Descent in rigid cohomology
(Padova – 2022)

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Sommaire

Introduction

Theorem

h-coverings satisfiy total descent with respect to constructible isocrystals.

History concerning *overconvergent* isocrystals:

- 2003 : Chiarellotto-Tsuzuki (étale cohomological descent) [CT03]
- 2003 : Tsuzuki (proper cohomological descent) [Tsu03]
- 2014 : Zureick-Brown (cohomological descent) [Zur14]
- 2019 : Shiho, Lazda (effective descent) [Laz19]

Ideas behind this new proof:

- cohomological and effective descent done simultaneously,
- étale and proper descent done simultaneously,
- valid for all constructible isocrystals,
- works on general (locally noetherian) formal schemes.

Formal schemes

A topological (noetherian) ring A is said to be *adic* if there exists an ideal I whose powers define a basis of neighborhoods of 0. A *(locally noetherian) formal scheme* is a topologically locally ringed space which is locally of the form

$$P = \mathrm{Spf}(A) = \{\text{open prime ideals in } A\}$$

where A is an adic ring. More precisely (for $f \in A$):

$$U = \{\mathfrak{p} \in P, f \notin \mathfrak{p}\} \quad \Rightarrow \quad \mathcal{O}_P(U) = \widehat{A[1/f]}.$$

Example

- A (locally noetherian) scheme is a formal scheme (discrete topology),
- Concretely: $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(\mathbb{F}_p)$ are formal schemes,
- If $X \hookrightarrow P$ is a closed embedding of (formal) schemes, we may consider the completion $P^{/X}$ of P along X which is a formal scheme,
- Concretely: the embedding $\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ provides $\mathrm{Spf}(\mathbb{Z}_p)$.

Adic spaces

A *Huber ring* (of noetherian type) is a topological ring A which is of finite type over some open adic subring A_0. It is called a *Tate ring* if there exists a topologically nilpotent unit π.

Example

- An adic ring A is a Huber ring,
- If A_0 is a π-adic ring, then $A_0[1/\pi]$ is a Tate ring.

An *adic (resp. analytic) space* is a valued topologically ringed space which is locally of the form

$$V = \mathrm{Spa}(A, A^+) = \{\text{continuous valuations on } A \text{ non negative on } A^+\}/\sim$$

where A is a Huber (resp. Tate) ring and $A^+ \subset A$ is power-bounded. More precisely, when $(f_0, \ldots, f_r)$ open in A:

$$U = \{v \in V, v(f_i) \geq v(f_0) \neq +\infty\} \quad \Rightarrow \quad \mathcal{O}_V(U) = \widehat{A[1/f_0]}$$

(with $A_0[f_1/f_0, \ldots, f_r/f_0]$ as open adic subring).

Example

- To any (usual) scheme X, one can associate an adic space X^{val} and a morphism $\mathrm{supp} : X^{\mathrm{val}} \to X$,
- Locally, if $X = \mathrm{Spec}(A)$, then $X^{\mathrm{val}} = \mathrm{Spa}(A, \emptyset)$ and $\mathrm{supp}(v) = \{f \in A, v(f) = +\infty\}$,
- To any formal scheme P, one can associate an adic space P^{ad} and a morphism $\mathrm{sp} : P^{\mathrm{ad}} \to P$,
- Locally, if $P = \mathrm{Spf}(A)$, then $P^{\mathrm{ad}} = \mathrm{Spa}(A, A)$ and $\mathrm{sp}(v) = \{f \in A, v(f) > 0\}$,
- If $X \hookrightarrow P$ is a closed embedding of formal schemes, its *tube* is $]X[_P := P^{/X,\mathrm{ad}} \subset P^{\mathrm{ad}}$,
- Concretely, the tube of $0 = \mathrm{Spec}(\mathbb{Z})$ in $\mathbb{A} = \mathrm{Spec}(\mathbb{Z}[T])$ is $\mathbb{D}^- = \mathrm{Spa}(\mathbb{Z}[[T]], \mathbb{Z}[[T]])$ (for the T-adic topology).

One can extend the notion of a tube to a locally closed embedding by boolean combination.

Overconvergent spaces

Definition

An *overconvergent space* $(X \hookrightarrow P \leftarrow V)$ is a locally closed embedding of formal schemes $X \hookrightarrow P$ together with a morphism of adic spaces $P^{\mathrm{ad}} \leftarrow V$. It is said to be *analytic* if V is analytic. The *tube* $]X[_V$ of X in V is then the inverse image of $]X[_P$ inside V.

Example

Let $\mathcal{V}$ be a discrete valuation ring with fraction (resp. residue) field K (resp. k). Then,

- $$\mathrm{Spec}(k) \hookrightarrow \mathrm{Spf}(\mathcal{V}) \leftarrow \mathrm{Spa}(K, \mathcal{V})$$

  is the usual basis for Berthelot's rigid cohomology (all spaces have a unique point),
- $$\mathrm{Spec}(k((t))) \hookrightarrow \mathrm{Spf}(\mathcal{V}[[t]]) \leftarrow \mathrm{Spa}(K \otimes_{\mathcal{V}} \mathcal{V}[[t]], \mathcal{V}[[t]])$$

  is the basis for Lazda-Pàl's rigid cohomology (the tube has two points $v \rightsquigarrow v^- \notin$ Berkovich).

Strict neighborhoods

Definition

A morphism of overconvergent spaces

$$\begin{array}{ccccc} X' & \hookrightarrow & P' & \leftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow u \\ X & \hookrightarrow & P & \leftarrow & V \end{array}$$

is called a *strict neighborhood* if f is an isomorphism, v is locally noetherian, u is an open embedding and $]f[_u$ is surjective (homeomorphism).

Example

- If $C \hookrightarrow S \leftarrow O$ is an analytic overconvergent space, then there is a sequence of strict neighborhoods (with suggestive notations):

  $(C \hookrightarrow \mathbb{A}_S^- \leftarrow \mathbb{D}_O^-) \to (C \hookrightarrow \mathbb{A}_S \leftarrow \mathbb{D}_O) \to (C \hookrightarrow \mathbb{P}_S \leftarrow \mathbb{P}_O)$.
- A formal blowing up centered outside X is a strict neitghborhood.

The overconvergent site

Definition

The *overconvergent site* $\mathbf{Ad}^\dagger$ is the category of overconvergent spaces localized at strict neighborhoods endowed with the topology inherited from adic spaces. We shall denote by (X, V) the object corresponding to $(X \hookrightarrow P \leftarrow V)$.

More generally, *an overconvergent site* is a fibred category T over $\mathbf{Ad}^\dagger$.

Example

- The category $\mathbf{Ad}^\dagger$ and the category $\mathbf{An}^\dagger$ of analytic overconvergent spaces,
- The fibred category represented by some $(X, V) \in \mathbf{Ad}^\dagger$,
- If $X \in \mathbf{FS}$ (category of formal schemes) is a formal scheme, then $X^\dagger := X \times_{\mathbf{FS}} \mathbf{Ad}^\dagger$ et $X^{\dagger,\mathrm{an}} := X \times_{\mathbf{FS}} \mathbf{An}^\dagger$.
- If $(C, O) \in \mathbf{Ad}^\dagger$ and $X \to C$ is a morphism of formal schemes, then

$$(X/O)^\dagger := X^\dagger \times_{C^\dagger} (C, O).$$

Crystals

If, for $(X, V) \in \mathbf{Ad}^\dagger$, we denote the inclusion of the tube by $i_X :]X[_V \hookrightarrow V$, then the $i_X^{-1}\mathcal{O}_V$-modules make a fibred category $\mathcal{M}\mathrm{od}$ over $\mathbf{Ad}^\dagger$.

Definition

A *(iso-) crystal* on an (analytic) overconvergent site T is a cartesian section of $\mathcal{M}\mathrm{od}_T : T \times_{\mathbb{A}^\dagger} \mathcal{M}\mathrm{od}$:

$$\mathrm{Cris}(T) = \mathrm{Hom}_{\mathbb{F}ib(T)}(T, \mathcal{M}\mathrm{od}_T).$$

One may always consider a crystal E as a sheaf on T via

$$E : s \in T(X, V) \mapsto \Gamma(]X[_V, E(s)).$$

Example

- The crystal $\mathcal{O}_T^\dagger$ corresponds to $s \mapsto i_X^{-1}\mathcal{O}_V$,
- A finitely presented $\mathcal{O}_T^\dagger$-module is automatically a crystal,
- If $(X, V) \in \mathbf{Ad}^\dagger$, then $\mathrm{Cris}(X, V) \simeq \mathrm{Mod}(i_X^{-1}\mathcal{O}_V)$.

Constructible crystals

Definition

A *crystal* E on T is said to be *constructible* if there exists a morphism $T \to X^\dagger$ and a locally finite covering of X by locally closed formal subschemes Y such that $E_{|Y}$ is finitely presented.

They provide us with a subcategory $\mathrm{Cris}_{\mathrm{cons}}(T) \subset \mathrm{Cris}(T)$.

If $(X, V) \in \mathbf{Ad}^\dagger$, then an $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ is said to be *constructible* when the corresponding crystal is.

Definition

Let $(X, V) \to (C, O)$ be a morphism of overconvergent spaces with V locally of finite type over O defined over $\mathbb{Q}$. An integrable connection on an $i_X^{-1}\mathcal{O}_V$-module is said to be *overconvergent* if its Taylor series converges on $]X[_{V(1)}$ where $V(1) := V \times_O V$.

We denote by $\mathrm{MIC}_{\mathrm{cons}}(X, V/O)^\dagger$ the category of constructible modules endowed with an overconvergent connection.

Materialization

Definition

A *geometric materialization* is a morphism

$$(X \hookrightarrow P \leftarrow V) \to (C \hookrightarrow S \leftarrow O)$$

of analytic overconvergent spaces which is right cartesian and such that P is partially proper and formally smooth over S in the neighborhood of X.

Example

If X is quasi-projective over C, one can chosse $P = \mathbb{P}^N_S$ and $V = \mathbb{P}^N_O$.

Theorem

If V is a geometric materialization of a formal scheme X over an analytic space O defined over $\mathbb{Q}$, then

$$\mathrm{Cris}_{\mathrm{cons}}(X/O)^\dagger \simeq \mathrm{MIC}_{\mathrm{cons}}(X, V/O)^\dagger.$$

Cohomology

If $(X,V) \to (C,O)$ is a morphism of overconvergent spaces, then there exist two morphisms of topoi (by considering the category of sheaves on both sides)

$$\varphi_V : \widetilde{(X,V)} \to \widetilde{]X[_V} \quad \text{et} \quad j_V : \widetilde{(X,V)} \to \widetilde{(X/O)}^\dagger.$$

Definition

If a module $\mathcal{F}$ on $]X[_V$ is endowed with an integrable connection, then its *(derived) linearization* is

$$\mathrm{R}L_{\mathrm{dR}}\mathcal{F} := \mathrm{R}j_{V*}\varphi_V^*\left(\mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega_V^\bullet\right).$$

Theorem (Poincaré lemma)

Let V be a geometric materialization of a formal scheme X over an analytic space O defrined over $\mathbb{Q}$. If E is a constructible isocrystal on $(X/O)^\dagger$ and $E_{X,V} := \varphi_{V*}j_V^{-1}E$, then $E \simeq \mathrm{R}L_{\mathrm{dR}}E_{X,V}$.

Corollary

$\forall k \in \mathbb{N}, \quad \mathcal{H}^k_{\mathrm{rig}}(X/O,E) := \mathcal{H}^k((X/O)^\dagger, E) \simeq \mathcal{H}^k_{\mathrm{dR}}(]X[_V, E_{X,V}).$

We consider a site $\mathbb{B}$ (a category endowed with a topology) with fibred products. If $f : X \to S$ is a morphism in $\mathbb{B}$, one may consider the standard simplicial complex

$$X(\bullet) : \cdots X(2) \overrightarrow{\underleftarrow{\overrightarrow{\underleftarrow{\to}}}} X(1) \overrightarrow{\underleftarrow{\to}} X(0)$$

where

$$X(i) := \underbrace{X \times_S \cdots \times_S X}_{i+1}.$$

One may then define a *simplicial sheaf* on $X(\bullet)$ as a family of sheaves $\mathcal{F}(i)$ on each $X(i)$ endowed with a compatible family of transition morphisms. The morphism f then induces a morphism of topoi

$$f_\epsilon : \widetilde{X(\bullet)} \to \widetilde{S}.$$

More precisely, if we denote by $f(i) : X(i) \to S$ the structural morphism, then

$$f_\epsilon^{-1}\mathcal{F} := f(\bullet)^{-1}\mathcal{F} \quad \text{and} \quad f_{\epsilon *}\mathcal{F}(\bullet) := \ker(f(0)_*\mathcal{F}(0) \rightrightarrows f(1)_*\mathcal{F}(1)).$$

Cohomological descent

Definition

The morphism f is said to satisfy *cohomological descent* with respect to an abelian sheaf $\mathcal{F}$ if the adjunction map

$$\mathcal{F} \simeq \mathrm{R}f_{\epsilon *}f_{\epsilon}^{-1}\mathcal{F}$$

is an isomorphism.

When this is the case, there exists a spectral sequence

$$E_1^{i,j} := \mathrm{H}^j(X(i), f(i)^{-1}\mathcal{F}) \Rightarrow \mathrm{H}^{i+j}(S, \mathcal{F}).$$

Let $\mathcal{M}$ be a fibred subcategory of the category of all abelian sheaves over $\mathbb{B}$.

Definition

The morphism f is said to satisfy *universal cohomological descent* with respect to $\mathcal{M}$ if each $\mathcal{F} \in \mathcal{M}(S)$ satisfies cohomological descent with respect to f and this still holds true after any base change $S' \to S$ in $\mathbb{B}$.

Effective descent

We denote by $p_1, p_2 : X(1) \to X(0)$ and $p_{12}, p_{13}, p_{23} : X(2) \to X(1)$ the first projections.

Definition

A *descent datum* on an abelian sheaf $\mathcal{F}$ on $X = X(0)$ is an isomorphism

$$\epsilon : p_2^{-1}\mathcal{F} \simeq p_1^{-1}\mathcal{F}$$

such that $p_{13}^{-1}(\epsilon) = p_{12}^{-1}(\epsilon) \circ p_{23}^{-1}(\epsilon)$.

With $\mathcal{M}$ as above, we denote by $\mathcal{M}(X \xrightarrow{f} S)$ the category of $\mathcal{F} \in \mathcal{M}(X)$ endowed with a descent datum with respect to f.

Definition

The morphism f is said to satisfy *universal effective descent* with respect to $\mathcal{M}$ if the pull-back

$$f^{-1} : \mathcal{M}(S) \simeq \mathcal{M}(X \to S)$$

is an equivalence and this still holds true after any base change $S' \to S$ in $\mathbb{B}$.

Total descent

Definition

The morphism f is said to satisfy *total descent* (with respect to $\mathcal{M}$) if it satisfies both universal cohomological descent and universal effective descent.

Example

A finite faithfully flat morphism of formal schemes or adic spaces satisfies total descent with respect to coherent sheaves.

Theorem (Giraud, Deligne, Saint-Donat, Conrad)

Morphisms that satisfy total descent (with respect to $\mathcal{M}$) define a topology which is finer than the original topology of $\mathbb{B}$:

- *A local epimorphism satisfies total descent,*
- *A morphism dominated by a morphism that satisfies total descent automatically satisfies total descent,*
- *The property is stable under composition,*
- *The property is stable under base change.*

Descent (on adic spaces)

We consider here the case where the base is $\mathbb{B} := \mathbf{Ad}^\dagger$ and $\mathcal{M}$ is a fibered category of crystals (a crystal on (X, V) is essentially an $i_X^{-1}\mathcal{O}_V$-module).

Lemma

A morphism $(Y, W) \to (X, V)$ such that $(]Y[_W, i_Y^{-1}\mathcal{O}_W) \simeq (]X[_V, i_X^{-1}\mathcal{O}_V)$ satifies total descent with respect to crystals.

Proposition

A morphism of analytic spaces

$$(Y \hookrightarrow Q \leftarrow W) \to (X \hookrightarrow P \leftarrow V)$$

which is left and right cartesian with $Q \to P$ finite faithfully flat satisfies total descent with respect to constructible isocrystals.

Proof.

Requires a delicate study of extensions of constructible isocrystals in order to reduce to the finitely presented case. □

Descent (on adic sites)

We consider now the case where $\mathbb{B} = \widehat{\mathbf{An}^\dagger}$ is the category of analytic overconvergent sites which are fibred in sets (or ideally, the 2-category $\mathbb{F}\mathrm{ib}(\mathbf{An}^\dagger)$ of all analytic overconvergent sites).

Lemma

If $X = \cup X_i$ is an open or closed Zariski covering, then $\coprod X_i^{\dagger,\mathrm{an}} \to X^{\dagger,\mathrm{an}}$ satisfies total descent with respect to isocrystals.

Now, we use Tsuzuki's induction method:

Lemma

A morphism $(Y/V)^\dagger \to (X,V)$, where $f : Y \to X$ is

1. *birational and partially proper, or*
2. *finite surjective*

and $\dim(Y) \leq d$, satisfies total descent with respect to constructible isocrystals if this is the case whenever f is partially proper surjective with $\dim(Y) < d$.

Proof of the lemma

Proof.

1. (birational and partially proper) We first reduce to the blowing up of a closed subscheme Z of X. It extends to a blowing up $Q \to P$ of the closure $\overline{Z}$ of Z. We may now assume that either $]\overline{Z}[_V =]\overline{X}[_V$ or $\overline{Z}_V = \emptyset$. The first case is obtained by induction. The other one comes from the fact that a blowing up being algebraic can only modify the fiber in the tube.
2. (finite surjective) We first reduce to the case f flat by birational arguments (using the first part). Raynaud-Gruson flatening techniques then provide a finite faithfully flat lift $Q \to P$. We can finally use our finite faithfully flat descent on analytic overconvergent spaces. □

The proof is purely formal and uses constructible isocrystals only for the last argument. Up to this point, we only relied on geometric constructions using the following:

- Morphism that satisfy total descent define a topology which is finer than the topology of the site,
- Open and closed Zariski coverings satisfy total descent.

Conclusion

The *h-topology* on the category of formal schemes is the topology generated by Zariski open coverings and partially proper surjective morphisms.

Theorem

If $\{X_i \to X\}_{i \in I}$ is an h-covering, then $\coprod_{i \in I} X_i^{\dagger,\mathrm{an}} \to X^{\dagger,\mathrm{an}}$ satisfies total descent for constructible isocrystals.

In other words, the topology of total descent with respect to constructible isocrystals is finer than the *h*-topology. Also, as a consequence, constructible isocrystals form a stack for the *h*-topology.

Corollary

Morphisms that are

- *faithfully flat locally formally of finite type, or*
- *partially proper surjective*

satisfy total descent with respect to constructible isocrystals.

Concretely

We give ourselves a morphism of formal schemes $X \to C$. We embed C into a formal scheme S. We give ourselves an analytic space O and a morphism $O \to S^{\mathrm{ad}}$.

Example

$C = \mathrm{Spec}(k)$ an X and algebraically variety over k $(\mathrm{Char}(k) = p > 0)$.
$S = \mathrm{Spf}(\mathcal{V})$ where $\mathcal{V}$ is a complete discrete valuation ring with residue field k.
$O = \mathrm{Spa}(K, \mathcal{V})$ where K is the fraction field of $\mathcal{V}$ $(\mathrm{Char}(K) = 0)$.

We give ourselves an h-covering $\{X_i \to X\}_{i \in I}$. Then, $\coprod_{i \in I} (X_i/O)^\dagger \to (X/O)^\dagger$ satisfies total descent with respect to constructible isocrystals.

Example

Overconvergent isocrystals satisfy total descent with respect to étale coverings, faithfully flat morphisms and proper surjective morphisms.

Remark: with a skeleton/coskeleton argument, in our setting, it is straightforward to extend all these results to hypercoverings.

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