The overconvergent site

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Singular cohomology

If X is a topological space, we may consider the complex of singular cochains

$$C^\bullet_{\text{sing}}(X) = Hom_{\text{sets}}(Hom_{\text{top}}(\triangle_\bullet, X), \mathbf{Z})$$

and define the singular cohomology of X:

$$H^*_{\text{sing}}(X) = H^*(C^\bullet_{\text{sing}}(X)).$$

Alternatively, this may be seen as the cohomology of a sheaf :

Theorem

If X is locally contractible, we have

$$H^*_{\text{sing}}(X) \simeq H^*(X, \mathbf{Z}).$$

This gives a very rich formalism in order to compute these spaces.

de Rham cohomology

One can compute cohomology by means of differential forms:

Theorem (de Rham)

If X is a real manifold, we have

$$H^*_{\mathrm{dR}}(X/\mathbf{R}) \simeq \mathbf{R} \otimes H^*(X, \mathbf{Z}).$$

Thus, if X is a complex manifold, we get

$$H^*_{\mathrm{dR}}(X/\mathbf{C}) \simeq \mathbf{C} \otimes H^*(X, \mathbf{Z}).$$

Now, if X is a non singular algebraic variety over a field K of characteristic zero, and $K \hookrightarrow \mathbf{C}$ any embedding, we have

$$\mathbf{C} \otimes_K H^*_{\mathrm{dR}}(X/K) \simeq H^*_{\mathrm{dR}}(X/\mathbf{C}) \simeq H^*_{\mathrm{dR}}(X^{an}/\mathbf{C}).$$

This gives a purely algebraic interpretation of cohomology.

Using sheaves

It is also possible to express de Rham cohomology as the cohomology of a sheaf:

Theorem (Grothendieck)

If X is an (non-singular) algebraic variety defined over a field K of characteristic zero, we have

$$H^*_{\mathrm{dR}}(X/K) \simeq H^*(\mathrm{inf}(X), \mathcal{O}_{\mathrm{inf}(X)}).$$

Here, $\mathrm{inf}(X)$ is not a topological space anymore but a site and $\mathcal{O}_{\mathrm{inf}(X)}$ denotes a sheaf on this site. Recall that a site is a category with given covering families, generalizing the category of open sets of a topological space and open coverings.

Anyway, we get a purely algebraic interpretation of cohomology as the cohomology of a sheaf.

Connexions

We want to give more details on this isomorphism and do it in a more general setting.

Let $p : X \to S$ be a morphism of schemes. An integrable connection on an $\mathcal{O}_X$-module $\mathcal{F}$ is a $\mathcal{O}_S$-linear map

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

satisfying Leibnitz rule:

$$\nabla(fm) = m \otimes df + f\nabla(m)$$

and $\nabla^2 = 0$. We may then form its de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}$. And we can define the absolute de Rham cohomology of $\mathcal{F}$ as

$$Rp_{dR}\mathcal{F} := Rp_*(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}).$$

The infinitesimal site

If $X \to S$ is a morphism of schemes, then $\inf(X/S)$ is the infinitesimal site whose objects are thickenings

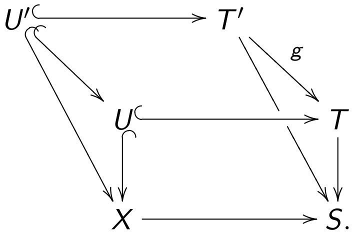
$$\begin{array}{ccc} U & \xhookrightarrow{\text{nilpotent}} & T \\ {\scriptstyle \text{open}}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & S. \end{array}$$

of open subsets of X over S. We will usually write $(U \subset T)$ for such an object.

The fundamental example is as follows: if X is defined by $\mathcal{I}$ inside $X \times_S X$, the thickening of order 1 of X is the subscheme $X(1)$ defined by $\mathcal{I}^2$. In particular the ideal of X inside $X(1)$ is nothing but $\Omega^1_{X/S}$.

Morphisms

A morphism in the category $\inf(X/S)$ is simply a commutative diagram



In practice, we will simply write $g : (U' \subset T') \to (U \subset T)$.

Covering families are $\{(U_i, T_i) \to (U, T)\}$ where $T = \cup T_i$ is a Zariski open covering.

The structural sheaf is defined as

$$\mathcal{O}_{\inf(X/S)} : (U \subset T) \mapsto \Gamma(T, \mathcal{O}_T).$$

Realizations

If we are given a thickening $(U \subset T)$ of an open subset U of X over S and a sheaf E on $\inf(X/S)$, we may consider the realization E_T of E on T which is the sheaf on T given by

$$\Gamma(V, E_T) = \Gamma(V \cap U \subset V, E).$$

Giving E is equivalent to giving all realizations E_T and transition maps

$$g^{-1}E_T \to E_{T'}$$

for all morphisms $g : (U', T') \to (U, T)$ (subject to a cocycle condition).

For example, the sheaf $\mathcal{O}_{\inf(X/S)}$ corresponds to the family of all sheaves $\mathcal{O}_T$ with natural transition maps

$$g^{-1}\mathcal{O}_T \to \mathcal{O}_{T'}.$$

Induced connections

There is a commutative diagram of thickenings

$$\begin{array}{ccc} X & \hookrightarrow & X(1) \\ \| & & p_1 \downarrow\downarrow p_2 \\ X & = & X. \end{array}$$

If E is a finitely presented $\mathcal{O}_{\mathrm{inf}(X)}$-module (or more generally, a crystal), we may consider the composite map

$$\eta : E_X \hookrightarrow p_2^* E_X \xrightarrow{\simeq} E_{X(1)} \xleftarrow{\simeq} p_1^* E_X.$$

The difference $\nabla := p_1^* - \eta$ takes values inside

$$E_X \otimes \Omega^1_X / S \subset p_1^* E_X$$

and defines an integrable connection on E_X.

Classical theorems

Let $p : X \to S$ be a smooth morphism between two algebraic varieties over a field K of characteristic zero. Then,

Theorem (part 1)

There is an equivalence of categories:

$$\begin{array}{ccc} \{\text{Finitely presented modules on } \inf(X/S)\} & \simeq & \{\text{Coherent modules with integrable connection on } X/S\} \\ E & \longmapsto & (E_X, \nabla) \end{array}$$

Theorem (part 2)

There exists a morphism of topos $p_{\inf} : (X/S)_{\inf} \to S$ with

$$Rp_{\mathrm{dR}} E_X \simeq Rp_{\inf *} E.$$

Rigid cohomology

If X is an algebraic variety over a field k of characteristic $p > 0$, de Rham cohomology has to be replaced with Berthelot's rigid cohomology

$$H^*_{\mathrm{rig}}(X/K),$$

where K is a complete ultrametric field of characteristic zero with residue field k.

More generally, let $\mathcal{V}$ denote the ring of integers of K and S a formal $\mathcal{V}$-scheme. One can define overconvergent isocristals E on X/S which are analogous to coherent modules with integrable connections in characteristic zero. Then, if $p : X \to S_k$ is a morphism of algebraic varieties, we can define the rigid cohomology $Rp_{\mathrm{rig}}E$.

Let us be more precise now:

Tubes

If P is a formal $\mathcal{V}$-scheme, we may consider

- its special fiber P_k which is an algebraic variety over k.
- its generic fiber P_K which is an analytic variety (we use Berkovich theory) over K.

This is functorial. Moreover, there is a nilpotent immersion and a specialization map

$$P_k \hookrightarrow P \xleftarrow{\mathrm{sp}} P_K.$$

Note that sp is not continuous. If X is an algebraic subvariety of P_k, we will consider its tube

$$]X[_P := \mathrm{sp}^{-1}(X) \subset P_K.$$

Note : we consider only formal schemes that have a locally finite covering by finitely presented $\mathfrak{m}$-adic formal schemes.

Local description

If $P = \mathrm{Spf} A$, $A_k := k \otimes_{\mathcal{V}} A$ et $A_K := K \otimes_{\mathcal{V}} A$, then

$$P_k = \mathrm{Spec}(A_k) \quad \text{and} \quad P_K = \mathcal{M}(A_K).$$

To $x \in P_k$, we associate

$$\ker(A \to A_k \to k(x)) \in P$$

and to $x \in P_K$, we associate its specialization

$$\ker(A \to \mathcal{V}(x) \to k(x)) \in P$$

(where the first application is induced by $A_K \to \mathcal{K}(x)$).

Finally, if $X \subset P_k$ is defined by

$$\forall i, \overline{f}_i(x) = 0 \quad \text{and} \quad \exists j, \overline{g}_j(x) \neq 0,$$

then $]X[_P \subset P_K$ is defined by

$$\forall i, |f_i(x)| < 1 \quad \text{and} \quad \exists j, |g_j(x)| = 1.$$

Example

We can consider the affine Legendre family

$$X : y^2 = x(x-1)(x-\lambda) \subset \mathbf{A}^3_k$$

over the affine line $S = \mathbf{A}^1_{\mathcal{V}}$ and embed it into the formal projective Legendre family

$$P : y^2 = x(x-z)(x-\lambda z) \subset \widehat{\mathbf{P}^2_S}.$$

Then, we will have

$$]X[_P = V^{\mathrm{an}} \cap \mathbf{B}^3(0, 1^+)$$

where V^{an} denotes the analytification of the affine Legendre family:

$$V : y^2 = x(x-1)(x-\lambda) \subset \mathbf{A}^3_K.$$

Formal embeddings

We fix a formal $\mathcal{V}$-scheme S. A formal embedding is a locally closed embedding $X \hookrightarrow P$ of an algebraic variety into a formal S-scheme. We will denote by $\overline{X}$ the closure of X inside P.

A morphism of formal embeddings is simply a commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{v} & P \\ \uparrow & & \uparrow \\ X' & \xrightarrow{f} & X \end{array}$$

The morphism is said to be flat, smooth or étale if v has this property in the neighborhood of X'. It is said to be separated, proper or finite if the induced map $\overline{X}' \to P_k$ is so. We will also say that the morphism v is flat, smooth, étale, separated, proper or finite at X'. We will apply this in particular to the canonical morphism $(X \subset P) \to (S_k \subset S)$.

Geometric versus analytic

Theorem

In the following cases, if a morphism of formal embeddings $(f, v) : (X' \subset P') \to (X \subset P)$ has the property (Geom), then then there exists neighborhoods V and V' of $]X[_P$ and $]X'[_{P'}$ respectively such that the induced map $v_K : V' \to V$ has the property (Ana).

1. *(Geom) = flat and (Ana) = universally flat*
2. *(Geom) = smooth and (Ana) = formally smooth*
3. *(Geom) = étale and (Ana) = formally étale*
4. *(Geom) = separated and (Ana) = locally separated*
5. *(Geom) = proper and (Ana) = boundaryless*
6. *(Geom) = proper smooth and (Ana) = smooth*
7. *(Geom) = finite étale and (Ana) = étale*

Overconvergent isocrystals

We fix a formal embedding $X \subset P$. In order to avoid problems with the translation into Berkovich theory, we will assume that all points of $]X[_P$ have an affinoid neighborhood in P_K (this is the case for example if S is affine and $\overline{X}$ is proper over S_k).

The functor of overconvergent sections on a neighborhood V of $]X[_P$ in $]\overline{X}[_P$ is defined as

$$j^\dagger \mathcal{F} := i_{X*} i_X^{-1} \mathcal{F}$$

where $i_X :]X[_P \hookrightarrow V$ denotes the inclusion map.

Let $\mathcal{F}$ be a coherent $j^\dagger \mathcal{O}_{]\overline{X}[_P}$-module. An integrable connection on $\mathcal{F}$ is overconvergent if its Taylor series comes from an isomorphism

$$j^\dagger p_2^* \mathcal{F} \simeq j^\dagger p_1^* \mathcal{F}$$

on $]\overline{X}[_{P \times_S P}$.

Berthelot's theorem 1

An overconvergent isocrystal on $X \subset P$ is a coherent $j^\dagger\mathcal{O}_{]\overline{X}[_P}$-module $\mathcal{F}$ with an overconvergent integrable connection.

Theorem (Berthelot)

1. *If P is smooth at X, this definition only depends on the pair $X \subset \overline{X}$.*
2. *If P is proper smooth at X, this definition only depends on X.*

Using this theorem, it is possible to define the category of overconvergent isocrystals on $(X \subset \overline{X})$, or even on X. In this case, we will call $\mathcal{F}$ the realization on P of the overconvergent isocrystal E.

Berthelot's theorem 2

In general, if we denote by $v : P \to S$ the structural morphism and write $v_K :]\overline{X}[_P \to S_K$ for the induced map, the absolute rigid cohomology of $\mathcal{F}$ is

$$Rp_{\mathrm{rig}}\mathcal{F} := Rv_{K*}(\mathcal{F} \otimes_{\mathcal{O}_{]\overline{X}[_P}} \Omega^{\bullet}_{]\overline{X}[_P/S_K})$$

Theorem (Berthelot)

1. *If P is smooth at X, this definition only depends on the pair $X \subset \overline{X}$.*
2. *If P is proper smooth at X, this definition only depends on X.*

We may therefore define the cohomology of an overconvergent isocrystal on $X \subset \overline{X}$ or the cohomology of an overconvergent isocrystal on X.

First theorem

Theorem (Part 1)

There exists a ringed site $(\mathrm{an}^\dagger(X/S), \mathcal{O}_{\mathrm{an}^\dagger(X/S)})$ *and an equivalence of categories*

$$\left\{\begin{array}{c}\text{Finitely presented}\\ \text{modules on } \mathrm{an}^\dagger(X/S)\end{array}\right\} \simeq \left\{\begin{array}{c}\text{Overconvergent isocrystals}\\ \text{on } X/S\end{array}\right\}$$

Actually, if we are given any immersion $X \hookrightarrow P$ into a formal S-scheme, there exists a realization functor that send any $\mathcal{O}_{\mathrm{an}^\dagger(X/S)}$-module E to an $i_X^{-1}\mathcal{O}_{]\overline{X}[_P}$-module E_P where $i_X :]X[_P \hookrightarrow]\overline{X}[_P$.

If E is finitely presented and P is proper smooth at X, then E_P is the realization on $X \subset P$ of the corresponding overconvergent isocrystal.

Second theorem

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Theorem (Part 2)

If X is an algebraic variety over S_k, there exists a morphism of ringed toposes

$$p_{\mathrm{an}^\dagger} : (X/S)_{\mathrm{an}^\dagger} \longrightarrow S_K$$

such that

$$Rp_{\mathrm{rig}} E \simeq Rp_{\mathrm{an}^\dagger *} E.$$

Corollary

If X is an algebraic variety over k and K is a complete ultrametric field of characteristic zero with residue field k, then

$$H^*_{\mathrm{rig}}(X/K) \simeq H^*(\mathrm{an}^\dagger(X/K), \mathcal{O}_{\mathrm{an}^\dagger(X/K)}).$$

This is the analog of Grothendieck theorem in positive characteristic.

OBJECTS

Let S be a formal $\mathcal{V}$-scheme as before and X an algebraic variety over S_k. An object of the (small) overconvergent site is a pair made of a formal embedding $X \subset P$ and an open subset $V \subset P_K$:

$$X \hookrightarrow P \overset{\mathrm{sp}}{\leftarrow} P_K \overset{\mathrm{open}}{\hookleftarrow} V.$$

The tube of X inside V is

$$]X[_V := V \cap]X[_P.$$

A morphism

$$(X \subset P' \leftarrow P'_K \hookleftarrow V') \to (X \subset P \leftarrow P_K \hookleftarrow V)$$

is simply a morphism $V' \dashrightarrow V$ defined on a neighborhood of $]X[_{V'}$ such that $\mathrm{sp}(u(x)) = \mathrm{sp}(x)$ for $x \in]X[_{V'}$, and the same holds after any isometric extension of K.

Alternative description

The big overconvergent site has a more "formal" defintion. A formal morphism is a commutative diagram

$$\begin{array}{ccccccc} U' & \hookrightarrow & P' & \leftarrow & P'_K & \leftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow v_K & & \downarrow u \\ U & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \end{array}$$

over X and S. We obtain a category $\mathrm{AN}(X/S)$.

A formal morphism is said to be a strict neighborhood if $f = \mathrm{Id}_U$ ans u is an open immersion that induces an isomorphism $]U[_{V'} \simeq]U[_V$ on the tubes.

The category $\mathrm{AN}(X/S)$ admits right calculus of fractions with respect to strict neighborhoods. We can define $\mathrm{AN}^\dagger(X/S)$ as the category of fractions with respect to strict neighborhoods.

Topology and sheaves

A covering in $\mathrm{an}^\dagger(X/S)$ is a family

$$(X \subset P \overset{\mathrm{sp}}{\leftarrow} P_K \supset V_i) \to (X \subset P \overset{\mathrm{sp}}{\leftarrow} P_K \supset V)$$

where each V_i is open into V and $\cup V_i$ is a neighborhood of $]U[_V$ into V. This defines a pretopology on $\mathrm{an}^\dagger(X/S)$ turning it into a site and giving rise to a topos $(X/S)_{\mathrm{an}^\dagger}$.

The structural sheaf is defined as

$$\mathcal{O}_{\mathrm{an}^\dagger(X/S)} : (X \subset P \leftarrow P_K \supset V) \mapsto \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V).$$

This way, we obtain a ringed site.

The relative site

If $X \hookrightarrow P$ is a formal immersion, we may consider the full subcategory of $\mathrm{an}^\dagger(X/S)$ of all objects who admit some morphism

$$(X \subset Q \leftarrow Q_K \supset V) \to (X \subset P \leftarrow P_K = P_K)$$

and we denote by $\mathrm{an}^\dagger(X_P/S)$ the corresponding ringed site. Then, we have:

Proposition

If P is smooth at X, there is an equivalence of categories

$$\left\{\begin{array}{c}\text{Finitely presented}\\ \text{modules on } \mathrm{an}^\dagger(X_P/S)\end{array}\right\} \simeq \left\{\begin{array}{c}\text{Overconvergent isocrystals}\\ \text{on } X \subset P/S\end{array}\right\}$$

The proof goes as in the classical case as we shall see:

Strategy of proof

We define an overconvergent stratification on an $i_X^{-1}\mathcal{O}_{P_K}$-module $\mathcal{F}$ as an isomorphism

$$i_X^{-1}p_2^* i_{X*}\mathcal{F} \simeq i_X^{-1}p_1^* i_{X*}\mathcal{F}$$

on $]X[_{P\times_S P}$ and we prove an equivalence

$$\left\{\begin{array}{c}\text{Crystals}\\ \text{on } (X_P/S)^\dagger_{\mathrm{an}}\end{array}\right\} \simeq \left\{\begin{array}{c} i_X^{-1}\mathcal{O}_{P_K}\text{-modules with}\\ \text{overconvergent stratification}\end{array}\right\}$$

Then, the difficult part is fulfaithfulness of

$$\left\{\begin{array}{c}\text{Coherent } i_X^{-1}\mathcal{O}_{P_K}\text{-modules}\\ \text{with overconvergent stratification}\end{array}\right\} \to \left\{\begin{array}{c}\text{Coherent stratified}\\ i_X^{-1}\mathcal{O}_{P_K}\text{-modules}\end{array}\right\}$$

Finally, we have

$$\left\{\begin{array}{c}\text{Stratified}\\ i_X^{-1}\mathcal{O}_{P_K}\text{-modules}\end{array}\right\} \simeq \left\{\begin{array}{c} i_X^{-1}\mathcal{O}_{P_K}\text{-modules}\\ \text{with integrable connection}\end{array}\right\}$$

The main theorem

Theorem

If $X \hookrightarrow P$ is a formal embedding with P proper and smooth at X, there is an equivalence of toposes

$$(X_P/S)^\dagger_{\mathrm{an}} \simeq (X/S)^\dagger_{\mathrm{an}}.$$

This theorem and the previous proposition implies theorem 1. And we get as a corollary the independence on $\overline{X}$ and P of the category of overconvergent isocrystals.

We can reformulate the conclusion of the theorem as follows:

Theorem

The object $(X \subset P \leftarrow P_K = P_K)$ is a covering of the final object of the topos $(X/S)^\dagger_{\mathrm{an}}$.

The proof

Using the diagonal embedding, it is sufficient to prove the following geometric result:

Proposition

Let $v : P' \to P$ be a morphism of formal embeddings of X which is proper and smooth at X. Then, the morphism

$$(X \subset P' \leftarrow P'_K = P'_K) \to (X \subset P \leftarrow P_K = P_K)$$

induced by v has locally a section in $\mathrm{an}^\dagger(X/S)$.

As already mentioned, we know that v induces a smooth morphism $v_K : V' \to V$ between two neighborhoods of the tubes.

The proof

The smoothness of v implies for all $x \in]X[_P$ an isomorphism

$$\phi_x : v_K^{-1}(x) \simeq \mathbf{B}^d_{\mathcal{K}(x)}(0, 1^-)$$

(this is a consequence of the weak fibration theorem).

One can then show that there exists two neighborhoods W of x in V and W' of $x' := \phi_x^{-1}(0)$ in V' such that ϕ_x extends to an open immersion

$$\phi : W' \hookrightarrow \mathbf{B}^d_W(0, 1^-).$$

We can then use the zero section s and we get the expected section:

$$(X \subset P \leftarrow P_K \supset W) \to (X \subset P' \leftarrow P'_K = P'_K).$$

Proof of theorem 1 is complete.

Strong fibration

In order to compare also the cohomology, we really need a strong fibration theorem:

Theorem (Berthelot)

Let $v : P' \to P$ be a morphism of formal embeddings of X that is proper and smooth at X. Then, locally for the Zariski topology on X and for the Grothendieck topology on P_K, there is an isomorphism

$$(X \subset P' \leftarrow P'_K = P'_K) \simeq (X \subset \widehat{\mathbf{A}}^n_P \leftarrow \mathbf{B}^n_{P_K} = \mathbf{B}^n_{P_K}).$$

The proof is a geometric reduction to the finite étale case. The finite étale case is a consequence of the weak fibration theorem and properties of the étale topology for (Berkovich) analytic varieties.

The Poincaré Lemma

From the strong fibration theorem, one can derive the Poincaré lemma:

Lemma (Berthelot)

Let $v : P' \to P$ be a morphism of formal embeddings of X that is proper and smooth at X. If $\mathcal{F}$ is a coherent $i_X^{-1}\mathcal{O}_{P_K}$-module and $v_K :]X[_{P'} \to]X[_P$ is the induced map, we have

$$\mathcal{F} \simeq Rv_{K*}(\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_{P_K}} i_X^{-1}\Omega^\bullet_{P'_K/P_K})$$

In order to prove this lemma, one uses the strong fibration theorem to reduce to the case of the projection of $\widehat{\mathbf{A}}^n_P$ onto P. Then, one uses induction on n to reduce to the case $n = 1$. The final computation is done directly.

Linearization

If $X \hookrightarrow P$ is a formal embedding , we may consider the site $\mathrm{an}^\dagger(X \subset P/S)$ of objects defined above $(X \subset P \leftarrow P_K = P_K)$. Restriction is the inverse image functor for a morphism of ringed toposes

$$j : (X \subset P/S)_{\mathrm{an}^\dagger} \to (X/S)_{\mathrm{an}^\dagger}.$$

On the other hand, realization is the direct image for a morphism of ringed sites

$$\varphi : \mathrm{an}^\dagger(X \subset P/S) \to]X[_P.$$

Then, if $\mathcal{F}^\bullet$ is a complex of $i^{-1}\mathcal{O}_{P_K}$-modules and differential operators, its derived linearization is

$$RL\mathcal{F}^\bullet := Rj_*\varphi^*\mathcal{F}^\bullet.$$

Poincaré Lemma

So far, I did not tell you what exactly a realization is: If E is a sheaf on $\mathrm{an}^\dagger(X/S)$ and $X \hookrightarrow P$ is a formal embedding, the realization of E on P is the sheaf on $]X[_P$ defined by

$$\Gamma(]X[_V, E_P) := \Gamma(X \subset P \leftarrow P_K \supset V, E)$$

From the Poincaré lemma, one deduces:

Lemma

Let $X \hookrightarrow P$ be a formal embedding with P proper and smooth at X over S. If E is a finitely presented $\mathcal{O}_{(X/S)^\dagger_{\mathrm{an}}}$-module, we have

$$E \simeq RL(E_P \otimes_{i_X^{-1}\mathcal{O}_{P_K}} i_X^{-1}\Omega^\bullet_{P_K/S_K}).$$

One easily sees that, if we are given another formal embedding $X \subset P'$, then $(RL\mathcal{F}^\bullet)_{P'} = Rp_{1*}i_X^{-1}p_2^* i_{X*}\mathcal{F}^\bullet$. The lemma follows.

END OF THE PROOF

LEMMA

Let $(X \subset P)$ be a formal embedding and $\mathcal{F}^\bullet$ a complex of $i_X^{-1}\mathcal{O}_{P_K}$-modules and differential operators. If $v : P \to S$ is the structural map and $v_K :]X[_P \to S_K$ the induced map. Then,

$$Rp_{\mathrm{an}^\dagger *}RL\mathcal{F}^\bullet = Rv_{K*}\mathcal{F}^\bullet.$$

It follows that if E is a finitely presented $\mathcal{O}_{\mathrm{an}^\dagger(X/S)}$-module, we obtain thanks to the Poincaré Lemma, a de Rham formula

$$Rp_{\mathrm{an}^\dagger *}E = Rp_{\mathrm{an}^\dagger}RL(E_P \otimes_{i_X^{-1}\mathcal{O}_{P_K}} i_X^{-1}\Omega^\bullet_{P_K/S_K})$$

$$= Rv_{K*}(E_P \otimes_{i_X^{-1}\mathcal{O}_{P_K}} i_X^{-1}\Omega^\bullet_{P_K/S_K}) = Rp_{\mathrm{rig}}E. \quad \square$$

We obtain as a corollary the independence of the rigid cohomology with respect to the formal embedding $X \subset P$.

Gelfand spectrum

If A is an affinoid algebra, then $V := \mathcal{M}(A)$ is the set of continuous semi-absolute values (or multiplicative semi-norms) $x : A \to \mathbf{R}$ over K. The "kernel" of x is a prime $\mathfrak{p}_x$ and $\mathcal{K}(x)$ will denote the completion of the fraction field of $A/\mathfrak{p}_x$. We write $f(x)$ for the image of $f \in A$ into $\mathcal{K}(x)$ so that $|f(x)| = x(f)$. We get a bijection

$$V_0 := \mathrm{Spm}(A) \simeq \{x \in V, \quad [\mathcal{K}(x) : K] < \infty\}$$

The set V is endowed with the simple convergence topology.

The simplest non trivial example is given by

$$\mathbf{D}(0, 1^+) = \mathcal{M}(K\{t\}).$$

Besides rigid points $\in \mathbf{D}(0, 1^+)_0$, there are many other points such as

$$|(\sum_i a_i t^i)(x)| = \sup |a_i| r^i \quad \text{with} \quad r \in [0, 1].$$

Affinoid varieties

An affinoid variety is a topological space V endowed with a homeomorphism $V \simeq \mathcal{M}(A)$. We will write $\mathcal{O}(W) = A$. Any homomorphism of affinoid algebra $A \to A'$ will induce an affinoid map

$$V' \simeq \mathcal{M}(A') \to \mathcal{M}(A) \simeq V.$$

A subspace W of $V := \mathcal{M}(A)$ is an affinoid domain if it is universal for affinoid maps: W has an affinoid structure and if a composite map $V' \to W \hookrightarrow V$ is affinoid, so is the first one.

For example, the disk

$$\mathbf{D}(0, \eta^+) := \{x \in \mathbf{D}(0, 1^+), \quad |t(x)| \ \leq \eta\}$$

is an affinoid domain for $\eta \in \sqrt{|\mathcal{V}|}$: if $\eta^k = |a|$, we have $\mathbf{D}(0, \eta^+) \simeq \mathcal{M}(K\{t, s\}/(t^k - as))$. Note that Berkovich theory can be enlarged in order to work with any $\eta \leq 1$: this is essential if we want to work with trivial valuations (which is not the case here).

Nets

A quasi-net on a topological space V is a collection τ of subsets W of V, the bricks, such that any point x of V has a neighborhood which is a finite union of bricks containing x. The subsets $[n, n+1]$ on the real line form a quasi-net.

It is called a net if whenever W is a finite intersection of bricks, all bricks contained in W form a quasi-net. The above example is not a net but the open subsets of a topological space or the affinoid domains of an affinoid variety form a net.

Then, a subset W of V will be called admissible if the bricks contained in W form a quasi-net on W. A covering of an admissible subset by admissible subsets will be admissible if it is a quasi-net. For example, the covering

$$\mathbf{D}(0, 1^-) = \cup_{\eta<1}\mathbf{D}(0, \eta^-)$$

is an admissible covering of an admissible subset.

Analytic varieties

An analytic variety is a locally Hausdorff topological space V endowed with a maximal net of affinoid domains: in other words, we want a net τ on V such that the category $(\tau, \subset)$ is a subcategory of affinoid varieties and inclusion of affinoid domains. We want it maximal: there is an obvious order by inclusions on such nets and τ must be maximal for this order.

Example: we denote by $\mathbf{A}^{1,\mathrm{an}}$ the set of semi-absolute values on $K[t]/K$. The subsets

$$D_R = \{x \in \mathbf{A}^{1,\mathrm{an}}, x(t) \leq R\}$$

define a net of affinoid domains. To make it maximal, it is sufficient to add all the affinoid domains W that sit inside some D_R.

Rigid versus Berkovich

Admissible subsets of an analytic variety are called analytic domains. Analytic domains and admissible coverings define a Grothendieck topology on V giving rise to a site V_G.

We can define the residue field $\mathcal{K}(x)$ at a point $x \in V$. If V is Hausdorff, the set

$$V_0 := \{x \in V, \quad [\mathcal{K}(x) : K] < \infty\}$$

has a unique structure of rigid analytic variety such that

1. The affinoid open subsets are the W_0 where W is an affinoid domain.
2. The admissible affinoid coverings are the coverings $V_0 = \cup W_{i0}$ where $V = \cup W_i$ is an admissible affinoid covering.

It follows that we have an equivalence of toposes $\widetilde{V_0} \simeq \widetilde{V_G}$.

Good analytic varieties

The presheaf $\mathcal{O}$ on τ extends uniquely to a sheaf $\mathcal{O}_{V_G}$ for the Grothendieck topology. It induces a sheaf $\mathcal{O}_V$ for the usual topology. There is a canonical morphism $\pi : V_G \to V$ which is continuous (this is just the identity on the underlying sets). It is a morphism of ringed spaces.

If any point of V has an affinoid neighborhood, V is said to be good, and we have equivalences

$$\left\{\begin{array}{c}\text{Coherent}\\ \mathcal{O}_{V_0}\text{-modules}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{Coherent}\\ \mathcal{O}_{V_G}\text{-modules}\end{array}\right\} \begin{array}{c}\xleftarrow{\pi_*}\\ \xrightarrow{\pi^*}\end{array} \left\{\begin{array}{c}\text{Coherent}\\ \mathcal{O}_{V}\text{-modules}\end{array}\right\}$$

Example: the analytic domain

$$V := \{(x,y) \in \mathbf{B}^2(0,1^+), \quad |x| = 1 \text{ or } |y| = 1\}$$

is not good. Note that this is the tube of $\mathbf{A}^2_k \setminus (0,0)$ for the natural embedding into $\widehat{\mathbf{P}}^2_{\mathcal{V}}$.