal{A}$ is a diagram on $(\mathbb{Z}, \geq)$ of subobjects of M:

$$M \supset \cdots \supset \mathrm{F}^n M \supset \mathrm{F}^{n+1} M \supset \cdots \supset 0.$$

We shall always assume that the filtration is *exhaustive*: $\bigcup_{n\in\mathbb{Z}} \mathrm{F}^n M = M$ and *separated*: $\bigcap_{n\in\mathbb{Z}} \mathrm{F}^n M = 0$. It is said to be *finite* if $\mathrm{F}^n M = M$ for $n << 0$ and $\mathrm{F}^n M = 0$ for $n >> 0$

We shall denote this category by $\mathbf{F}(\mathcal{A})$ and write for each $n \in \mathbb{Z}$, $\mathrm{Gr}^n M = \mathrm{F}^n M/\mathrm{F}^{n+1} M$.

Exercise 2.3.33 Show that $\mathbf{F}(\mathbf{C}(\mathcal{A})) \simeq \mathbf{C}(\mathbf{F}(\mathcal{A}))$.

Definition 2.3.34 A spectral sequence

$$E_{r_0}^{p,q} \Rightarrow H^{p+q}$$

is

1. a family of complexes with $p, q, r \in \mathbb{Z}$ and $r \geq r_0$

$$E_r := \cdots \longrightarrow E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q-r+1} \longrightarrow \cdots$$

such that

$$E_{r+1}^{p,q} = H^{p,q}(E_r) := \ker(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1}),$$

2. a family of filtered[a] objects H^n for $n \in \mathbb{Z}$ such that

$$\forall p, q \in \mathbb{N}, \quad \mathrm{Gr}^p H^{p+q} = E_r^{p,q} \quad \text{for } r >> 0.$$

It is called a *first quadrant* spectral sequence if $E_r^{p,q} = 0$ unless $p, q \in \mathbb{N}$.

[a]Recall that all filtrations are assumed to be separated and exhausted.

Exercise 2.3.35 Represent E_0, E_1 and E_2 with p, q as coordinates.

Solution. We have

$$\begin{array}{c} E_0^{p,q+1} \\ {\scriptstyle d}\uparrow \\ E_0^{p,q} \end{array}, \quad E_1^{p,q} \xrightarrow{d} E_1^{p+1,q} \quad \text{and} \quad \begin{array}{ll} E_2^{p,q} & \\ \big|{\scriptstyle 1} \quad \searrow^{d} & \\ \underline{\quad\ 2 \quad\ } & E_2^{p+2,q-1}. \end{array}$$

■

Exercise 2.3.36 Show that
1. if $r \geq 1$ and $\forall p \neq 0, E_r^{p,q} = 0$, then $\forall n \in \mathbb{Z}, H^n \simeq E_r^{0,n}$ and
2. if $r \geq 2$ and $\forall q \neq 0, E_r^{p,q} = 0$, then $\forall n \in \mathbb{Z}, H^n = E_r^{n,0}$.

Solution. Assume $E_r^{p,q} = 0$ unless $p = 0$. If $r \geq 1$, then $d_r^{p,q} = 0$ for all p, q. It follows that $E_{r+1}^{p,q} = 0$ when $p \neq 0$ and $E_{r+1}^{0,q} = E_r^{0,q}$. Thus we see that $\mathrm{Gr}^p H^{p+q} = 0$ for $p \neq 0$ and $\mathrm{Gr}^0 H^q = E_r^{0,q}$. Said differently, $\mathrm{Gr}^k H^n = 0$ for $k \neq 0$ and $\mathrm{Gr}^0 H^n = E_r^{0,n}$. This means that $H^n = E_r^{0,n}$.

The second assertion is proven exactly in the same way. ■

Exercise 2.3.37 Show that, in a first quadrant spectral sequence, the sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0}$$

is exact.

Theorem 2.3.38 Let $K^\bullet$ is a filtered complex. Assume $K^\bullet$ is bounded below. Then there exists a spectral sequence

$$E_0^{p,q} = \mathrm{Gr}^p K^{p+q} \Rightarrow \mathrm{H}^{p+q} K^\bullet.$$

Proof. To do. ■

Exercise 2.3.39 Show that $E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p K^\bullet)$.

Definition 2.3.40 A *bicomplex* (in an additive category) is a complex of complexes.

In other words, a bicomplex is a diagram $(K^{p,q}, d^{p,q}, d'^{p,q})$ on $(\mathbb{Z}, \leq)^2$ such that for all $p, q \in \mathbb{Z}$,

$$d^{p+1,q} \circ d^{p,q} = 0, \quad d'^{p,q+1} \circ d'^{p,q} = 0 \quad \text{and} \quad d'^{p+1,q} \circ d^{p,q} = d^{p,q+1} \circ d'^{p,q}.$$

Exercise 2.3.41 Show that if $K^{\bullet,\bullet}$ is a bicomplex, and we endow $K^n := \bigoplus_{p+q=n} K^{p,q}$ with $d^n := \oplus_{p+q=n}(d^{p,q} + (-1)^p d'^{p,q})$, then $K^\bullet$ is a complex.

It is called the *simple complex* associated to the bicomplex or the *total complex* of the bicomplex.

Proposition 2.3.42 Let $K^{\bullet,\bullet}$ be a bicomplex. Assume that it is bounded below (in both variables). Then there exists a spectral sequence

$$E_0^{p,q} = K^{p,q} \Rightarrow \mathrm{H}^{p+q}(K^\bullet).$$

Proof. Apply theorem 2.3.38 with $\mathrm{F}^p K^n := \bigoplus_{i+j=n, i \geq p} K^{p,q}$. ■

Exercise 2.3.43 Show that $E_1^{p,q} = \mathrm{H}^q(K^{p,\bullet})$ and $E_2^{p,q} = \mathrm{H}^p(\mathrm{H}^q(K^{\bullet,\bullet}))$.

Lemma 2.3.44 Assume $\mathcal{A}$ has enough injectives. If $K^\bullet$ is bounded below, then there exists a bounded below bicomplex $I^{\bullet,\bullet}$ with $I^{\bullet,q} = 0$ for $q < 0$ and a morphism of complexes $K^\bullet \to I^{\bullet,0}$ such that each $K^p \to I^{p,\bullet}$ and $H^p(K^\bullet) \to H^p(I^{\bullet,\bullet})$ are injective resolutions.

Proof. To do. ■

This is called a *Cartan-Eilenberg* resolution.

Exercise 2.3.45 Show that if $I^\bullet$ denotes the associated simple complex, then $K^\bullet \to I^\bullet$ is also an injective resolution.

Proposition 2.3.46 Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be an additive functor. Assume $\mathcal{A}$ has enough injectives. If $K^\bullet$ is bounded below, then there exists two spectral sequences

$$'E_1^{p,q} = R^q F K^p \Rightarrow R^{p+q} F K^\bullet \quad \text{and} \quad ''E_2^{p,q} = R^p F(H^q(K^\bullet)) \Rightarrow R^{p+q} F K^\bullet.$$

Proof. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $K^\bullet$. Then, there exists a spectral sequence

$$'E_1^{p,q} = \mathrm{H}^q(F I^{p,\bullet}) \Rightarrow \mathrm{H}^{p+q}(F I^\bullet).$$

But we may also exchange the rôle of p and q and consider the spectral sequence

$$''E_2^{p,q} = \mathrm{H}^q(F\mathrm{H}^p(I^{\bullet,\bullet})) \Rightarrow \mathrm{H}^{p+q}(F I^\bullet).$$ ■

Example With $h_M : \mathbf{A}\mathrm{b} \to \mathbf{A}\mathrm{b}$, the second spectral sequence reads

$$E_2^{p,q} = \mathrm{Ext}^p(H^q(K^\bullet), M)) \Rightarrow \mathrm{Ext}^{p+q}(K^\bullet, M).$$

But we have $E_2^{p,q} = 0$ for $p \neq 0, 1$. It follows that $\mathrm{Gr}^p = 0$ for $p \neq 0, 1$, $\mathrm{Gr}^0 H^q = E_2^{0,q}$ and $\mathrm{Gr}^1 H^{q+1} = E_2^{1,q}$. Thus we see that $\mathrm{F}^0 H^q = H^q$, $\mathrm{F}^1 H^q = E_2^{1,q-1}$, $\mathrm{F}^2 H^q = 0$. The spectral sequence "degenerates" into short exact sequences

$$0 \to \mathrm{Ext}(H^{q-1}(K^\bullet), M)) \to \mathrm{Ext}^q(K^\bullet, M) \to \mathrm{Hom}(H^q(K^\bullet), M)) \to 0.$$

In the case $K^\bullet = \mathrm{C}_\bullet(X)$, we obtain the universal coefficient theorem: the sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}^{\mathrm{sing}}(X), M)) \to \mathrm{H}^n_{\mathrm{sing}}(X, M) \to \mathrm{Hom}(\mathrm{H}_n^{\mathrm{sing}}(X), M)) \to 0$$

is exact.

Corollary 2.3.47 Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}'$ be a sequence of additive functors between abelian categories. Assume $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and FI is G-acyclic whenever I is injective. If $K^\bullet$ is bounded below, then there exists a spectral sequence

$$E_2^{p,q} = R^p G(R^q F(K^\bullet)) \Rightarrow R^{p+q}(G \circ F)(K^\bullet).$$

Proof. We may assume that $K^\bullet \in \mathbf{C}^+(\mathcal{I})$ and apply the proposition to $FK^\bullet$. ■

Exercise 2.3.48 Show that, in the situation of the corollary, when F (resp. G) is exact, there is an isomorphism

$$R^nG(F(K^\bullet)) \simeq R^n(G \circ F)(K^\bullet) \quad (\text{resp. } G(R^nF(K^\bullet)) \simeq R^n(G \circ F)(K^\bullet)).$$

3. Cohomology of sheaves

From now on, for simplicity and unless otherwise specified, all rings are assumed to be commutative. Also, to avoid confusion, we shall write $\mathrm{Hom}(-,-)$ and not $\mathcal{F}(-,-)$ anymore to denote the set of maps from one set to another.

3.1 Presheaves

3.1.1 Definition/Examples

We let X be a topological space.

Definition 3.1.1 A *presheaf* $\mathcal{F}$ with values in a category $\mathcal{D}$ (we shall say "of $\mathcal{D}$") on X is a contravariant functor $\mathcal{F} : \mathbf{O}\mathrm{pen}(X)^{\mathrm{op}} \to \mathcal{D}$. A *morphism of presheaves* $\alpha : \mathcal{F} \to \mathcal{G}$ is a natural transformation.

They form a category[1]

$$\widehat{X}(\mathcal{D}) := \mathbf{H}\mathrm{om}(\mathbf{O}\mathrm{pen}(X)^{\mathrm{op}}, \mathcal{D}).$$

We shall mostly concentrate on the category $\widehat{X} := \widehat{X}(\mathbf{S}\mathrm{et})$ of presheaves of sets.

Examples
1. A presheaf of sets $\mathcal{F}$ on X is the following data:
   (a) a set $\mathcal{F}(U)$ of *sections* on any open subset U of X and
   (b) *restriction maps* $\mathcal{F}(U) \to \mathcal{F}(V), s \mapsto s_{|V}$ for $V \subset U$ satisfying $(s_{|V})_{|W} = s_{|W}$ when $W \subset V$ and $s_{|U} = s$.

   A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a family of maps $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for U open in X satisfying $\alpha_U(s)_{|V} = \alpha_V(s_{|V})$ whenever $V \subset U$ and $s \in \mathcal{F}(U)$.
2. A presheaf of abelian groups $\mathcal{M}$ on X is a presheaf of sets such that
   (a) $\mathcal{M}(U)$ has the structure of an abelian group when $U \subset X$ is open and

[1] That notation may be confusing when some completion is involved and should then be changed.

(b) restrictions maps are group homomorphisms.

A morphism $\varphi : \mathcal{M} \to \mathcal{N}$ is a morphism of presheaves of sets such that $\varphi_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is a group homomorphism when $U \subset X$ is open.

3. A presheaf of rings $\mathcal{A}$ on X is a presheaf of sets such that
   (a) $\mathcal{A}(U)$ as the structure of a ring when $U \subset X$ is open and
   (b) restrictions maps are ring homomorphisms.

   A morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of presheaves of sets such that $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ is a ring homomorphism when $U \subset X$ is open.

Exercise 3.1.2 Show that[a] $\widehat{X}$ is a cartesian category with final object given by $1(U) = 1 := \{0\}$ for all open subsets U of X and such that

$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$

for all presheaves of sets $\mathcal{F}$ and $\mathcal{G}$.

[a]This will be generalized later in proposition 3.1.9.

We introduced in section 1.5.1 the notion of an algebraic structure in a cartesian category. It provides an alternative way to describe some categories of presheaves:

Lemma 3.1.3 There exists an equivalence $\widehat{X}(\mathbf{A}\mathrm{b}) \simeq \mathbf{A}\mathrm{b}(\widehat{X})$.

Proof. This is a particular instance of exercise 1.5.5. ∎

It means that it is equivalent to give ourselves a presheaf of abelian groups on X or an abelian group of the category $\widehat{X}$ and we shall use these equivalences to identify both categories. There exists an exact analog for other algebraic structures such as rings for example. This alternative way to describe presheaves makes it possible to define the notion of a presheaf of modules:

Definition 3.1.4 If $\mathcal{A}$ is a presheaf of rings on X (seen as a ring of $\widehat{X}$), then a *presheaf of $\mathcal{A}$-modules* is an $\mathcal{A}$-module of the category $\widehat{X}$.

They form a category $\widehat{\mathcal{A}\text{-}\mathbf{M}\mathrm{od}} := \mathcal{A}\text{-}\mathbf{M}\mathrm{od}(\widehat{X})$ with an obvious forgetful functor

$$\widehat{\mathcal{A}\text{-}\mathbf{M}\mathrm{od}} \to \widehat{X}(\mathbf{A}\mathrm{b})$$

and we shall denote by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ the set morphisms from $\mathcal{M}$ to $\mathcal{N}$. There exists also the similar notion of *presheaf of $\mathcal{A}$-algebras*.

Exercise 3.1.5 Show that, if $\mathcal{A}$ is a presheaf of rings on X, then a presheaf of $\mathcal{A}$-modules is (the same thing as) a presheaf of sets $\mathcal{M}$ together with a structure of $\mathcal{A}(U)$-module on $\mathcal{M}(U)$ for U open in X such that

1. $\forall s, s' \in \mathcal{M}(U), \quad (s + s')_{|V} = s_{|V} + s'_{|V}$ and
2. $\forall f \in \mathcal{A}(U), \forall s \in \mathcal{M}(U), \quad (fs)_{|V} = f_{|V} s_{|V}$

for V open in U.

Show that a morphism of presheaves of $\mathcal{A}$-modules $\alpha : \mathcal{M} \to \mathcal{N}$ is a morphism of presheaves of sets such that $\alpha_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{A}(U)$-linear whenever U is an open subset of X.

Examples 1. If E is any object of a category $\mathcal{D}$ with final object 1, then there exists

(a) the *constant* presheaf $\widehat{E}_X$ with $\widehat{E}_X(U) = E$,

(b) the presheaf $\widehat{E}^!_X$ with $\widehat{E}^!_X(X) = E$ and $\widehat{E}^!_X(U) = 1$ otherwise,

(c) for $x \in X$, the *skyscraper* $E^!_x$ with $E^!_x(U) = E$ if $x \in U$ and 1 otherwise.

2. If k is a (commutative) ring and M is a k-module, then $\widehat{M}_X$, $\widehat{M}^!_X$ and $M^!_x$ are presheaves of $\widehat{k}_X$-modules.
3. If E is a topological space (or topological group or topological ring or topological k-module or topological k-algebras), then there exists a presheaf of sets (or groups or rings or $\widehat{k}_X$-modules or $\widehat{k}_X$-algebras) $\mathcal{C}^X_E : U \mapsto \mathcal{C}(U, E)$ on X such that $f_{|V}$ is usual restriction if $f : U \to E$ is a continuous map and $V \subset U$.
4. If $p : E \to X$ is a continuous map, then there exists a presheaf of sets $\mathcal{C}_{E/X}$ of *continuous sections* of E on X where $\mathcal{C}_{E/X}(U)$ is the set of continuous maps $s : U \to E$ such that $p \circ s : U \rightarrowtail X$ is the inclusion map and $s_{|V}$ is restriction:

$$\begin{array}{ccccc} & & & & E \\ & {\scriptstyle s_{|V}} \nearrow & & {\scriptstyle s} \nearrow & \downarrow {\scriptstyle p} \\ V & \rightarrowtail & U & \rightarrowtail & X. \end{array}$$

Note that $\mathcal{C}^X_E \simeq \mathcal{C}_{E \times X/X}$.

5. If X is a topological space and M an abelian group, then there exists a presheaf of singular cochains $\widehat{\mathrm{C}}^n_M : U \mapsto \mathrm{C}^n(U, M)$: composition $\Delta^n \to V \rightarrowtail U$ provides $\mathrm{S}_n(V) \to \mathrm{S}_n(U)$ and composition $\mathrm{S}_n(V) \to \mathrm{S}_n(U) \to M$ then provides restriction $\mathrm{C}^n(U, M) \to \mathrm{C}^n(V, M)$.
6. If X is a $\mathcal{C}^k$-differentiable (resp. complex) manifold, then there exists a presheaf $\mathcal{O}_X$ of $\widehat{\mathbb{R}}_X$- (resp. $\widehat{\mathbb{C}}_X$-) algebras on X with $\mathcal{O}_X(U) = \mathcal{C}^k(U, \mathbb{R})$ (resp. $\mathcal{C}^{\mathrm{hol}}(U, \mathbb{C})$).
7. If E is a vector bundle on a (topological, differentiable or complex) manifold X, then the assignment $U \mapsto E(U) = \Gamma(U, E)$ (together with restriction) defines a presheaf of $\mathcal{O}_X$-modules.
8. If $X = \mathrm{Spec}(A)$ (resp. and M is an A-module), then

$$\mathcal{O}_X(U) := \varprojlim_{D(f) \subset U} A_f \qquad \left(\text{resp. } \widetilde{M}(U) := \varprojlim_{D(f) \subset U} M_f\right)$$

defines a presheaf of rings for the Zariski topology of X (resp. a presheaf of $\mathcal{O}_X$-modules). $(X, \mathcal{O}_X)$ is called an *affine scheme* and $\widetilde{M}$ is said to be *quasi-coherent*.

There exists the following version of Yoneda lemma:

Exercise 3.1.6 Show that, if $\mathcal{F}$ is a presheaf of sets (resp. $\mathcal{M}$ is a presheaf of $\mathcal{A}$-modules), then there exists a bijection (resp. an isomorphism)

$$\mathrm{Hom}(1_U, \mathcal{F}) \simeq \mathcal{F}(U) \qquad \left(\text{resp. } \mathrm{Hom}_{\mathcal{A}}(\widehat{\mathcal{A}}_U, \mathcal{M}) \simeq \mathcal{M}(U)\right)$$

with

$$1_U(V) = \begin{cases} 1 := \{0\} \text{ if } V \subset U, \\ 0 := \emptyset \text{ otherwise} \end{cases} \quad \left(\text{resp. } \widehat{\mathcal{A}}_U(V) = \begin{cases} \mathcal{A}(U) \text{ if } V \subset U, \\ \{0\} \text{ otherwise} \end{cases}\right).$$

A presheaf of abelian groups on X is the same thing as a presheaf of $\widehat{\mathbb{Z}}_X$-modules as one can easily check:

Exercise 3.1.7 Show that the forgetful functor $\widehat{\widehat{\mathbb{Z}}_X\text{-}\mathbf{Mod}} \to \widehat{X}(\mathbf{Ab})$ is an isomorphism of categories.

We shall not make much use of this fact but any functor $T : \mathcal{D} \to \mathcal{E}$ extends to a functor

$$T_* : \widehat{X}(\mathcal{D}) \to \widehat{X}(\mathcal{E}), \quad \mathcal{F} \mapsto T \circ \mathcal{F}$$

so that $(T_*\mathcal{F})(U) = T(\mathcal{F}(U))$. This is functorial and any adjunction extends to an adjunction at the presheaves level. This applies for example to forgetful functors such as $\mathbf{Ab} \to \mathbf{Set}$ which provides a forgetful functor $\widehat{X}(\mathbf{Ab}) \to \widehat{X}$ with an adjoint in the other direction. This generalizes as follows:

Exercise 3.1.8 Show that the forgetful functor $\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{X}$ has an adjoint.

Solution. Given a presheaf of sets $\mathcal{F}$, one considers the presheaf $\mathcal{A} \hat{\cdot} \mathcal{F}$ of $\mathcal{A}$-modules that sends U to the free $\mathcal{A}(U)$-modules generated by $\mathcal{F}(U)$. ∎

3.1.2 Sections and stalk

We denote by X a topological space and shall concentrate on presheaves of sets on X. We also fix a presheaf of rings $\mathcal{A}$ and will systematically indicate the analog statements in the case of presheaves of $\mathcal{A}$-modules (which contains the case of presheaves of abelian groups).

Proposition 3.1.9 All limits and colimits exist in $\widehat{X}$ and are preserved by the functors

$$\widehat{X} \to \mathbf{Set}, \quad \mathcal{F} \mapsto \mathcal{F}(U)$$

for U open in X.

Proof. This is a particular instance of exercise 1.3.48. ∎

It means that limits or colimits are computed "argument by argument":

$$\left(\varprojlim_I \mathcal{F}_i\right)(U) = \varprojlim_I \mathcal{F}_i(U) \quad \text{and} \quad \left(\varinjlim_I \mathcal{F}_i\right)(U) = \varinjlim_I \mathcal{F}_i(U).$$

Proposition 3.1.9 holds more generally with presheaves with values in any bicomplete category $\mathcal{D}$ or in the presence of an algebraic structure and we have for example:

Corollary 3.1.10 All limits and colimits exist in $\widehat{\mathcal{A}\text{-}\mathbf{Mod}}$ and are preserved by the functors

$$\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \mathcal{A}(U)\text{-}\mathbf{Mod}, \quad \mathcal{M} \mapsto \mathcal{M}(U).$$

for U open in X.

Proof. It is sufficient to show that limits and colimits of sets commute with products. For limits, this is completely formal and and for colimits, this follows from the fact that they are stable by pull back in the category of sets (exercise 1.3.44). ∎

Any formal property for presheaves of sets can be checked in the category of sets:

Exercise 3.1.11 Show that the functor

$$\widehat{X} \to \mathbf{Set}^{\mathrm{Open}(X)}, \quad \mathcal{F} \to (\mathcal{F}(U))_{U \in \mathrm{Open}(X)}$$

1. is faithful and conservative,
2. preserves (and reflects) all limits and colimits.

As a direct application, we have the following:

Examples
1. A morphism of presheaves of sets $\mathcal{F} \to \mathcal{G}$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. surjective, resp. bijective) whenever U is an open subset of X.
2. Any monomorphism/epimorphism of presheaves of sets is regular.
3. The category of presheaves of sets is balanced.
4. Any morphism of presheaves of sets is strict.
5. All colimits of presheaves of sets are stable under pullback.
6. All filtered colimits of presheaves of sets are exact.

There exists an exact analog to exercise 3.1.11 with

$$\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \prod_{U \in \mathrm{Open}(X)} \mathcal{A}(U)\text{-}\mathbf{Mod} \quad \mathcal{M} \to (\mathcal{M}(U))_{U \in \mathrm{Open}(X)}.$$

Exercise 3.1.12 Show that $\widehat{\mathcal{A}\text{-}\mathbf{Mod}}$ is an abelian category (satisfying AB6 and AB4*).

Proposition 3.1.13 The functor

$$\widehat{X} \to \mathbf{Set}, \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

has adjoint $E \mapsto \widehat{E}_X$ and coadjoint $E \mapsto \widehat{E}^!_X$.

Proof. It means that there exists natural bijections

$$\mathrm{Hom}(\widehat{E}_X, \mathcal{F}) \simeq \mathrm{Hom}(E, \mathcal{F}(X)) \quad \text{and} \quad \mathrm{Hom}(\mathcal{F}(X), E) \simeq \mathrm{Hom}(\mathcal{F}, \widehat{E}^!_X).$$

This is readily checked. ∎

In the case $X = \emptyset$, we obtain an isomorphism of categories

$$\widehat{\emptyset} \simeq \mathbf{Set}, \quad \mathcal{F} \to \mathcal{F}(\emptyset)$$

with inverse $E \mapsto \widehat{E}_\emptyset = \widehat{E}^!_\emptyset$ and we may *identify* these two categories.

The analog of proposition 3.1.13 is straightforward for abelian groups but needs some refinement in the case of $\mathcal{A}$-modules with adjoint (resp. coadjoint) sending M to $U \mapsto A(U) \otimes_{A(X)} M$ (resp. $\mathrm{Hom}_{A(X)}(A(U), M)$).

The following makes sense as long as the colimit exists:

Definition 3.1.14 If $\mathcal{F}$ is a presheaf on X, then the *stalk* of $\mathcal{F}$ at $x \in X$ is[a].

$$\mathcal{F}(x) := \varinjlim_{x \in U} \mathcal{F}(U).$$

[a]The usual notation $\mathcal{F}_x$ conflcts with notation E_x for the fiber in the important case of a vector bundle.

This is functorial and we have for example

$$\widehat{X} \to \mathbf{Set}, \quad \mathcal{F} \mapsto \mathcal{F}(x) \quad (\text{or} \quad \widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \mathcal{A}(x)\text{-}\mathbf{Mod}, \quad \mathcal{M} \mapsto \mathcal{M}(x)). \qquad (3.1)$$

By definition, if U is an open neighborhood of x, then there exists a structural map

$$\mathcal{F}(U) \to \mathcal{F}(x), \quad s \mapsto s_x$$

and s_x is called the *germ* of s at x. Conversely, any element the stalk is the germ of some section s in some neighborhood U of x. Moreover, another section s' in another neighborhood U' has the same germ if and only if they coincide on a smaller neighborhood U'':

$$s_x = s'_x \Leftrightarrow \exists U'' \subset U \cap U', s_{|U''} = s'_{|U''}.$$

Exercise 3.1.15 Show that the stalk functor $\mathcal{F} \mapsto \mathcal{F}(x)$ on $\widehat{X}$ is exact and preserves all colimits.

Examples
1. We have $\widehat{E}_X(x) = E$ whatever E and x.
2. We have $E^!_x(y) = \begin{cases} E \text{ if } y \in \overline{\{x\}}, \\ 1 \text{ otherwise.} \end{cases}$
3. If E is a vector bundle on a manifold X, then $E(x)$ is a free $\mathcal{O}_X(x)$-module of finite rank.
4. If $X = \mathrm{Spec}(A)$, $x = \mathfrak{p}$ and $\widetilde{M}$ is an A-module M, then $\widetilde{M}(x) = M_{\mathfrak{p}}$.

Proposition 3.1.16 There exists an adjunction

$$\mathrm{Hom}(\mathcal{F}(x), E) \simeq \mathrm{Hom}(\mathcal{F}, E^!_x) \quad (\text{and } \mathrm{Hom}_{\mathcal{A}(x)}(\mathcal{M}(x), N) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, N^!_x)).$$

Proof. Giving a morphism $\mathcal{F} \to E^!_x$ amounts to giving a compatible family of maps $\mathcal{F}(U) \to E$ when $x \in U$. ∎

Exercise 3.1.17 Show that $\mathcal{O}_X(x)$ is a local ring when X is a manifold.

Solution. If U is an open subset of X and $x \in U$, then there exists an obvious surjective morphism of rings

$$\mathcal{O}_X(U) := \mathcal{C}^k(U, \mathbb{R}) \twoheadrightarrow \mathbb{R}, \quad f \mapsto f(x).$$

Since filtered limits of rings are exact, we obtain a surjective morphism of rings

$$\mathcal{O}_X(x) \twoheadrightarrow \mathbb{R}, \quad f \mapsto f(x)$$

and we need to show that the kernel $\mathfrak{m}_X(x)$ is the unique maximal ideal. Assume that $f \in \mathcal{O}_X(x) \setminus \mathfrak{m}_X(x)$. Then, f is defined on some open subset $U \subset X$ and $f(x) \neq 0$. Since f is continuous, $f^{-1}(0)$ is closed in U. After shrinking U if necessary (removing $f^{-1}(0)$), we may therefore assume that $f^{-1}(0) = \emptyset$ and consider the function $1/f$ on U. It is a differentiable function that provides an inverse for f in $\mathcal{O}_X(x)$ (showing that $\mathfrak{m}_X(x)$ is indeed the biggest proper ideal). ■

3.1.3 Internal constructions

We let again X be a topological space and $\mathcal{A}$ a presheaf of (commutative) rings on X.

Definition 3.1.18 If $\mathcal{F}$ is a presheaf on X, then the *restriction* of $\mathcal{F}$ to an open subset U is the presheaf $\mathcal{F}_{|U}$ on U given by $\mathcal{F}_{|U}(V) = \mathcal{F}(V)$ if $V \subset U$ is an open subset.

This is functorial and provides a restriction functor

$$\widehat{X} \to \widehat{U}, \quad \mathcal{F} \mapsto \mathcal{F}_{|U} \quad (\text{and } \widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{\mathcal{A}_{|U}\text{-}\mathbf{Mod}}, \quad \mathcal{M} \mapsto \mathcal{M}_{|U}).$$

Exercise 3.1.19 Show that, if U is an open subset of X, then the restriction has and adjoint and a coadjoint given respectively by[a]

$$\mathcal{F} \mapsto \left(V \mapsto \begin{cases} \mathcal{F}(V) \text{ if } V \subset U \\ \emptyset \text{ otherwise} \end{cases}\right) \quad \text{and} \quad \mathcal{F} \mapsto (V \mapsto \mathcal{F}(U \cap V)).$$

[a]Use $\{0\}$ in place of $\emptyset$ in the algebraic case.

Definition 3.1.20 If $\mathcal{F}$ and $\mathcal{G}$ are two presheaves of sets on X, then the *presheaf of morphisms from $\mathcal{F}$ to $\mathcal{G}$* is defined by

$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U}).$$

when U is an open subset of X.

This defines a functor

$$\widehat{X}^{\mathrm{op}} \times \widehat{X} \to \widehat{X}, \quad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}).$$

Examples 1. $\mathcal{H}\mathrm{om}(1, \mathcal{F}) \simeq \mathcal{F}$,
2. $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(X) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

Proposition 3.1.21 If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are three presheaves of sets on X, then there exists a natural isomorphism

$$\mathcal{H}\mathrm{om}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})).$$

Proof. We have to show that, if U is an open subset of X, then

$$\mathrm{Hom}((\mathcal{F} \times \mathcal{G})_{|U}, \mathcal{H}_{|U}) \simeq \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})_{|U}).$$

But, clearly, by definition

$$(\mathcal{F} \times \mathcal{G})_{|U} = \mathcal{F}_{|U} \times \mathcal{G}_{|U} \quad \text{and} \quad \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})_{|U} = \mathrm{Hom}(\mathcal{G}_{|U}, \mathcal{H}_{|U}).$$

After replacing X by U, we are therefore reduced to showing that

$$\mathrm{Hom}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})). \tag{3.2}$$

A morphism $\mathcal{F} \to \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})$ is a compatible family of applications

$$\mathcal{F}(U) \to \mathrm{Hom}(\mathcal{G}_{|U}, \mathcal{H}_{|U})$$

for $U \subset X$ or equivalently of applications

$$\mathcal{F}(U) \to \mathrm{Hom}(\mathcal{G}(V), \mathcal{H}(V))$$

for $V \subset U \subset X$. By currying, this is the same as applications

$$\mathcal{F}(U) \times \mathcal{G}(V) \to \mathcal{H}(V).$$

Compatibility implies that they are uniquely determined from the case $U = V$. This corresponds to a morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. ■

Property (3.2), that can also easily be recovered from the statement by taking global sections, shows that, for fixed $\mathcal{G}$, the functor $\mathcal{F} \mapsto \mathcal{F} \times \mathcal{G}$ is adjoint to the functor $\mathcal{H} \mapsto \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{H})$. In particular, the first one preserves all (limits and) colimits and the second one preserves all limits.

Even if $\mathcal{H}\mathrm{om}$ makes sense for presheaves with values in any category $\mathcal{D}$, proposition 3.1.21 is obviously specific to presheaves of sets. There exists however a variant for presheaves of abelian groups or more generally of $\mathcal{A}$-modules.

Definition 3.1.22 Given two presheaves of $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$, the *presheaf of $\mathcal{A}$-linear morphisms from $\mathcal{M}$ to $\mathcal{N}$* is defined by

$$\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})(U) := \mathrm{Hom}_{\mathcal{A}_{|U}}(\mathcal{M}_{|U}, \mathcal{N}_{|U})$$

when U is an open subset of X.

Examples
1. $\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}) \simeq \mathcal{M}$,
2. $\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})(X) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$.

This provides a functor

$$\widehat{\mathcal{A}\text{-}\mathbf{Mod}}^{\mathrm{op}} \times \widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{\mathcal{A}\text{-}\mathbf{Mod}}, \quad (\mathcal{M}, \mathcal{N}) \mapsto \mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$$

Definition 3.1.23 The *tensor product presheaf* of presheaves $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{A}$-modules is defined by[a]

$$(\mathcal{M} \widehat{\otimes}_{\mathcal{A}} \mathcal{N})(U) := \mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$$

for U open in X.

[a] This is a bad notation that could be confused with completion.

We get a functor

$$\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \times \widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{\mathcal{A}\text{-}\mathbf{Mod}}, \quad (\mathcal{M},\mathcal{N}) \mapsto \mathcal{M}\widehat{\otimes}_{\mathcal{A}}\mathcal{N}.$$

Exercise 3.1.24 Show that $(\mathcal{M}\widehat{\otimes}_{\mathcal{A}}\mathcal{N})(x) = \mathcal{M}(x) \otimes_{\mathcal{A}(x)} \mathcal{N}(x)$ for $x \in X$

Proposition 3.1.25 If $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ are three presheaves of $\mathcal{A}$-modules on X, then there exists a natural isomorphism

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{M}\widehat{\otimes}_{\mathcal{A}}\mathcal{N},\mathcal{P}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{H}om_{\mathcal{A}}(\mathcal{N},\mathcal{P})).$$

Proof. This is shown exactly as proposition 3.1.21. ∎

As above, for fixed $\mathcal{N}$, the functor $\mathcal{M} \mapsto \mathcal{M}\widehat{\otimes}_{\mathcal{A}}\mathcal{N}$ is adjoint to the functor $\mathcal{P} \mapsto \mathcal{H}om_{\mathcal{A}}(\mathcal{N},\mathcal{P})$. When $\mathcal{A} \to \mathcal{B}$ is a morphism of presheaves of rings and $\mathcal{M}$ is a presheaf of $\mathcal{A}$-modules, then $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{M}$ has a natural structure of presheaf of $\mathcal{B}$-modules and we get a functor

$$\widehat{\mathcal{A}\text{-}\mathbf{Mod}} \to \widehat{\mathcal{B}\text{-}\mathbf{Mod}}, \quad \mathcal{M} \mapsto \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{M}.$$

Exercise 3.1.26 Show that if $\mathcal{A} \to \mathcal{B}$ is a morphism of presheaves of rings and $\mathcal{M}$ (resp. $\mathcal{N}$) is a presheaf of $\mathcal{A}$-modules (resp. $\mathcal{B}$-modules), then there exists a natural isomorphism

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{M},\mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{N}).$$

One can also define in the same way the presheaves of (graded) rings $\widehat{\mathrm{T}}(\mathcal{M})$, $\widehat{\mathrm{S}}(\mathcal{M})$ and $\widehat{\Lambda}(\mathcal{M})$.

Exercise 3.1.27 Show that $\widehat{\mathrm{T}}(\mathcal{M})(x) = \widehat{\mathrm{T}}(\mathcal{M}(x))$, $\widehat{\mathrm{S}}(\mathcal{M})(x) = \widehat{\mathrm{S}}(\mathcal{M}(x))$ and $\widehat{\Lambda}(\mathcal{M})(x) = \widehat{\Lambda}(\mathcal{M}(x))$.

Hint. For the first two, one may simply use the fact that both T and S have a coadjoint. ∎

3.1.4 Direct and inverse image of presheaves

We fix a continuous map $f : X \to Y$ as well as a a presheaf of rings $\mathcal{A}$ (resp. $\mathcal{B}$) on X (resp. Y).

By composition, the functor $f^{-1} : \mathbf{Open}(Y) \to \mathbf{Open}(X)$ yields a functor

$$\widehat{f}_* : \widehat{X} \to \widehat{Y}, \quad \mathcal{F} \mapsto \mathcal{F} \circ f^{-1}$$

so that

$$(\widehat{f}_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)).$$

Proposition 3.1.28 The functor $\widehat{f}_* : \widehat{X} \to \widehat{Y}$ has both an *exact* adjoint

$$\widehat{f}^{-1} : \widehat{Y} \to \widehat{X} \quad \text{with} \quad (\widehat{f}^{-1}\mathcal{G})(U) = \varinjlim_{f(U)\subset V} \mathcal{G}(V)$$

and a coadjoint

$$\widehat{f}^! : \widehat{Y} \to \widehat{X} \quad \text{with} \quad (\widehat{f}^!\mathcal{G})(U) = \varprojlim_{V\subset f(U)} \mathcal{G}(V).$$

Proof. [2] Since filtered limits of sets are exact, we have to describe adjunctions

$$\mathrm{Hom}(\widehat{f}^{-1}\mathcal{G},\mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G},\widehat{f}_*\mathcal{F}) \quad \text{and} \quad \mathrm{Hom}(\widehat{f}_*\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F},\widehat{f}^!\mathcal{G})$$

[2] One may also rely on exercise 1.4.26 with $g = f^{-1}$.

but we shall only do the first one, leaving the other which is totally similar as an exercise. Giving a morphism $\widehat{f}^{-1}\mathcal{G} \to \mathcal{F}$ amounts to giving a compatible family of maps

$$\varinjlim_{f(U)\subset V} \mathcal{G}(V) \to \mathcal{F}(U)$$

for U open in X. This is the same thing as a compatible family of maps $\mathcal{G}(V) \to \mathcal{F}(U)$ for U open in X, V open in Y and $f(U) \subset V$. Such a family is uniquely determined by the compatible family of maps $\mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ for V open in Y. In other words: a morphism $\mathcal{G} \to \widehat{f}_*\mathcal{F}$. ■

As a consequence, $\widehat{f}^{-1}$ preserves all finite limits and all colimits, $\widehat{f}_*$ preserves all limits and colimits and $\widehat{f}^!$ preserves all limits. Moreover, this is functorial in the sense that

$$\widehat{(g\circ f)}^{-1} = \widehat{f}^{-1}\circ\widehat{g}^{-1}, \quad \widehat{(g\circ f)}_* = \widehat{g}_*\circ\widehat{f}_* \quad \text{and} \quad \widehat{(g\circ f)}^! = \widehat{f}^!\circ\widehat{g}^!$$

if $g : Y \to Z$ is another continuous map.

In general, for presheaves with values in a given category $\mathcal{D}$, one can always set

$$\widehat{f}_* : \widehat{X}(\mathcal{D}) \to \widehat{Y(\mathcal{D})}, \quad \mathcal{F} \mapsto \mathcal{F}\circ f^{-1}$$

but the existence of $\widehat{f}^{-1}$ requires filtered colimits (and dual) and $\widehat{f}^{-1}$ need not be left exact. However, since in the case of presheaves of sets, the maps $\widehat{f}^{-1}, \widehat{f}_*, \widehat{f}^!$ preserve finite products, they preserve all algebraic structures and we have the following adjunctions:

$$\mathrm{Hom}_{\widehat{f}^{-1}\mathcal{B}}(\widehat{f}^{-1}\mathcal{N},\mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{B}}(\mathcal{N},\widehat{f}_*\mathcal{M})$$

and

$$\mathrm{Hom}_{\widehat{f}_*\mathcal{A}}(\widehat{f}_*\mathcal{M},\mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\widehat{f}^!\mathcal{N}).$$

In the rest of the section, we shall concentrate on presheaves of sets and only mention it when there is something to worry about in general.

Exercise 3.1.29 Show that, for a set E, we have $\widehat{f}^{-1}E_Y = E_X$.

Solution. Both presheaves share the same universal property:

$$\mathrm{Hom}(\widehat{f}^{-1}E_Y,\mathcal{F}) \simeq \mathrm{Hom}(E_Y,\widehat{f}_*\mathcal{F}) \simeq \mathrm{Hom}(E,(\widehat{f}_*\mathcal{F})(Y)) = \mathrm{Hom}(E,\mathcal{F}(X)).$$ ■

Exercise 3.1.30 Show that, if $i : Z \rightarrowtail X$ is the inclusion of a subset (with the induced topology), then $\widehat{i}_*$ is fully faithful.

Solution. We use the argument of exercise 1.4.10. We have

$$(\widehat{i}_*\mathcal{G})(U) = \mathcal{G}(U\cap Z) \quad \text{and} \quad (\widehat{i}^{-1}\mathcal{F})(V) = \varinjlim_{V=U\cap Z} \mathcal{F}(U)$$

and therefore

$$((\widehat{i}^{-1}\circ\widehat{i}_*)\mathcal{G})(V) = \varinjlim_{V=U\cap Z} \mathcal{G}(U\cap Z) = \mathcal{G}(V).$$ ■

Exercise 3.1.31 Show that

1. if $j : U \rightarrowtail X$ is the inclusion of a open subset and $\mathcal{F}$ is a presheaf of sets on X, then $\widehat{j}^{-1}\mathcal{F} = \mathcal{F}_{|U}$,
2. $\widehat{j}^{-1}$ has an adjoint $\widehat{j}_!$ given by $(j_!\mathcal{G})(V) = \mathcal{G}(V)$ if $V \subset U$ and $\emptyset$ otherwise[a],
3. $\widehat{j}_!$ is fully faithful and preserves all limits and all colimits.

[a]For presheaves of $\mathcal{A}$-modules, we must replace $\emptyset$ with $\{0\}$.

Exercise 3.1.32 Consider the empty map by $\emptyset_X : \emptyset \rightarrowtail X$. Make $\widehat{\emptyset}_X^{-1}$, $\widehat{\emptyset}_{X*}$, and $\widehat{\emptyset}_X^!$ explicit (using the identification $\widehat{\emptyset(\mathbf{Set})} = \mathbf{Set}$ so that $E(\emptyset) = E$).

Solution. First of all, we have $(\emptyset_{X*}E)(U) = E(\emptyset) = E$ so that $\emptyset_{X*}E = \widehat{E}_X$ (constant presheaf on X). By adjunction, $\emptyset_X^!\mathcal{F} = \mathcal{F}(X)$. Finally, we have

$$\mathrm{Hom}(\mathcal{F}(\emptyset), E) \simeq \mathrm{Hom}(\mathcal{F}, E_X)$$

showing that $\emptyset_X^{-1}\mathcal{F} = \mathcal{F}(\emptyset)$ (global sections on $\emptyset$). ■

Exercise 3.1.33 Show that, if $\mathcal{F}$ is a presheaf of sets on a topological space X and $i_x : x \rightarrowtail X$ denotes the inclusion of a point, then[a]

$$\mathcal{F}(x) = (i_x^{-1}\mathcal{F})(x).$$

[a]This is an a priori justification for notation $\mathcal{F}(x)$ in place of $\mathcal{F}_x$.

Exercise 3.1.34 Show that
1. if $f : X \to Y$ is a continuous map, then $(\widehat{f}^{-1}\mathcal{G})(x) \simeq \mathcal{G}_{f(x)}$,
2. if $i : Z \rightarrowtail X$ is an inclusion map, then $(\widehat{i}_*\mathcal{F})(x) \simeq \mathcal{F}(x)$,
3. if $j : U \rightarrowtail X$ is the inclusion of an open subset, then $(\widehat{j}_!\mathcal{F})(x) \simeq \mathcal{F}(x)$ if $x \in U$ and $\emptyset$ otherwise[a].

[a]Or $\{0\}$ in the case of presheaves of modules for example.

3.2 Sheaves

3.2.1 Definition/Examples

We let X be a topological space.

We shall use the notion of a covering sieve that we believe is relevant but is also suitable to further generalizations. Everything however could be done in terms of open coverings.

Definition 3.2.1 A *sieve* of X is a subset R of $\mathrm{Open}(X)$ such that

$$\forall V, V' \in \mathrm{Open}(X), \quad V' \subset V \in R \Rightarrow V' \in R.$$

It is called a *covering sieve* if R is a covering of X, i.e. $X = \bigcup_{V \in R} V$.

We shall systematically use the obvious fact that an arbitrary union or intersection of sieves is always a sieve.

Examples 1. The sieves of $\emptyset$ are $\emptyset$ and $\{\emptyset\}$. They are both coverings sieves of $\emptyset$.
2. If U is an open subset of X, then $\mathrm{Open}(U)$ is a sieve of X and it is a covering sieve if and only if $U = X$.
3. More generally, there always exists a smallest sieve

$$R := \bigcup_{U \in E} \mathrm{Open}(U)$$

containing some given $E \subset \mathrm{Open}(X)$ (and same for families). Moreover, R a covering sieve if and only if E is a covering of X.

Sieves may be turned into a poset (by inclusion) and then a category.

Exercise 3.2.2 Show that there exists an equivalence between the category of sieves of X and the category of presheaves with values in $[\mathbf{1}]$ sending R to 1_R with

$$1_R(V) := \begin{cases} 1 \text{ if } V \in R, \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 3.2.3 The set $J(X)$ of all covering sieves of X is a *filter* on the poset of all sieves. More precisely,

1. $J(X) \neq \emptyset$,
2. if $R_1, R_2 \in J(X)$, then $R_1 \cap R_2 \in J(X)$,
3. if $R \subset R'$ and $R \in J(X)$, then $R' \in J(X)$.

Proof. Only the second assertion needs a proof. Given $x \in X$, there exists by definition $V_i \in R$ such that $x \in V_i$ for $i = 1, 2$. It follows that $x \in V := V_1 \cap V_2$. But $V \subset V_i \in R_i$ and then, by definition again, $V \in R_i$. It follows that $V \in R_1 \cap R_2$. ■

Exercise 3.2.4 Show that, if $R \in J(X)$ and U is an open subset of X, then $R \cap \text{Open}(U) \in J(U)$.

Proposition 3.2.5 Let R be a sieve of X, for all $V \in R$, $S_V \in J(V)$ and $S := \bigcup_{V \in R} S_V$. Then, $R \in J(X) \Leftrightarrow S \in J(X)$.

Proof. If $R \in J(X)$ and $x \in X$, then there exists $V \in R$ such that $x \in V$, and then $W \in S_V$ such that $x \in W$ with $W \in S$. The reverse implication follows from the fact that $S \subset R$. ■

Definition 3.2.6 A *sheaf* $\mathcal{F}$ on X with values in a complete category $\mathcal{D}$ is a presheaf such that the morphism

$$\mathcal{F}(U) \to \mathcal{F}(R) := \varprojlim_{V \in R} \mathcal{F}(V) \tag{3.3}$$

is an isomorphism whenever $R \in J(U)$. When the map $\mathcal{F}(U) \to \mathcal{F}(R)$ is only a monomorphism, then $\mathcal{F}$ is said to be *separated*.

They form a full subcategory $\widetilde{X}(\mathcal{D})$ of the category of all presheaves and we shall consider the embedding

$$\mathcal{H} : \widetilde{X}(\mathcal{D}) \rightarrowtail \widehat{X}(\mathcal{D}).$$

In practice, one often simply say *sheaf* for sheaf of sets and denote by $\widetilde{X}$ their category. It is also common to say *abelian sheaf* instead of "sheaf of abelian groups".

Exercise 3.2.7 Show that if $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}(\emptyset) = 1$.

Exercise 3.2.8 Show that a sheaf $\mathcal{F}$ is uniquely determined by $\mathcal{F}(V)$ when V runs through a basis $\mathcal{B}$ of open subsets of X.

Hint. If U is an open subset of X, then the set $E = \{V \in \mathcal{B}, V \subset U\}$ is stable under finite intersection. If we denote by R the sieve generated by E, we will then have

$\mathcal{F}(U) = \varprojlim_{V\in R} \mathcal{F}(V) = \varprojlim_{V\in E} \mathcal{F}(V)$ (this may be checked directly but also results from proposition 1.3.9). ■

The definition in terms of coverings is as follows:

Proposition 3.2.9 A presheaf $\mathcal{F}$ on X is a sheaf if and only if the sequence

$$\mathcal{F}(U) \to \prod_{V\in E} \mathcal{F}(V) \rightrightarrows \prod_{V,W\in E} \mathcal{F}(V\cap W)$$

is left exact whenever E is a covering of an open subset U of X (and idem with families).

Proof. If we denote by R the sieve generated by E, then we have an isomorphism (use proposition 1.3.9 or do it directly):

$$\varprojlim_{V\in R} \mathcal{F}(V) \simeq \ker\left(\prod_{V\in E} \mathcal{F}(V) \rightrightarrows \prod_{V,W\in E} \mathcal{F}(V\cap W)\right)$$

(and conversely: any sieve is generated by its elements). ■

For an $\mathcal{A}$-module $\mathcal{M}$, the condition means that the sequence

$$0 \to \mathcal{M}(U) \to \prod_{V\in E} \mathcal{M}(V) \to \prod_{V,W\in E} \mathcal{M}(V\cap W)$$

is (left) exact (where second map sends (s_V) to $(s_{V|V\cap W} - s_{W|V\cap W})$). Note also that $\mathcal{F}$ is separated if and only if the map

$$\mathcal{F}(U) \to \prod_{V\in E} \mathcal{F}(E)$$

is always injective (for any covering).

We recover the classical definition of a sheaf of sets in terms of covering families:

Corollary 3.2.10 A presheaf of sets $\mathcal{F}$ is a sheaf if and only if, given any open covering $U = \bigcup_{i\in I} U_i$ and any sections $s_i \in \mathcal{F}(U_i)$ for $i \in I$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ whenever $i, j \in I$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$ whenever $i \in I$. ■

As usual, uniqueness may checked on the side and we can actually split the condition in two:

1. If $s, s' \in \mathcal{F}(U)$ satisfy $s_{|U_i} = s'_{|U_i}$ whenever $i \in I$, then $s = s'$ and
2. given $s_i \in \mathcal{F}(U_i)$ when $i \in I$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ whenever $i, j \in I$, there exists $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$ whenever $i \in I$.

Exercise 3.2.11 Show[a] that a presheaf $\mathcal{F}$ with values in a complete category $\mathcal{D}$ is a sheaf if and only if the presheaf $U \mapsto \mathrm{Hom}(E, \mathcal{F}(U))$ is a sheaf of sets whenever $E \in \mathcal{D}$.

[a] This provides a way to extend the definition of a sheaf even when there are not enough

limits in $\mathcal{D}$.

Exercise 3.2.12 Show that a presheaf of abelian groups or rings is a sheaf if and only if the underlying presheaf of sets is sheaf.

Modeled on this last result, we can do the following:

Definition 3.2.13 If $\mathcal{A}$ is a sheaf of rings on X, then an $\mathcal{A}$-*module* is a presheaf of $\mathcal{A}$-modules whose underlying presheaf of sets is a sheaf.

They form a full subcategory $\mathcal{A}$-**Mod** and we shall also denote by

$$\mathcal{H} : \mathcal{A}\text{-}\mathbf{Mod} \rightarrowtail \widehat{\mathcal{A}\text{-}\mathbf{Mod}}$$

the embedding. There exists an alternative definition for an $\mathcal{A}$-module:

Exercise 3.2.14 Show that if $\mathcal{A}$ is a sheaf of rings on X, then

$$\mathcal{A}\text{-}\mathbf{Mod} \simeq \mathcal{A}\text{-}\mathbf{Mod}(\widetilde{X})$$

(a sheaf of $\mathcal{A}$-modules is essentially the same thing as an $\mathcal{A}$-module in the category of sheaves of sets).

Examples 1. The presheaves $\widehat{E}_X$ and $\widehat{E}^!_X$ are not sheaves in general.

2. A skyscraper $E^!_x$ is always a sheaf.
3. If E is a topological space (or topological abelian group or topological ring or topological A-module or topological k-algebras), then the presheaf $\mathcal{C}^X_E$ is actually a sheaf on X.
4. If $p : E \to X$ is a continuous map, then the presheaf $\mathcal{C}_{E/X}$ of continuous sections of p is a sheaf (of sets).
5. The presheaf of singular cochains $\widehat{\mathrm{C}}^n_M$ is not a sheaf in general.
6. If X is a manifold, then $\mathcal{O}_X$ is a sheaf.
7. If E is a vector bundle on a manifold X, then the presheaf $U \mapsto E(U) := \Gamma(U, E)$ is a sheaf (of $\mathcal{O}_X$-modules).
8. If X is a smooth manifold, then $\Omega^\bullet_X$ is a complex of $\mathbb{R}_X$-modules (complex made of $\mathcal{O}_X$-modules but only $\mathbb{R}_X$-linear maps).
9. If $X = \mathrm{Spec}(A)$ and M is an A-module, then $\mathcal{O}_X$ is a sheaf of rings for the Zariski topology and $\widetilde{M}$ is an $\mathcal{O}_X$-module (see exercise 3.2.15).

Exercise 3.2.15 Show that $\mathcal{O}_X$ is a sheaf on $X := \mathrm{Spec}(A)$ and that $\widetilde{M}$ is an $\mathcal{O}_X$-module when M is an A-module.

Hint. We have to show that, if U is an open subset of X and $R \in J(U)$, then

$$\varprojlim_{D(f)\subset U} M_f \simeq \varprojlim_{D(f)\subset V \in R} M_f.$$

Since the special subsets $D(f)$ form a basis and X is compact, this blows down to showing that if $X = D(f) \cup D(g)$, then

$$M = \ker\left(M_f \times M_g \xrightarrow{-} M_{fg}\right).$$

The property translates into f and g being comaximal and our assertion therefore follows from exercise 2.1.39. ■

Exercise 3.2.16 Show that, if $X = \mathrm{Spec}(A)$, then there exists an adjunction

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{N}) \simeq \mathrm{Hom}_A(M, \Gamma(X, \mathcal{N}))$$

if M is an A-module and $\mathcal{N}$ an $\mathcal{O}_X$-module.

Hint. If $U = \mathrm{D}(f) \subset X$, then any A-linear map $M \to \Gamma(X, \mathcal{N})$ will induce

$$\Gamma(U, \widetilde{M}) = M_f = A_f \otimes_A M \to A_f \otimes_A \Gamma(X, \mathcal{N}) \to \Gamma(U, \mathcal{N}).$$ ■

Exercise 3.2.17 An $\mathcal{A}$-module is said to be *locally free of rank n* if it is locally isomorphic to $\mathcal{A}^n$. Show that the category of vector bundles on a manifold X is equivalent to the category of locally free $\mathcal{O}_X$-modules of finite rank.

As we usually do, we concentrate now on the case of sheaves of sets and sometimes mention the $\mathcal{A}$-module analog when it holds. There exists a sheaf analog to exercise 3.1.13 which is slightly more involved:

Proposition 3.2.18 The global section functor[a]

$$\widetilde{X} \to \mathbf{Set}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

has adjoint $E \mapsto \mathcal{C}_E^X$ where E has the discrete topology.

[a]There exists an exact analog for abelian sheaves but not for $\mathcal{A}$-modules.

Solution. We have to show that there exists a natural isomorphism

$$\mathrm{Hom}(\mathcal{C}_E^X, \mathcal{F}) \simeq \mathrm{Hom}(E, \mathcal{F}(X)).$$

Since E has the discrete topology, if U is an open subset of X, then $\mathcal{C}_E^X(U) := \mathcal{C}(U, E)$ is the set of locally constant maps. There exists a map $c_U : E \to \mathcal{C}(U, E)$ sending $a \in E$ to the constant map $x \mapsto a$. Given a sheaf of sets and a morphism $\varphi : \mathcal{C}_E^X \to \mathcal{F}$, we may then consider the composition $E \overset{c_X}{\to} \mathcal{C}_E^X(X) \overset{\varphi_X}{\to} \mathcal{F}(X)$. Conversely, assume given $\psi : E \to \mathcal{F}(X)$ and a locally constant map $f : U \to E$ on an open subset of X. If we write $U_a := f^{-1}(a)$ for $a \in E$, then $U = \coprod_{a\in E} U_a$ is a disjoint open covering and since $\mathcal{F}$ is a sheaf, we have $\mathcal{F}(U) = \prod_{a\in E} \mathcal{F}(U_a)$. We may then set

$$\varphi_U(f) := \left(\psi(a)_{|U_a}\right) \in \prod_{a\in E} \mathcal{F}(U_a).$$

We can then check that this does define a morphism $\varphi : \mathcal{C}_E^X \to \mathcal{F}$ and that our maps are inverse to each other. ■

Note also that proposition 3.1.16 makes also sense at the sheaf level: there exists an adjunction

$$\mathrm{Hom}(\mathcal{F}(x), E) \simeq \mathrm{Hom}(\mathcal{F}, E_x^!) \quad (\text{and } \mathrm{Hom}_{\mathcal{A}(x)}(\mathcal{M}(x), N) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, N_x^!)).$$

Finally, there exists the following characterization of separated presheaves:

Proposition 3.2.19 A presheaf of sets (or $\mathcal{A}$-moudles) $\mathcal{F}$ is separated if and only if all maps

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}(x)$$

for U open in X are injective.

Proof. If the condition is satisfied and R is a covering sieve of an open subset U, then the map

$$\mathcal{F}(R) := \varprojlim_{V \in R} \mathcal{F}(V) \to \varprojlim_{V \in R} \prod_{x \in V} \mathcal{F}(x) = \prod_{x \in U} \mathcal{F}(x) \simeq \mathcal{F}(U)$$

is injective. Conversely, if $s, s' \in \mathcal{F}(U)$ satisfy $s_x = s'_x$ for all $x \in U$, then $R := \{V \subset U, s_{|V} = s'_{|V}\}$ is a covering sieve of U. If $\mathcal{F}$ is separated, we have an injection $\mathcal{F}(U) \rightarrowtail \mathcal{F}(R) \rightarrowtail \prod_{V \in R} \mathcal{F}(V)$ and therefore $s = s'$. ∎

3.2.2 Sheafification

We fix a topological space X and a sheaf of rings $\mathcal{A}$ on X. From now on, we only say *(pre) sheaf* for a *(pre) sheaf of sets* and make it clear when we consider another situation.

Let $\mathcal{F}$ be a presheaf on X. If U is an open subset of X and $R, R' \in J(U)$ with $R \subset R'$, then there exists a *restriction* map

$$\mathcal{F}(R') := \varprojlim_{V \in R'} \mathcal{F}(V) \to \mathcal{F}(R) := \varprojlim_{V \in R} \mathcal{F}(V).$$

We then set

$$\check{\mathcal{H}}(\mathcal{F})(U) := \varinjlim_{R \in J(U)} \mathcal{F}(R) \tag{3.4}$$

(be careful that we mix limits and – fortunately filtered – colimits). This defines a presheaf on X and we obtain an endofunctor $\check{\mathcal{H}} : \widehat{X} \to \widehat{X}$ together with a natural transformation $\mathcal{F} \to \check{\mathcal{H}}(\mathcal{F})$ induced by the natural maps $\mathcal{F}(U) \to \mathcal{F}(R)$.

Proposition 3.2.20
1. The functor $\check{\mathcal{H}}$ is left exact.
2. If $\mathcal{F}$ is a presheaf (resp. a separated presheaf), then $\check{\mathcal{H}}(\mathcal{F})$ is separated (resp. a sheaf).
3. A presheaf $\mathcal{F}$ is separated (resp. a sheaf) if and only if $\mathcal{F} \to \check{\mathcal{H}}(\mathcal{F})$ is a monomorphism (resp. an isomorphism).

Proof. The first assertion follows from the fact that both limits and filtered colimits are left exact (and that they are computed termwise on presheaves).

To prove that $\check{\mathcal{H}}(\mathcal{F})$ is always separated, we have to show that the map

$$\check{\mathcal{H}}(\mathcal{F})(U) \to \check{\mathcal{H}}(\mathcal{F})(R) \tag{3.5}$$

is injective whenever U is an open subset of X and $R \in J(U)$. Thus, we give ourselves two sections of $\check{\mathcal{H}}(\mathcal{F})$ on U that coincide in $\check{\mathcal{H}}(\mathcal{F})(R)$ and we must show that they already coincide in $\check{\mathcal{H}}(\mathcal{F})(U)$. By definition, they come from two sections of $\mathcal{F}(R')$ for some covering sieve R' of U and it is sufficient to show that they coincide after possibly refining R'. In particular, after a quick look at the commutative diagram

$$\begin{array}{ccc}\mathcal{F}(R') & \longrightarrow & \check{\mathcal{H}}(\mathcal{F})(R) \\ \downarrow & & \downarrow \\ \mathcal{F}(R \cap R') & \longrightarrow & \check{\mathcal{H}}(\mathcal{F})(R \cap R'),\end{array}$$

we can replace both R and R' by $R \cap R'$ and therefore assume that $R' = R$ so that both sections now come from $\mathcal{F}(R)$. Since they coincide in $\check{\mathcal{H}}(\mathcal{F})(R)$, then for all $V \in R$, they must also coincide in $\check{\mathcal{H}}(\mathcal{F})(V)$. Since

$$\check{\mathcal{H}}(\mathcal{F})(V) = \varinjlim_{S \in J(V)} \mathcal{F}(S),$$

there must exist $S_V \in J(V)$ such that they coincide in $\mathcal{F}(S_V)$ and therefore also on $\mathcal{F}(W)$ whenever $W \in S_V$. Since $R \in J(U)$, then we know from proposition 3.2.5 that $S := \bigcup_{V \in R} S_V$ is a refinement of R and both sections will coincide in $\mathcal{F}(S)$ since they do on any $\mathcal{F}(W)$ whenever $W \in S$. This is it.

Since limits are left exact, if $\mathcal{F} \rightarrowtail \mathcal{G}$ is a monomorphism and $\mathcal{G}$ is separated, then $\mathcal{F}$ is separated. In particular, if $\mathcal{F} \rightarrowtail \check{\mathcal{H}}(\mathcal{F})$ is a monomorphism, then $\mathcal{F}$ is separated. The converse assertion follows from the fact that filtered colimits of sets are exact.

Now, the proof that $\check{\mathcal{H}}(\mathcal{F})$ is a sheaf when $\mathcal{F}$ is separated is quite similar to the proof that $\check{\mathcal{H}}(\mathcal{F})$ is always separated. Thus, we assume that $\mathcal{F}$ is separated and we have to show that the maps (3.5) are always surjective. We give ourselves $s \in \check{\mathcal{H}}(\mathcal{F})(R)$. By definition, if $V \in R$, then the image of s in $\check{\mathcal{H}}(\mathcal{F})(V)$ comes from an element of $\mathcal{F}(S_V)$ for some $S_V \in J(V)$ that gives in turn some $s_W \in \mathcal{F}(W)$ whenever $W \in S_V$. We shall consider as before the refinement $S := \bigcup_{V \in R} S_V$ of R. Since $\mathcal{F}$ is separated, we know that the map $\mathcal{F} \rightarrowtail \check{\mathcal{H}}(\mathcal{F})$ is a monomorphism. It follows that, if $W' \subset W \in S$, then there exists a commutative diagram with injective horizontal maps

$$\begin{array}{ccc} & & \check{\mathcal{H}}(\mathcal{F})(U) \ni s \\ & & \downarrow \\ s_W \in \mathcal{F}(W) & \rightarrowtail & \check{\mathcal{H}}(\mathcal{F})(W) \\ \downarrow & & \downarrow \\ s_{W'} \in \mathcal{F}(W') & \rightarrowtail & \check{\mathcal{H}}(\mathcal{F})(W') \end{array}$$

showing that s_W is sent to $s_{W'}$. They define an element of $\mathcal{F}(S)$ which is sent to $s \in \check{\mathcal{H}}(\mathcal{F})(U)$.

The proof of the remaining assertions is now quite formal. First of all, by definition, if $\mathcal{F}$ is sheaf, then $\mathcal{F} \simeq \check{\mathcal{H}}(\mathcal{F})$. Conversely, since $\check{\mathcal{H}}(\mathcal{F})$ is always separated, such an isomorphism implies that $\mathcal{F}$ is separated in which case $\check{\mathcal{H}}(\mathcal{F})$ is a sheaf and the isomorphism shows that $\mathcal{F}$ also is a sheaf. ■

Theorem 3.2.21 The embedding $\mathcal{H} : \widetilde{X} \rightarrowtail \widehat{X}$ has an exact adjoint

$$\widehat{X} \to \widetilde{X}, \quad \mathcal{F} \mapsto \widetilde{\mathcal{F}} := \check{\mathcal{H}}(\check{\mathcal{H}}(\mathcal{F})).$$

Proof. The functor $\mathcal{H}$ is fully faithful and there exists a natural map $\mathcal{F} \to \check{\mathcal{H}}(\mathcal{F}) \to \check{\mathcal{H}}(\check{\mathcal{H}}(\mathcal{F})) =: \widetilde{\mathcal{F}}$ which is an isomorphism when $\mathcal{F}$ is a sheaf. We can then use exercise 1.4.20 and the fact that $\check{\mathcal{H}}$ is left exact. ■

The adjunction reads

$$\forall \mathcal{F} \in \widehat{X}, \mathcal{G} \in \widetilde{X}, \quad \mathrm{Hom}(\widetilde{\mathcal{F}}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

In more down to earth terms, if $\mathcal{F}$ is a presheaf, then there exists a sheaf $\widetilde{\mathcal{F}}$ and a morphism $\mathcal{F} \to \widetilde{\mathcal{F}}$, and if $\mathcal{G}$ is a sheaf, then any morphism $\mathcal{F} \to \mathcal{G}$ extends uniquely to $\widetilde{\mathcal{F}} \to \mathcal{G}$.

Proposition 3.2.22 If $\mathcal{F}$ is a presheaf on X and $x \in X$, then $\mathcal{F}(x) \simeq \check{\mathcal{H}}(\mathcal{F})(x)$ and (therefore) $\mathcal{F}(x) \simeq \widetilde{\mathcal{F}}(x)$.

Hint. Thus, we have to show that the map

$$\check{\mathcal{H}}(\mathcal{F})(x) = \varinjlim_{x \in U} \mathcal{F}(U) \to \varinjlim_{x \in U} \varinjlim_{R \in J(U)} \varprojlim_{V \in R} \mathcal{F}(V) \check{\mathcal{H}}(\mathcal{F})(x)$$

is bijective. Assume given two germs s_{1x} and s_{2x} defined on some open subsets U_1 and U_2 with same image in $\check{\mathcal{H}}(\mathcal{F})(x)$. Then, there exists U of X contained in U_1 and U_2 and $R \in J(U)$ such that for all $V \in R$, we have $s_{1|V} = s_{2|V}$. Since R is a covering of U, we may assume that $x \in V$ and therefore $s_{1x} = s_{2x}$. Assume conversely, given an open neighborhood U of x, a covering sieve R of U and a compatible family of sections s_V for $V \in R$. Then, there exists $W \in R$ such that $x \in W$ and one may consider the germ s_{Wx}. Its image is represented by the compatible family $s_{V \cap W}$ for the sieve $R \cap \mathrm{Open}(W)$ of W. ■

Exercise 3.2.23 Show that, if $\mathcal{F} \to \widetilde{\mathcal{F}}$ is an epimorphism of presheaves, then $\widetilde{\mathcal{F}} = \check{\mathcal{H}}(\mathcal{F})$.

Definition 3.2.24 The *constant sheaf* associated to a set E is the sheafification E_X of the constant presheaf $\widehat{E}_X : U \mapsto E$.

We have $\widehat{E}_X \neq E_X$ unless E has exactly one element. However, $E_X(x) = E$ for all $x \in X$.

Exercise 3.2.25 Show that, if E is a set, then $E_X \simeq \mathcal{C}_E^X$ when E is endowed with the discrete topology, and that there exists an adjunction

$$\mathrm{Hom}(E_X, \mathcal{F}) \simeq \mathrm{Hom}(E, \Gamma(X, \mathcal{F}))$$

if $\mathcal{F}$ is a sheaf.

Exercise 3.2.26 Show that if x is a point, then there exists an isomorphism of categories $\widetilde{x} \simeq \mathbf{Set}$ given by $\mathcal{F} \mapsto E := \mathcal{F}(x)$ (and we shall identify both categories).

Theorem 3.2.21 has the following immediate consequence:

Corollary 3.2.27
1. All limits and colimits exist in $\widetilde{X}$,
2. the embedding $\mathcal{H} : \widetilde{X} \rightarrowtail \widehat{X}$ preserves all limits,
3. sheafification $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ is exact and preserves all colimits.

Proof. The argument for existence of limits relies on theorem 1.4.17 which is hard to prove but our assertion may also be shown directly. It means that a presheaf $\mathcal{F}$ which is a limit of a diagram of sheaves is automatically a sheaf. This is just a commutation of limits. ∎

Thus, limits of sheaves are computed termwise. For colimits, one first compute termwise limit and then sheafify. It is important to emphasize for example the fact that an epimorphism of sheaves $\mathcal{F} \to \mathcal{G}$ is usually not an epimorphism of presheaves so that the maps $\mathcal{F}(U) \to \mathcal{G}(U)$ may not all be surjective (but see exercise 3.2.43 below).

Since $\check{\mathcal{H}}$ is left exact, it preserves all algebraic structures and everything above extends. For example, we have:

Corollary 3.2.28 The functor $\mathcal{H} : \mathcal{A}\text{-}\mathbf{Mod} \rightarrowtail \widehat{\mathcal{A}\text{-}\mathbf{Mod}}$ has an exact adjoint $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$. ∎

For the sake of completeness it is worth mentioning explicitly the adjunction:

$$\forall \mathcal{M} \in \widehat{\mathcal{A}\text{-}\mathbf{Mod}}, \mathcal{N} \in \mathcal{A}\text{-}\mathbf{Mod}, \quad \mathrm{Hom}_{\mathcal{A}}(\widetilde{\mathcal{M}}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}).$$

Exercise 3.2.29 Show that, if $\mathcal{M}$ is an $\mathcal{A}$-module, then there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_U, \mathcal{M}) \simeq \mathcal{M}(U)$$

where $\mathcal{A}_U$ is the sheafification of $\widehat{\mathcal{A}}_U$.

Let us also mention that we can identify an abelian sheaf with a $\mathbb{Z}_X$-module:

Exercise 3.2.30 Show that the forgetful functor $\mathbb{Z}_X\text{-}\mathbf{Mod} \to \widetilde{X}(\mathbf{Ab})$ is an isomorphism.

Proposition 3.2.31 $\mathcal{A}$-$\mathbf{Mod}$ is an abelian category (satisfying AB5 and AB3*).

Proof. Recall from exercise 3.1.12, that $\widehat{\mathcal{A}\text{-}\mathbf{Mod}}$ is abelian. It then follows from exercise 2.1.54 and corollary 3.2.28 that $\mathcal{A}$-$\mathbf{Mod}$ is abelian. ∎

Exercise 3.2.32 Show that the forgetful functor $\mathcal{A}\text{-}\mathbf{Mod} \to \widetilde{X}$ has an adjoint $\mathcal{F} \mapsto \mathcal{A} \cdot \mathcal{F}$.

Definition 3.2.33 An *étale space* over X is a topological space E endowed with a local homeomorphism $E \to X$.

They form a full subcategory $\mathrm{Et}(X) \subset \mathbf{Top}_{/X}$. Note that any morphism of étale spaces is itself a local homeomorphism. We shall soon need the following technical property:

Exercise 3.2.34 Show that, if $\mathcal{F}$ is a presheaf on X, $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, then $\{x \in U \cap V, s_x = t_x\}$ is open in X.

If $\mathcal{F}$ is a presheaf on X, then the set $E(\mathcal{F}) := \coprod_{x \in X} \mathcal{F}(x)$ is endowed with the topology generated by all $\widetilde{s}(U)$ for $s \in \mathcal{F}(U)$ with

$$\widetilde{s} : U \to \mathcal{F}(x) \subset E(\mathcal{F}), \quad x \mapsto s_x.$$

Then there exists an alternative definition for the associated sheaf:

Exercise 3.2.35 Show that, if $\mathcal{F}$ is a presheaf on X, then

1. $E(\mathcal{F})$ is an étale space over X,
2. there exists a morphisms of presheaves $\alpha_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}_{E(\mathcal{F})/X}, s \mapsto \widetilde{s}$,
3. it induces an isomorphism $\widetilde{\mathcal{F}} \simeq \mathcal{C}_{E(\mathcal{F})/X}$.

Theorem 3.2.36 There exists an adjunction between

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \mathbf{Top}_{/X} \\ \mathcal{F} & \longmapsto & E(\mathcal{F}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Top}_{/X} & \longrightarrow & \widehat{X} \\ Y & \longmapsto & \mathcal{C}_{Y/X}. \end{array}$$

It induces an equivalence $\mathrm{Et}(X) \simeq \widetilde{X}$.

Adjunction means that there exists a natural bijection

$$\mathcal{C}_{/X}(E(\mathcal{F}), Y) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{C}_{Y/X}).$$

Proof. We only outline the strategy: according to lemma 1.4.8, we have to show that there exists natural transformations $\alpha_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}_{E(\mathcal{F})/X}$ and $\beta_Y : E(\mathcal{C}_{Y/X}) \to Y$ making commutative

$$\circlearrowright\ E(\mathcal{F}) \underset{\beta_{E(\mathcal{F})}}{\overset{E(\alpha_{\mathcal{F}})}{\rightleftarrows}} E(\mathcal{C}_{E(\mathcal{F})/X}) \ \text{ and } \ \mathcal{C}_{E(\mathcal{C}_{Y/X})/X} \underset{\alpha_{\mathcal{C}_{Y/X}}}{\overset{\mathcal{C}_{\beta_{Y/X}}}{\rightleftarrows}} \mathcal{C}_{Y/X}\ \circlearrowleft$$

■

3.2.3 Properties

Let X be a topological space and $\mathcal{A}$ a sheaf of rings on X. We shall focus on sheaves of sets. There also exists an algebraic analog.

A morphism of sheaves which is at the same time a monomorphism and an epimorphism is automatically an isomorphism:

Lemma 3.2.37 The category $\widetilde{X}$ is balanced.

Proof. Given a morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$, we can make the cocartesian diagram of presheaves

$$\begin{array}{ccc} \mathcal{H}(\mathcal{F}) & \xrightarrow{\mathcal{H}(\alpha)} & \mathcal{H}(\mathcal{G}) \\ \downarrow{\scriptstyle \mathcal{H}(\alpha)} & & \downarrow \\ \mathcal{H}(\mathcal{G}) & \longrightarrow & \mathcal{Q}. \end{array}$$

If α is a monomorphism then $\mathcal{H}(\alpha)$ also is a monomorphism and it follows that the diagram of presheaves is also cartesian (use exercises 3.1.11 and 1.3.30). If we sheafify, we obtain a diagram which is both cartesian and cocartesian

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \\ {\scriptstyle\alpha}\downarrow & \ulcorner \quad \lrcorner & \downarrow \\ \mathcal{G} & \longrightarrow & \widetilde{\mathcal{Q}}. \end{array}$$

If α is also an epimorphism, then by definition, the fact that the diagram is cocartesian implies $\mathcal{G} \simeq \widetilde{\mathcal{Q}}$. But then, the fact that the diagram is cartesian implies $\alpha : \mathcal{F} \simeq \mathcal{G}$. ■

The next result could have been stated earlier but deserves a special mention:

Proposition 3.2.38 If $x \in X$, then the functor

$$\widetilde{X} \to \mathbf{Set}, \quad \mathcal{F} \to \mathcal{F}(x)$$

is exact and preserves all colimits.

Proof. Preservation of finite limits is automatic by decomposition $\widetilde{X} \rightarrowtail \widehat{X} \to \mathbf{Set}$ and preservation of colimits results from adjunction

$$\mathrm{Hom}(\mathcal{F}(x), E) \simeq \mathrm{Hom}(\mathcal{F}, E_x^!).$$ ■

Proposition 3.2.39 The functor

$$\widetilde{X} \to \mathbf{Set}^X, \quad \mathcal{F} \to (\mathcal{F}(x))_{x \in X} \tag{3.6}$$

is faithful and conservative. It preserves (and reflects) all finite limits and all colimits.

Proof. Faithfulness follows immediately from proposition 3.2.19. Since a faithful functor reflects both monomorphisms and epimorphisms (exercise 1.3.52, part 3) and the category $\widetilde{X}$ is balanced (lemma 3.2.37), it formally follows (exercise 1.3.53) that our functor is conservative. On the other hand, preservation of finite limits and colimits follow from proposition 3.2.38. Reflection is then an automatic consequence by exercise 1.3.52 again, but part 2 this time. ■

Exercise 3.2.40 Show that the functor $\mathcal{F} \to (\mathcal{F}(x))_{x \in X}$ is adjoint to the functor $(E_x)_{x \in X} \mapsto \prod_{x \in X} E_x^!$.

Exercise 3.2.41 Show that the functor $\mathcal{F} \to (\mathcal{F}(x))_{x \in X}$ is an equivalence when X is discrete.

Proposition 3.2.42
1. Any monomorphism/epimorphism of sheaves is regular,
2. any morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ is strict,
3. colimits of sheaves are stable under pullback and filtered colimits of sheaves

are exact.

Proof. Since the functor of 3.6 preserves and reflects all finite limits and all colimits, it is sufficient to check the property in $\mathbf{Set}^X$. This may be checked termwise and reduces to $\mathbf{Set}$ where we know that it holds. ■

Exercise 3.2.43 Show that the following are equivalent:

1. $\alpha : \mathcal{F} \to \mathcal{G}$ is an epimorphism,
2. for all U open in X and all $t \in \mathcal{G}(U)$, there exists an $R \in J(U)$ and $s_v \in \mathcal{F}(V)$ for all $V \in R$ such that $t_{|V} = \alpha_V(s_V)$,
3. for all U open in X and all $t \in \mathcal{G}(U)$, there exists an open covering $U = \bigcup_{i\in I} U_i$ and for all $i \in I$, $s_i \in \mathcal{F}(U_i)$ such that $t_{|U_i} = \alpha_{U_i}(s_i)$.

Hint. One can denote by $\mathcal{I}$ the image of $\mathcal{H}(\alpha)$ and notice that α is an epimorphism if and only if $\check{\mathcal{H}}(\mathcal{I}) \simeq \mathcal{G}$. It is then a matter of writing down this condition explicitly. Alternatively, one can use the fact that α is an epimorphism if and only if $\alpha_x : \mathcal{F}(x) \to \mathcal{G}(x)$ is surjective for all $x \in X$. ■

Exercise 3.2.44 Show that

1. if $\mathcal{F}$ is a presheaf on X and U is an open subset, then $(\widetilde{\mathcal{F}})_{|U} \simeq \widetilde{\mathcal{F}_{|U}}$,
2. $\mathcal{F}_{|U}$ is automatically a sheaf when $\mathcal{F}$ is a sheaf,
3. the corresponding functor $\widetilde{X} \to \widetilde{U}$ has both an adjoint and a coadjoint.

Proposition 3.2.45 If $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, then $\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})$ is a sheaf and

$$\mathcal{H}\mathrm{om}(\widetilde{\mathcal{F}},\mathcal{G}) \simeq \mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})$$

Proof. We give ourselves a compatible family of morphisms $\mathcal{F}_{|V} \to \mathcal{G}_{|V}$, or equivalently $s_V : \mathcal{F}(V) \to \mathcal{G}(V)$, for $V \in R$, $R \in J(U)$ and U open in X. If U' is an open subset of U, then $R' := \{V \in R, V \subset U'\} \in J(U')$. If $x \in \mathcal{F}(U')$, then we have a compatible family $s_V(x_{|V}) \in \mathcal{G}(V)$ for $V \in R'$. Since $\mathcal{G}$ is a sheaf, there exists a unique $s(x) \in \mathcal{G}(U')$ such that $s(x)_{|V} = s_V(x_{|V})$. This defines a compatible family of maps $s_{U'} : \mathcal{F}(U') \to \mathcal{G}(U')$ for $U' \subset U$ or equivalently a morphism $\mathcal{F}_{|U} \to \mathcal{G}_{|U}$ which is the unique solution to our problem. The second assertion is then an immediate consequence of exercise 3.2.44. ■

Currying (proposition 3.1.21) applies in particular to sheaves so that

$$\mathcal{H}\mathrm{om}(\mathcal{F} \times \mathcal{G},\mathcal{H}) \simeq \mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{H}\mathrm{om}(\mathcal{G},\mathcal{H}))$$

when $\mathcal{F},\mathcal{G},\mathcal{H} \in \widetilde{X}$. The algebraic analog however needs some care:

Definition 3.2.46 The *tensor product* of a right $\mathcal{A}$-modules $\mathcal{M}$ with a left $\mathcal{A}$-modules $\mathcal{N}$ is the sheafification $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ of $\mathcal{M}\widehat{\otimes}_{\mathcal{A}}\mathcal{N}$.

Be careful that $(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N})(U) \neq \mathcal{M}(U)\otimes_{\mathcal{A}}\mathcal{N}(U)$ in general for U open in X but $(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N})(x) = \mathcal{M}(x) \otimes_{\mathcal{A}(x)} \mathcal{N}(x)$ for $x \in X$.

Proposition 3.2.47 If $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ are three $\mathcal{A}$-modules on X, then there exists a natural isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N},\mathcal{P}) \simeq \mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{M},\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{N},\mathcal{P})).$$

Proof. This is a formal consequence of sheafification (corollary 3.2.28) and the presheaf analog (proposition 3.1.25). ■

Exercise 3.2.48 Show that if $\mathcal{A} \to \mathcal{B}$ is a morphism of sheaves of rings and $\mathcal{M}$ (resp. $\mathcal{N}$) is an $\mathcal{A}$-module (resp. $\mathcal{B}$-module), then there exists a natural isomorphism

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N}).$$

Actually, there also exists a coadjoint to the forgetful functor, that in consequence, preserves all limits and all colimits (and this holds also at the presheaf level).

Finally, if $\mathcal{M}$ is an $\mathcal{A}$-module, we shall denote by $\mathrm{T}(\mathcal{M})$ (resp. $\mathrm{S}(\mathcal{M})$, resp. $\Lambda(\mathcal{M})$) the sheafification of $\widehat{\mathrm{T}}(\mathcal{M})$ (resp. $\widehat{\mathrm{S}}(\mathcal{M})$, resp. $\widehat{\Lambda}(\mathcal{M})$). We shall have $\mathrm{T}(\mathcal{M})(x) = \mathrm{T}(\mathcal{M}(x))$ (resp. $\mathrm{S}(\mathcal{M})(x) = \mathrm{S}(\mathcal{M}(x))$, resp. $\Lambda(\mathcal{M})(x) = \Lambda(\mathcal{M}(x))$).

3.2.4 Direct and inverse image of sheaves

We fix a continuous map $f : X \to Y$ and a sheaf of rings $\mathcal{A}$ (resp. $\mathcal{B}$) on X (resp. Y).

Exercise 3.2.49 Show that

1. if S is a sieve on Y, then
$$f^{-1}(S) := \{V \in \mathrm{Open}(X), \exists W \in S, f(V) \subset W\}$$
is a sieve on X,
2. if S is a covering sieve of U, then $f^{-1}(S)$ is a covering sieve of $f^{-1}(U)$,
3. $\{f^{-1}(W), W \in S\}$ is *final* in $f^{-1}(S)$.

Proposition 3.2.50 The functor $\widehat{f}_* : \widehat{X} \to \widehat{Y}$ induces a functor $f_* : \widetilde{X} \to \widetilde{Y}$ that has an exact adjoint f^{-1}.

Proof. If $\mathcal{F}$ is a sheaf on X and S is a covering sieve of an open subset U of Y, then

$$\varprojlim_{W \in S} (\widehat{f}_*\mathcal{F})(W) = \varprojlim_{W \in S} \mathcal{F}(f^{-1}(W)) = \varprojlim_{V \in f^{-1}(S)} \mathcal{F}(V) = \mathcal{F}(f^{-1}(U)) = (\widehat{f}_*\mathcal{F})(U)$$

(the middle equality relies on proposition 1.3.9 but may also be checked by hand). This proves the existence of f_* and we define $f^{-1}\mathcal{G}$ as the sheafification of $\widehat{f}^{-1}\mathcal{G}$ for a sheaf $\mathcal{G}$ on Y. Then, if $\mathcal{F}$ is a sheaf on X, we have

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}(\widehat{f}^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G}, \widehat{f}_*\mathcal{F}) = \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

The functor f^{-1} is right exact as an adjoint and left exact as a composition of left exact functors ($\mathcal{H}$, $\widehat{f}^{-1}$ and sheafification). ∎

Definition 3.2.51 The functors f^{-1} and f_* are called *inverse* and *direct image* respectively.

It is worth mentioning the following commutative diagrams

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f_*} & \widetilde{Y} \\ {\scriptstyle\sim}\uparrow\downarrow & & {\scriptstyle\sim}\uparrow\downarrow \\ \widehat{X} & \xrightarrow{\widehat{f}_*} & \widehat{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{Y} & \xrightarrow{f^{-1}} & \widetilde{X} \\ {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow \\ \widehat{Y} & \xrightarrow{\widehat{f}^{-1}} & \widehat{X}. \end{array}$$

As a consequence, f^{-1} preserves all finite limits and all colimits and $\widehat{f}_*$ preserves all limits. Also, this is functorial in the sense that

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

if $g : Y \to Z$ is another continuous map.

Since our functors f^{-1} and f_* preserve finite products, they preserve all algebraic structures and there exists

$$f^{-1} : \mathcal{B}\text{-}\mathrm{Mod} \to f^{-1}(\mathcal{B})\text{-}\mathrm{Mod} \quad \text{and} \quad f_* : \mathcal{A}\text{-}\mathrm{Mod} \to f_*\mathcal{A}\text{-}\mathrm{Mod}.$$

Exercise 3.2.52 Show that there exists an adjunction

$$\mathrm{Hom}_{f^{-1}\mathcal{B}}(f^{-1}\mathcal{N}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, f_*\mathcal{M})$$

with exact adjoint.

Up to the end of the section, we focus on presheaves of sets.

Exercise 3.2.53 Show that

1. if $i_x : x \rightarrowtail X$ denotes the inclusion of a point, then $i_{x*}E = E^!(x)$ and $i(x)^{-1}\mathcal{F} = \mathcal{F}(x)$,
2. if $p : X \twoheadrightarrow x$ denotes the projection onto a point, then $p_*\mathcal{F} = \Gamma(X, \mathcal{F})$ and $p^{-1}E = E_X$.

Exercise 3.2.54 Show that the inverse image for identity $X^{\mathrm{disc}} \to X$ identifies with (3.6).

Exercise 3.2.55 Show that, if $i : Y \rightarrowtail X$ is the inclusion of a subset (with the induced topology), then i_* is fully faithful.

Exercise 3.2.56 Show that

1. if $j : U \rightarrowtail X$ is the inclusion of a open subset and $\mathcal{F}$ is a sheaf of sets on X, then $j^{-1}\mathcal{F} = \mathcal{F}_{|U}$,
2. j^{-1} has an adjoint $j_!$ given by $(j_!\mathcal{G})(V) = \mathcal{G}(V)$ if $V \subset U$ and $\emptyset$ otherwise[a],
3. $j_!$ is fully faithful and preserves all limits and all colimits.

[a]For sheaves of modules, we must replace $\emptyset$ with $\{0\}$.

Exercise 3.2.57 Show that

1. if $f : X \to Y$ is a continuous map, then $(f^{-1}\mathcal{G})(x) \simeq \mathcal{G}_{f(x)}$,
2. if $i : Y \rightarrowtail X$ is an inclusion map, then $(i_*\mathcal{F})(x) \simeq \mathcal{F}(x)$,
3. If $j : U \rightarrowtail X$ is the inclusion of an open subset, then $(j_!\mathcal{F})(x) \simeq \mathcal{F}(x)$ if $x \in U$ and $\emptyset$ otherwise[a].

[a]Or $\{0\}$ in the case of presheaves of modules.

3.2.5 Ringed spaces

Definition 3.2.58 A *ringed space* is a pair $(X, \mathcal{O}_X)$ made of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.

3.3 Sheaf cohomology

3.3.1 Definition/Examples

We let X be a topological space, U an open subset of X, $\mathcal{A}$ a sheaf of rings on X and $\mathcal{M}^\bullet$ a complex of $\mathcal{A}$-modules. Note that we can always choose $\mathcal{A} = \mathbb{Z}_X$ in which case an $\mathcal{A}$-module is simply an abelian sheaf on X.

Theorem 3.3.1 The category of $\mathcal{A}$-modules has enough injectives.

Proof. Proposition 3.2.39 (or more precisely its algebraic version) shows that the stalks functor

$$\mathcal{A}\text{-}\mathrm{Mod} \to \prod_{x \in X} \mathcal{A}(x)\text{-}\mathrm{Mod}$$

is faithful exact with coadjoint. We may then use exercises 2.2.54, 1.5.16 and proposition 2.2.55. ■

Be careful however that there does not exist enough projectives in general (and $\mathcal{A}$ itself need not be projective).

Exercise 3.3.2 Show that if $\mathcal{I}$ is an injective $\mathcal{A}$-module, then $\mathcal{I}_{|U}$ also is injective.

Recall now that there exists a left exact functor

$$\mathcal{A}\text{-}\mathbf{Mod} \to \mathcal{A}(U)\text{-}\mathbf{Mod}, \quad \mathcal{M} \mapsto \Gamma(U, \mathcal{M}) := \mathcal{M}(U).$$

Definition 3.3.3 The *nth cohomology* of $\mathcal{M}^\bullet$ on U is

$$\mathrm{H}^n(U, \mathcal{M}^\bullet) := \mathrm{R}^n\Gamma(U, \mathcal{M}^\bullet).$$

Of course this applies to a single $\mathcal{A}$-module $\mathcal{M}$ seen as a complex concentrated in degree 0 in which case, we have $\mathrm{H}^0(U, \mathcal{M}) = \Gamma(U, \mathcal{M})$.

Examples We give (without details at this point) a list of examples that show that sheaf cohomology agrees with any other theory:

1. If X is a paracompact Hausdorff space (and unconditionally when $n = 1$) and $\mathcal{M}$ a sheaf of $\mathcal{A}$-moudules on X, then[3]

$$\check{\mathrm{H}}^n(X, \mathcal{M}) \simeq \mathrm{H}^n(X, \mathcal{M}).$$

2. If X is a locally contractible topological space and M an abelian group, then

$$\mathrm{H}^n(X, M_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X, M).$$

3. If X is a smooth manifold, then

$$\mathrm{H}^n_{\mathrm{sing}}(X, \mathbb{R}) \simeq \mathrm{H}^n(X, \mathbb{R}_X) \simeq \mathrm{H}^n(X, \Omega^\bullet_{X/\mathbb{R}}) \simeq \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{R}).$$

4. If a *smooth algebraic variety* over $\mathbb{C}$, then there exists an *algebraic de Rham complex* $\Omega^\bullet_{X/\mathbb{C}}$ on X and

$$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}) := \mathrm{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \mathbb{C}).$$

Cohomology can be computed directly on U:

Proposition 3.3.4 $\mathrm{H}^n(U, \mathcal{M}^\bullet) = \mathrm{H}^n(U, \mathcal{M}^\bullet_{|U})$.

Proof. Follows from the fact that restriction to U is an exact functor that preserves injectives (and corollary 2.3.29). ■

Cohomology can be computed at the abelian group level: if we denote with a superscritpt “ab” the underlying abelian group or the underlying complex of abelian sheaves, then

[3] One can define *Čech cohomology* as $\check{\mathrm{H}}^n(X, \mathcal{M}^\bullet) := (\mathrm{R}^n\check{\mathcal{H}}\mathcal{M}^\bullet)(X)$.

Proposition 3.3.5 $\mathrm{H}^n(U, \mathcal{M}^\bullet)^{\mathrm{ab}} = \mathrm{H}^n(U, \mathcal{M}^{\bullet,\mathrm{ab}})$.

Proof. There exists a commutative diagram

$$\begin{array}{ccc} \mathcal{A}\text{-}\mathbf{Mod} & \longrightarrow & \Gamma(X,\mathcal{A})\text{-}\mathbf{Mod} \\ \downarrow & & \downarrow \\ \widetilde{X}(\mathbf{Ab}) & \longrightarrow & \mathbf{Ab} \end{array}$$

with exact vertical functors that preserve injectives. We may then apply corollary 2.3.29 (and exercise 2.3.20). ■

Exercise 3.3.6 Show that $\mathrm{H}^n(X, \mathcal{M}^\bullet) \simeq \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{M}^\bullet)$ for all $n \in \mathbb{Z}$.

Exercise 3.3.7 Show that, if $X = \coprod_{i\in I} X_i$, then $\mathrm{H}^n(X, \mathcal{M}^\bullet) \simeq \prod_{i\in I} \mathrm{H}^n(X_i, \mathcal{M}^\bullet_{|X_i})$.

3.3.2 Acyclicity

We let X be a topological space.

Definition 3.3.8 An abelian sheaf $\mathcal{M}$ is said to be *acyclic* if $\mathrm{H}^n(X, \mathcal{M}) = 0$ for all $n > 0$.

Of course it means that $\mathcal{M}$ is acyclic with respect to the functor $\Gamma(X, -)$. In particular (the abelian sheaf underlying) an injective $\mathcal{A}$-module is automatically acyclic. We shall provide various other examples in this section.

Definition 3.3.9 A presheaf $\mathcal{F}$ on X is said to be *flabby* if $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective whenever U is open in X.

Equivalently, if we are given open subsets $V \subset U \subset X$, then $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Clearly, if $\mathcal{F}$ is flabby and $U \subset X$ is an open subset, then $\mathcal{F}_{|U}$ also is flabby.

Example
1. If X is discrete, then any sheaf is flabby.
2. An injective $\mathcal{A}$-module $\mathcal{I}$ is flabby (proposition 3.3.11 below).
3. A presheaf of singular cochains $\widehat{\mathrm{C}}^n_M : U \mapsto \mathrm{C}^n(U, M)$ is flabby.
4. A constant sheaf or a vector bundle is not flabby (in general).

Exercise 3.3.10 Show that if M is an abelian group, then the presheaf of singular cochains $\widehat{\mathrm{C}}^n_M$ is flabby.

Proposition 3.3.11 If $\mathcal{A}$ is a sheaf of rings, then any injective $\mathcal{A}$-module $\mathcal{I}$ is flabby.

Proof. Let U be an open subset of X. Since $\mathcal{A}_{Ux} = \mathcal{A}(x)$ when $x \in U$ and $\{1\}$ otherwise, there is a monomorphism $\mathcal{A}_U \hookrightarrow A$. It follows that the map

$$\mathcal{I}(X) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_U, \mathcal{I}) \simeq \mathcal{I}(U)$$

is surjective. ■

Lemma 3.3.12 If $0 \to \mathcal{M}' \xrightarrow{i} \mathcal{M} \xrightarrow{\pi} \mathcal{M}'' \to 0$ is an exact sequence of abelian sheaves with $\mathcal{M}'$ flabby, then the sequence

$$0 \to \mathcal{M}'(X) \to \mathcal{M}(X) \to \mathcal{M}''(X) \to 0$$

also is exact.

Proof. The sequence is automatically left exact. Let $s'' \in \mathcal{M}''(X)$. By Zorn's lemma, there exists an open subset V of X and $s \in \mathcal{M}(V)$ maximal for $\pi(s) = s''_{|V}$. Since π is an epimorphism, if $x \in X$, there exists an open neighborhood W of x and $t \in \mathcal{M}(W)$ such that $\pi(t) = s''_{|W}$. Since the sequence is left exact, there exists $\tau' \in \mathcal{M}'(V \cap W)$ such that $i(\tau') = (s-t)_{|V\cap W}$. Since $\mathcal{M}'$ is flabby, we may even find $\tau' \in \mathcal{M}'(W)$ such that $i(\tau_{|V\cap W}) = (s-t)_{|V\cap W}$. We can then replace t with $t + i(\tau')$ and glue s and t along $V \cap W$. The maximality of V then implies that $W \subset V$ so that $x \in V$ and finally $V = X$. ■

Exercise 3.3.13 Show that, if $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of abelian sheaves with $\mathcal{M}'$ flabby, then $\mathcal{M}$ is flabby if and only if $\mathcal{M}''$ is.

Proposition 3.3.14 A flabby abelian sheaf is acyclic.

Proof. We can embed our flabby abelian sheaf $\mathcal{M}$ into an injective abelian sheaf $\mathcal{I}$ and denote by $\mathcal{N}$ the quotient. Since both $\mathcal{M}$ and $\mathcal{I}$ (thanks to lemma 3.3.11) are flabby, then exercise 3.3.13 shows that $\mathcal{N}$ is flabby and we can proceed by induction on n. Since $\mathcal{I}$ is acyclic, the long exact sequence in cohomology provides an isomorphism $\mathrm{H}^n(X, \mathcal{N}) \simeq \mathrm{H}^{n+1}(X, \mathcal{M})$ for all $n > 1$ but also $\mathrm{H}^1(X, \mathcal{M}) = 0$ thanks to lemma 3.3.12. ■

The notion of a flabby sheaf is not sufficient to understand vector bundles in analytic geometry and it is necessary to introduce a more subtle variant. We assume up the end of this section that X is *paracompact Hausdorff*.

Definition 3.3.15 The *set of sections* of a sheaf $\mathcal{F}$ on a closed subset Z of X is

$$\mathcal{F}(Z) := \varinjlim_{Z \subset U} \mathcal{F}(U)$$

when U runs among all open neighborhoods of Z in X.

The above definition makes sense on any topological space but needs to be refined in general. We send the reader to section 3.2.4 for the details but, if Z is a subset of X and we set $(i_*\mathcal{F})(U) := \mathcal{F}(U \cap Z)$, we then obtain a functor $i_* : \widetilde{Z} \to \widetilde{X}$. This functor has an adjoint $\mathcal{F} \mapsto \mathcal{F}_{|Z}$ and $\mathcal{F}(Z) := \mathcal{F}_{|Z}(Z)$. It is not a trivial matter to check that both definitions coincide in the case X is paracompact Hausdorff (proposition 2.5.1 in [KS90]). Let us notice however that $\mathcal{F} \mapsto \mathcal{F}(Z)$ is a left exact functor.

Definition 3.3.16 A sheaf $\mathcal{F}$ is said to be *soft* if $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective whenever Z is closed in X.

Example 1. A flasque sheaf is soft.
2. A vector bundle on a $\mathcal{C}^k$-differentiable manifold is soft when $k \leq \infty$ (see proposition 3.3.19 below).
3. An analytic/holomorphic vector bundle is usually not soft.

Exercise 3.3.17 Show that $\mathcal{F}$ is soft if and only if whenever U is an open subset of X, $s \in \mathcal{F}(U)$ and Z is a closed subset of X with $Z \subset U$, then there exists $s' \in \mathcal{F}(X)$ and U' open in X with $Z \subset U' \subset U$ such that $s'_{|U'} = s_{|U'}$

Exercise 3.3.18 Show that a flabby sheaf (on X paracompact Hausdorff) is soft.

Proposition 3.3.19 If X is a $\mathcal{C}^k$-differentiable manifold with $k \leq \infty$, then any $\mathcal{O}_X$-module is soft.

Proof. By definition, any section of our $\mathcal{O}_X$-module $\mathcal{M}$ on a closed subset Z comes from some $s \in \mathcal{M}(U)$ where U is an open neighborhood of Z. Since X is normal, there exists a sequence of inclusions of open and closed subsets $Z \subset U' \subset Z' \subset U'' \subset Z'' \subset U$ and we set $V := X \setminus Z''$. By Urysohn's lemma, there exists $f \in \mathcal{O}(X)$ such that $f_{|Z'} = 1$ and $f_{|X \setminus U''} = 0$. Since $(fs)_{|U \cap V} = 0$, there exists a unique $s' \in \mathcal{O}(X)$ such that $s'_{|U} = (fs)_{|U}$ and $s'_{|V} = 0$. Then, $s'_{|U'} = s_{|U}$. ■

Exercise 3.3.20 Show that, if $0 \to \mathcal{M}' \xrightarrow{i} \mathcal{M} \xrightarrow{\pi} \mathcal{M}'' \to 0$ is an exact sequence of abelian sheaves with $\mathcal{M}'$ soft, then the sequence

$$0 \to \mathcal{M}'(X) \to \mathcal{M}(X) \to \mathcal{M}''(X) \to 0$$

also is exact. Show that $\mathcal{M}'$ is soft if and only if $\mathcal{M}''$ is.

Exercise 3.3.21 Show that a soft sheaf is acyclic.

3.3.3 Comparison theorems

We still denote by X a topological space.

Recall that, if $\mathcal{F}$ is a presheaf on X, then

$$\mathcal{F}(R) := \varprojlim_{V \in R} \mathcal{F}(V)$$

if R is a sieve of X,

$$\check{\mathcal{H}}(\mathcal{F})(U) := \varinjlim_{R \in J(U)} \mathcal{F}(R)$$

if U is an open subset of X and $\widetilde{\mathcal{F}} := \check{\mathcal{H}}(\check{\mathcal{H}}(\mathcal{F}))$.

Lemma 3.3.22 Assume X paracompact Hausdorff. Let $\mathcal{F}$ be a presheaf on X. If the maps $\mathcal{F}(X) \to \mathcal{F}(R)$ are surjective when $R \in J(X)$, then both maps $\mathcal{F}(X) \to \check{\mathcal{H}}(\mathcal{F})(X)$ and $\check{\mathcal{H}}(\mathcal{F})(X) \to \widetilde{\mathcal{F}}(X)$ are also surjective.

Proof. The case of the first map results from exactness of filtered colimits. Assume now given $s \in \widetilde{\mathcal{F}}(X)$. It comes from some $s_R \in \check{\mathcal{H}}(\mathcal{F})(R)$ for some $R \in J(X)$ that gives rise to $s_V \in \check{\mathcal{H}}(\mathcal{F})(V)$ when $V \in R$. This s_V in turns comes from some $s_{R_V} \in \mathcal{F}(R_V)$ with $R_V \in J(V)$ giving rise to $s_W \in \mathcal{F}(W)$ when $W \in S_V$:

$$\begin{array}{ccccc}
\widetilde{\mathcal{F}}(X) = \varinjlim \check{\mathcal{H}}(\mathcal{F})(R) & & & & \\
\uparrow & & & & \\
\check{\mathcal{H}}(\mathcal{F})(R) & \twoheadrightarrow & \check{\mathcal{H}}(\mathcal{F})(V) = \varinjlim \mathcal{F}(R_V) & & \\
& & \uparrow & & \\
& & \mathcal{F})(R_V) & \twoheadrightarrow & \mathcal{F}(W)
\end{array}$$

By construction, the image of s_W in $\widetilde{\mathcal{F}}(W)$ is $s_{|W}$. After replacing R with $\bigcup_{V\in R} R_V$, we may therefore assume that, for all $V \in R$, there exists $s_V \in \mathcal{F}(V)$ with image $s_{|V} \in \widetilde{\mathcal{F}}(V)$ and we fix such an s_V. Since X is paracompact Hausdorff, we can further assume that R is generated by set E of open subsets of X such that, for all $x \in X$, there exists an open neighborhood U_x of x such that $U_x \subset V$ for finitely many $V \in E$ and $U_x \cap V = \emptyset$ otherwise. Moreover, since the stalk $s_{Vx} = s_x$ does not depend on $V \ni x$ and this only happens a finite number of times, we may assume after possibly shrinking U_x that $s_{U_x} := s_{V|U_x}$ only depends on x and not on V. Now, if $x, y \in X$ and $U_x \cap U_y \neq \emptyset$, then there exists $V \in E$ such that both $U_x, U_y \subset V$. Thus we see that $s_{U_x|U_x\cap U_y} = s_{V|U_x\cap U_y} = s_{U_y|U_x\cap U_y}$. After replacing R by the sieve generated by $\{U_x\}_{x\in X}$, we may therefore assume that s comes from $\mathcal{F}(R)$ and obtain expected surjectivity. ■

Definition 3.3.23 An *epipresheaf* is a presheaf $\mathcal{F}$ such that the canonical map $\mathcal{F}(U) \to \mathcal{F}(R)$ is surjective whenever R is a covering sieve of an open subset U.

Note that this is the dual notion to a separated sheaf and that a sheaf is the same thing as a separated epipresheaf.

Lemma 3.3.24 For all $n \in \mathbb{N}$, the presheaf $\widehat{\mathrm{C}}^n_M$ of n-singular cochains with values in an abelian group M is an epipresheaf.

Proof. Recall that if R is a covering sieve of an open subset U of X, then we have

$$S_n(R) \subset S_n(U) \quad \text{with} \quad S_n(R) := \bigcup_{V\in R} S_n(V) = \varinjlim_{V\in R} S_n(V).$$

Now, we have

$$\begin{aligned}
\widehat{\mathrm{C}}^n_M(R) &= \varprojlim_{V\in R} \widehat{\mathrm{C}}^n_M(V) \\
&= \varprojlim_{V\in R} \mathrm{Hom}(S_n(V), M) \\
&= \mathrm{Hom}\left(\varinjlim_{V\in R} S_n(V), M\right) \\
&= \mathrm{Hom}(S_n(R), M),
\end{aligned}$$

and therefore (since a set is aways projective) a surjection

$$\widehat{\mathrm{C}}^n_M(U) := \mathrm{Hom}(S_n(U), M) \twoheadrightarrow \mathrm{Hom}(S_n(R), M) = \widehat{\mathrm{C}}^n_M(R). \qquad \blacksquare$$

Proposition 3.3.25 If X is hereditary paracompact[a] Hausdorff and $\mathcal{F}$ is an epipresheaf on X, then

1. $\mathcal{F} \to \widetilde{\mathcal{F}}$ is an epimorphism of presheaves,
2. if $\mathcal{F}$ is flabby, then $\widetilde{\mathcal{F}}$ also is flabby.

[a]Any (open) subset is paracompact.

Proof. The second assertion is an immediate consequence of the first that is obtained by applying lemma 3.3.22 to any open subset. ■

If we denote by $\mathrm{C}^\bullet_M$ the sheaf associated to $\widehat{\mathrm{C}}^\bullet_M$, then the previous proposition implies that $\mathrm{C}^\bullet_M$ is flabby when X is hereditary paracompact Hausdorff. Under this hypothesis, we also have the following:

Proposition 3.3.26 If X is hereditarily paracompact Hausdorff, then the morphism $\widehat{\mathrm{C}}^\bullet_M(X) \twoheadrightarrow \mathrm{C}^\bullet_M(X)$ is a quasi-isomorphism.

Proof. It follows from exercise 3.2.23 and proposition 3.3.25 that

$$\mathrm{C}^n(X, M) = \varinjlim_{R \in J(X)} C^n(R, M)$$

when $n \in \mathbb{N}$. Since directed colimits are exact, our assertion then follows from the small simplices theorem (more precisely exercise 2.2.61). ■

The augmentation maps $M \to \mathrm{C}^\bullet(U, M)$ for U open in X provide a morphism of presheaves $\widehat{M_X} \to \widehat{\mathrm{C}}^\bullet_M$ that sheafifies to $M_X \to \mathrm{C}^\bullet_M$.

Lemma 3.3.27 If X is locally contractible, then $M_X \to \mathrm{C}^\bullet_M$ is a resolution.

Proof. If $x \in X$, we have

$$\mathrm{C}^\bullet_M(x) = \varinjlim_{x \in U} C^\bullet(U, M)$$

when U runs through contractible open neighborhoods of M and we can rely on exercise 2.2.63. ■

We shall only prove the following theorem under the unnecessary condition that X is *hereditary paracompact Hausdorff*:

Theorem 3.3.28 If X is a locally contractible topological space and M an abelian group, then

$$\forall n \in \mathbb{N}, \quad \mathrm{H}^n(X, M_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X, M).$$

Proof. First of all, it follows from lemma 3.3.27 that

$$\mathrm{H}^n(X, M_X) \simeq \mathrm{H}^n(X, \mathrm{C}^\bullet_M).$$

Also, we know from lemma 3.3.24 that $\widehat{\mathrm{C}}^n_M$ which is a flabby presheaf, is an epipresheaf. We assume in the rest of the proof that X is *hereditary paracompact Hausdorff*. Proposition 3.3.25 then implies C^n_M is flabby. It follows that

$$\mathrm{H}^n(X, \mathrm{C}^\bullet_M) = \mathrm{H}^n(\mathrm{C}^\bullet_M(X)).$$

But we can also apply proposition 3.3.26 that provides

$$\mathrm{H}^n_{\mathrm{sing}}(X, M)) := \mathrm{H}^n(\widehat{\mathrm{C}}^\bullet_M(X)) \simeq \mathrm{H}^n(\mathrm{C}^\bullet_M(X)).$$ ■

Theorem 3.3.29 If X is a smooth manifold, then

$$\mathrm{H}^n(X, \mathbb{R}_X) \simeq \mathrm{H}^n_{\mathrm{dR}}(X).$$

Proof. First of all, since each sheaf (associated to) Ω^k_X is soft, it is acyclic and it follows that

$$\mathrm{H}^n_{\mathrm{dR}}(X) := \mathrm{H}^n(\Omega^\bullet(X)) = \mathrm{H}^n(X, \Omega^\bullet_X).$$

Now, the canonical morphisms $\mathbb{R} \to \Omega^\bullet(U)$ for U open in X provide a morphism of sheaves $\mathbb{R}_X \to \Omega^\bullet_X$ and we shall show that it is a quasi-isomorphism so that

$$\mathrm{H}^n(X, \mathbb{R}_X) \simeq \mathrm{H}^n(X, \Omega^\bullet_X).$$

This can be checked on the stalks. Given any $x \in X$, there exists an basis of open neighborhoods U of x that admit a smooth diffeomorphism $U \simeq \mathbb{R}^n$. It follows from exercise 2.2.64 that each $\mathbb{R} \to \Omega^\bullet(U)$ is a quasi-isomorphism and so is $\mathbb{R} \to \Omega^\bullet(x)$. ■

The isomorphism

$$\mathrm{H}^n_{\mathrm{dR}}(X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X, \mathbb{R}) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\mathrm{H}^{\mathrm{sing}}_n(X), \mathbb{R})$$

corresponds by currying to a bilinear map

$$\mathrm{H}^n_{\mathrm{dR}}(X) \times \mathrm{H}^{\mathrm{sing}}_n(X) \to \mathbb{R}, \quad ([\omega], [\gamma]) \mapsto \int_\gamma \omega.$$

In the simple case X is an open subset of $\mathbb{R}^d$, $\gamma : [0,1] \simeq \Delta^1 \to X$ is a smooth loop ($\gamma(0) = \gamma(1)$) and $\omega = \sum_{i=1}^d f_i \mathrm{d}x_i$ is integrable ($\partial f_i/\partial x_j = \partial f_j/\partial x_i$), we recover the usual integral

$$\int_\gamma \omega = \sum_{i=1}^d \int_0^1 f_i(\gamma(t))(x_i \circ \gamma)'(t)\mathrm{d}t$$

that only depend on the (co)homology classes !

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