Constructible isocrystals
(London – 2015)

Bernard Le Stum

Université de Rennes 1

March 30, 2015

Contents

The geometry behind

Overconvergent connections

Construtibility

A correspondance

Valuations (additive notation)

When $f = \sum_i a_i T^i \in \mathbb{Q}[T]$ and p is a prime number, we may consider

- *Tate* points: e.g. if $c \in \mathbb{Z}$, $v_c(f) = v_p(\sum a_i c^i) \in \overline{\mathbb{Z}}$.
- *Berkovich* points: e.g. $v_\xi(f) = \min v_p(a_i) \in \overline{\mathbb{Z}}$.
- *Huber* points: e.g. $v_{\xi_-}(f) = (v_\xi(f), \min\{i, v_\xi(f) = v_p(a_i)\}) \in \overline{\mathbb{Z} \oplus \mathbb{Z}}$ with lexicographic order.

We could use any theory of *analytic spaces* but we choose Huber's : e.g. $\mathbb{D}(0, 1^+)$ is the set of continuous ($v_x(p^k) \to +\infty$) bounded ($v_x(T) \geq 0$) valuations on $\mathbb{Q}[T]$ (up to equivalence). Then we have a disjoint union

$$\mathbb{D}(0, 1^+) = \mathbb{D}(0, 1^-) \cup \{\xi_-\} \cup \mathbb{A}(0; 1)$$

where

$$\mathbb{D}(0, 1^-) := \{x, v_x(T^k) \to +\infty\} \text{ and } \mathbb{A}(0; 1) := \{x, v_x(T) = 0\}$$

are both *open* in $\mathbb{D}(0, 1^+)$ and "hooked" by the *closed* point ξ_-.

Tubes

Let K be a complete non archimedean field of characteristic 0 with valuation ring $\mathcal{V}$, maximal ideal $\mathfrak{m}$ and residue field k. It is convenient to fix some non zero $\pi \in \mathfrak{m}$.

Then, any formal $\mathcal{V}$-scheme P (with locally finitetely generated ideal of definition) has a *special fiber* P_k which is a scheme over k and a *generic fibre* P_K which is an analytic space over K.

More precisely, there exists a *fully faithful* functor $P \mapsto P^{\mathrm{ad}}$ from formal $\mathcal{V}$-schemes to $\mathcal{V}$-adic spaces, and $P_K := P_K^{\mathrm{ad}} = P^{\mathrm{ad}} \otimes_{\mathcal{V}} K$ is just the generic fiber of P^{ad} in the category of adic spaces over $\mathcal{V}$.

Actually, if Z is a *closed* subscheme of P_k (locally defined by a finitely generated ideal modulo $\mathfrak{m}$), we may consider the completion $\mathcal{Z} := \widehat{P}^{/Z}$ of P along Z and define the *tube* of Z in P as $]Z[_P := \mathcal{Z}_K$. This definition extends to constructible subsets X of P_k by boolean combination. For example, we have $]\mathcal{U}_k[_P = \overline{\mathcal{U}_K}$ if $\mathcal{U}$ is a (retrocompact) open subset of P.

Local description

If A is a topological $\mathcal{V}$-algebra (with finitely generated ideal of definition), then $P := \mathrm{Spf}(A)$ is the set of open prime ideals in A and $P^{\mathrm{ad}} = \mathrm{Spa}(A)$ is the set of (everywhere) bounded continuous valuations on A.

When A is π-adically of finite type, then $P_K = \mathrm{Spa}(A \otimes_{\mathcal{V}} K)$.

If Z is the closed subscheme defined by f modulo $\mathfrak{m}$, then $\mathcal{Z}$ is the completion of P along (f, π) and $\mathcal{Z}^{\mathrm{ad}}$ (resp. $]Z[_P$) is the set of $x \in P^{\mathrm{ad}}$ such that $v_x(f^k) \to +\infty$ (and $v_x(\pi) \neq \infty$). Note that this condition is *stronger* than $v_x(f) > 0$ when the height of the valuation is ≥ 2.

Example

1. $\mathrm{Spa}(\mathcal{V}) \simeq \mathrm{Spec}(\mathcal{V})$ and $\mathrm{Spa}(K) \simeq \mathrm{Spec}(K)$.
2. If $P := \widehat{\mathbb{A}}^1_{\mathcal{V}}$, then $P_K = \mathbb{D}(0, 1^+)$ (as above).
3. If $P := \widehat{\mathbb{P}}^n_{\mathcal{V}}$, then $P_K = \mathbb{P}^{n,\mathrm{an}}_K$.
4. If $X := \{0\} \subset P := \widehat{\mathbb{A}}^1_{\mathcal{V}}$, then $]X[_P = \mathbb{D}(0, 1^-)$ (as above).

Overconvergent functions

We consider a pair

$$(X \hookrightarrow P, P_K \xleftarrow{\lambda} V)$$

where the first map is the embedding of a constructible subscheme X into a formal $\mathcal{V}$-scheme P and λ is any morphism from an analytic K-space V.

Definition

The *tube* of X in V is $]X[_V := \lambda^{-1}(]X[_P)$.

We denote by

$$i_X :]X[_V \hookrightarrow V$$

the inclusion map (so that $i_{X*}i_X^{-1} =: j_X^\dagger$).

Definition

The sheaf $i_X^{-1}\mathcal{O}_V$ is the sheaf of *overconvergent* functions on the tube.

Example (Berthelot)

If

$$A := \mathcal{V}[t_1, \ldots, t_n]/(f_1, \ldots, f_r),$$

and we set $A_k := k \otimes_{\mathcal{V}} A$ and $A_K := K \otimes_{\mathcal{V}} A$, we can consider

$$\begin{array}{ccc} (X := \mathrm{Spec}(A_k) & \hookrightarrow & \widehat{\mathbb{P}^n_{\mathcal{V}}} \\ \| & & \\ \{\bar{f}_i = 0\} \subset \mathbb{A}^n_k & & \end{array} \quad , \qquad \begin{array}{ccc} \mathbb{P}^{n,\mathrm{an}}_K & \hookleftarrow & (\mathrm{Spec}(A_K))^{\mathrm{an}} =: V) \\ & & \| \\ & & \{f_i = 0\} \subset \mathbb{A}^{n,\mathrm{an}}_K \end{array}$$

We have

$$]X[_V = V \cap \overline{\mathbb{B}^n(0, 1^+)} = \{f_i = 0, v(t_j^k) >> -\infty\}.$$

One easily sees that the subsets

$$V_\lambda := V \cap \mathbb{B}^n(0, \lambda^+) := \{f_i = 0, v(\pi t_j^N) \geq 0\}$$

Example (Continuing)

with $\lambda = |\pi|^{-1/N}$, form a fundamental system of affinoid neighborhoods of $]X[_V$ in V. Actually, if we set

$$A_\lambda := \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}/(f_1, \ldots, f_r),$$

then we have $A_{\lambda K} := K \otimes_{\mathcal{V}} A = \Gamma(V_\lambda, \mathcal{O}_{V_\lambda})$. If we define

$$\mathcal{V}[t_1, \ldots, t_n]^\dagger = \left\{ \sum_{\underline{i} \in \underline{\mathbb{N}}} a_{\underline{i}} \underline{t}^{\underline{i}}, \quad \exists \lambda > 1, |a_{\underline{i}}| \lambda^{|\underline{i}|} \to 0 \right\}$$

and set $A^\dagger := \mathcal{V}[t_1, \ldots, t_n]^\dagger/(f_1, \ldots, f_r)$, then we will have

$$\Gamma(]X[_P, i_X^{-1}\mathcal{O}_V) = \varinjlim A_{\lambda K} = A_K^\dagger := K \otimes_{\mathcal{V}} A^\dagger.$$

Example (Lazda-Pal)

We endow $\mathcal{V}[[t]]$ with the π-adic topology (and *not* the (π, t)-adic topology) and consider

$$\left(X := \mathrm{Spec}(k((t))) \hookrightarrow P := \mathrm{Spf}(\mathcal{V}[[t]]) \quad , \quad P_K \stackrel{\lambda}{=} V\right).$$

Note that $V = P_K = \mathbb{D}(0, 1^{\mathrm{b}}) := \mathrm{Spa}(K[[t]]^{\mathrm{b}})$ with

$$K[[t]]^{\mathrm{b}} := \left\{\sum_{i \in \mathbb{N}} a_i t^i, \quad |a_i| << +\infty\right\}.$$

As topological space, we can write (where ξ is an *open* point):

$$\mathbb{D}(0, 1^{\mathrm{b}}) = \mathbb{D}(0, 1^{-}) \cup \{\xi_{-}\} \cup \{\xi\}.$$

Example (Continuing)

We have

$$]X[_P = \{x \in V, v_x(t^k) << +\infty\}$$

and the affinoid open subsets

$$V_\lambda := \{x \in P_K, v_x(t^N/\pi) \leq 0\},$$

with $\lambda = |\pi|^{-1/N}$, form a fundamental system of affinoid neighborhoods of $]X[_P$. Moreover,

$$\Gamma(V_\lambda, \mathcal{O}_{V_\lambda}) = \mathcal{R}^{\mathrm{b}}_\lambda := \left\{ \sum_{i \in \mathbb{Z}} a_i t^i, |a_i| << +\infty, |a_i|/\lambda^i \to 0 \right\},$$

and it follows that

$$\Gamma(]X[_P, i_X^{-1}\mathcal{O}_V) = \varinjlim \mathcal{R}^{\mathrm{b}}_\lambda = \mathcal{R}^{\mathrm{b}}$$

is the *bounded Robba ring*.

Geometric setting

For more generality, we work in a relative situation

$$\begin{array}{ccc} X & \hookrightarrow & P \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} \\ C & \hookrightarrow & S \end{array} \quad , \qquad \begin{array}{ccc} P_K & \leftarrow & V \\ \downarrow{\scriptstyle u_k} & & \downarrow{\scriptstyle v} \\ S_K & \leftarrow & O \end{array}$$

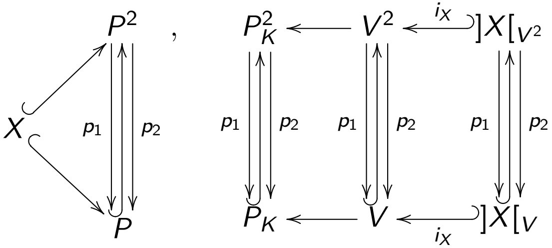
where the upper line is as above, the lower line is of the same type, and the diagrams are commutative.

We may also embed diagonally X into $P^2 = P \times_S P$. Then, we will have a map

$$P_K \times_{S_K} P_K = P_K^2 \leftarrow V^2 = V \times_O V$$

and we may consider the following commuative diagrams:

Diagonal embedding



where the climbing arrows Δ denote the diagonal embeddings and the p_i's denote the projections.

Note that we can also draw the same kind of diagram with the triple products and the projections $p_{ij} : P \times_S P \times_S P \to P \times_S P$.

Overconvergence

Definition

An *overconvergent stratification* on an $i_X^{-1}\mathcal{O}_V$-module E is an isomorphism of $i_X^{-1}\mathcal{O}_{V^2}$-modules on $]X[_{V^2}$

$$\epsilon : p_2^\dagger E := i_X^{-1} p_2^* i_{X*} E \simeq i_X^{-1} p_1^* i_{X*} E =: p_1^\dagger E$$

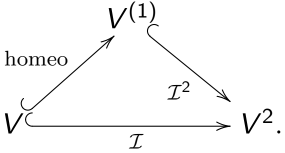
satisfying

$$p_{13}^\dagger(\epsilon) = p_{12}^\dagger(\epsilon) \circ p_{23}^\dagger(\epsilon) \quad \text{on} \quad]X[_{V^3} \quad \text{and} \quad \Delta^\dagger(\epsilon) = \mathrm{Id} \quad \text{on} \quad]X[_V.$$

These are called the *cocycle* and *normalization* conditions.

Infinitesimal neighborhoods

If V is defined by the ideal $\mathcal{I}$ in V^2, the *first infinitesimal neighborhood* of V is the subscheme $V^{(1)}$ defined by $\mathcal{I}^2$:



There exists an exact sequence

$$0 \to \Omega^1_V \to \mathcal{O}_{V^{(1)}} \to \mathcal{O}_V \to 0.$$

We will denote by $p_1^{(1)}, p_2^{(1)} : V^{(1)} \to V$ the maps induced by the projections. Then the derivation on $\mathcal{O}_V$ is defined by

$$\mathrm{d}f = p_2^{(1)*}(f) - p_1^{(1)*}(f) \in \Omega^1_V.$$

Overconvergent connection

If an $i_X^{-1}\mathcal{O}_V$-module E is endowed with an overconvergent stratification $\epsilon : p_2^\dagger E \simeq p_1^\dagger E$, we can restrict it to $V^{(1)}$ and get

$$\begin{array}{ccccccc} E & \hookrightarrow & p_2^{(1)\dagger}E & \xrightarrow{\simeq} & p_1^{(1)\dagger}E & = & E \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(1)}} \\ m & \longmapsto & 1 \otimes m & & \longmapsto & & m \otimes 1 + \nabla(m) \end{array}$$

where

$$\nabla : E \to E \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_V$$

happens to be an *integrable connection* on E that will be said to be *overconvergent*.

We call E an *overconvergent isocrystal* if *moreover* E is locally finitely presented as an $i_X^{-1}\mathcal{O}_V$-module (Berthelot). We want to relax this condition a little bit in order to get more flexibility.

Constructible modules

If Y is a constructible subscheme of X, we may consider

$$
\begin{array}{ccc}
Y & \hookrightarrow & P \\
{\scriptstyle\alpha}\downarrow & & \| \\
X & \hookrightarrow & P \\
\downarrow & & \downarrow \\
C & \hookrightarrow & S
\end{array}
,\qquad
\begin{array}{ccccc}
P_K & \longleftarrow & V & \hookleftarrow &]Y[_V \\
\| & & \| & & \downarrow{\scriptstyle]\alpha[} \\
P_K & \longleftarrow & V & \hookleftarrow &]X[_V \\
\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle p} \\
S_K & \longleftarrow & O & \hookleftarrow &]C[_O
\end{array}
$$

Definition

An $i_X^{-1}\mathcal{O}_V$-module E is *constructible* if there exists a locally finite covering of X by constructible subschemes Y such that, if $\alpha : Y \hookrightarrow X$ denotes the inclusion map, then $]\alpha[^{-1}E$ is a locally finitely presented $i_Y^{-1}\mathcal{O}_V$-module.

Constructible isocrystals

Definition

A *constructible isocrystal* on X/O is a constructible $i_X^{-1}\mathcal{O}_V$-module E endowed with an overconvergent integrable connection.

A constructible isocrystal can be described by an exact sequence

$$0 \longrightarrow]\beta[_! E' \longrightarrow E \longrightarrow]\alpha[_* E'' \longrightarrow 0$$

(when X is noetherian) where E' is a constructible isocrystal on a closed subscheme Z of X, E'' is an overconvergent isocrystal on its dense open complement U and $\alpha : U \hookrightarrow X, \beta : Z \to X$ denote the inclusion maps. Such an exact sequence corresponds to a unique morphism

$$E'' \to]\alpha[^{-1}]\beta[_* E'$$

through a pull back of the standard exact sequence

$$0 \longrightarrow]\beta[_! E' \longrightarrow]\beta[_* E' \longrightarrow]\alpha[_*]\alpha[^{-1}]\beta[_* E' \longrightarrow 0\ .$$

overconvergent connection again

In order to give a more down to earth description of constructible isocrystals in standard situations, we recall the following:

Definition

Let A be a smooth affine $\mathcal{V}$-algebra. An integrable connection ∇ on a coherent $A_K^\dagger$-module M is said to be *overconvergent* if it satisfies locally

$$\lim_{\lambda \to 1} \inf_{s \in M} \inf\{\lambda, \underline{\lim}_k \| \frac{1}{\underline{k}!} \nabla^{\underline{k}}(s) \|_\lambda^{-\frac{1}{k}}\} = 1.$$

Such an M is then called an *overconvergent isocrystal* on $A_K^\dagger$.

As a particular case, if K' is a finite extension of K, an *(overconvergent) isocrystal* on K' is simply a finite dimensional K'-vector space H.

Constructible isocrystals on a curve

We assume from now on that $\mathcal{V}$ is a discrete valuation ring, that $S = \mathrm{Spf}\ \mathcal{V}$, that $O = \mathrm{Spa}(K)$, that k is perfect and that P is proper smooth in the neighborhood of X.

If x is a closed point of X, then we denote by $K(x)$ an unramified lifting of $k(x)$ and by $\mathcal{R}_x$ be the *Robba ring* over $K(x)$.

Theorem

A constructible isocrystal on a non singular curve X is given by:

1. *An overconvergent isocrystal M on $A_K^\dagger$, where A is a smooth affine $\mathcal{V}$-algebra such that $U = \mathrm{Spec}\ A_k$ is an affine open subset of X.*
2. *An isocrystal H_x on $K(x)$ when x is not in U.*
3. *A horizontal morphism $\mathcal{R}_x \otimes_{A_K^\dagger} M \to \mathcal{R}_x \otimes_{K(x)} H_x$ for any such x.*

Morphisms are defined after shrinking U if necessary (and

Frobenius structure

We assume from now on that k has characteristic $p > 0$ and that K is endowed with a lifting σ of the Frobenius of k.

We recall the following :

Definition

Let A be a smooth affine $\mathcal{V}$-algebra and F^* a σ-linear lifting to $A^\dagger$ of the Frobenius of A_k. Then, an *F-isocrystal* on $A_K^\dagger$ is a coherent $A_K^\dagger$-module M endowed with an integrable connection ∇ and a horizontal isomorphism $F_K^* M \simeq M$.

Note that the connection will automatically be overconvergent.

As a particular case, if K' is a finite extension of K and σ' a σ-linear lifting of Frobenius to K', an *F-isocrystal* on K' is just a finite dimensional K'-vector space H endowed with an isomorphism $\sigma'^* H \simeq H$.

Constructible F-isocrystals

When X is locally of finite type, the notion of constructible isocrystal is functorial in X. Therefore, if we denote by $F_{X/k}$ the relative Frobenius of X/k, we can make the following definition:

Definition

A *constructible F-isocrystal* is a constructible isocrystal E endowed with a *Frobenius* isomorphism $\Phi : F^*_{X/k}E \simeq E$.

Theorem

A constructible F-isocrystal on a non singular curve X is given by:

1. *An F-isocrystal M on $A^\dagger_K$, where A is a smooth affine $\mathcal{V}$-algebra such that $U = \mathrm{Spec}\ A_k$ is an affine open subset of X.*
2. *An F-isocrystal H_x on $K(x)$, when x is not in U.*
3. *A horizontal morphism $\mathcal{R}_x \otimes_{A^\dagger} M \to \mathcal{R}_x \otimes_{K(x)} H_x$ for any such x, which is compatible with the Frobenius structures.*

Comparison theorem

Conjecture

Derived specialization induces an equivalence between constructible F-isocrystals and perverse holonomic arithmetic $\mathcal{D}$-modules.

Theorem

Conjecture holds when X is a non-singular curve as above.

Let us be more precise. We may assume that X is projective and choose $P := \mathcal{X}$, a smooth lifting of X.

Then, if E is a constructible F-isocrystal on $\mathcal{X}_K$, we first have to show that $\mathcal{E} := \mathrm{Rsp}_* E$ is *perverse* in the sense that

- $\mathcal{H}^0(\mathcal{E})$ is flat over $\mathcal{O}_{\mathcal{X}\mathbb{Q}}$,
- $\mathcal{H}^1(\mathcal{E})$ has finite support
- $\mathcal{H}^i(\mathcal{E}) = 0$ for $i \neq 0, 1$.

More

We must also show that $\mathcal{E}$ has a structure of $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-module; the latest is locally the set of

$$P = \sum_{k=0}^{\infty} \frac{1}{k!} f_k \frac{\partial^k}{\partial t^k}, \quad \exists c > 0, \quad \exists \eta < 1, \quad \|f_k\| \leq c\eta^{|k|}$$

with $f_k \in \mathcal{O}_{\mathcal{X}\mathbb{Q}}$.

Next, one must show (and we need the local monodromy theorem at that point) that $\mathcal{E}$ is holonomic in the sense that it is $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-coherent and that $\mathcal{E}(^\dagger Z)$ is $\mathcal{O}_{\mathcal{X}\mathbb{Q}}(^\dagger Z)$-coherent for some finite set Z; the latest is locally the set of

$$\sum_{l=0}^{\infty} \frac{s_l}{g^l}, \quad \exists c > 0, \quad \exists \eta < 1, \quad \|s_l\| \leq c\eta^l,$$

where $g = 0 \mod \mathfrak{m}$ is a local equation for Z.

Finally, it will remain to show that we do obtain an equivalence.

Proof of the theorem

How does it work exactly ?

Note that there exists

- a finite set Z of closed points of X
- for each $x \in Z$, a finite dimensional vector space H_x
- an overconvergent isocrystal E_U on the complement U of Z, and
- an exact sequence

$$0 \to \sum_{x \in Z} h_{x!}(\mathcal{O}_{]x[} \otimes_{K(x)} H_x) \to E \to E_U \to 0$$

where $h_x :]x[\hookrightarrow \mathcal{X}_K$ denotes the inclusion of the residue class at x.

Berthelot proved that $\mathrm{Rsp}_* E_U$ is a $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-module and one can check that

Proof again

$$\mathrm{Rsp}_*(h_{x!}(\mathcal{O}_{]x[}\otimes_{K(x)} H)) \simeq \tilde{h}_{x+}H$$

with $\tilde{h}_x : \mathrm{Spf}\mathcal{V}(x) \hookrightarrow \mathcal{X}$.

It easily follows that $\mathcal{E}$ is a perverse $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-module. And the Frobenius structure will make it holonomic: thanks to the local monodromy theorem, $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}$-coherent.

Then, in order to get an equivalence of categories, it is sufficient to prove that if E_1 and E_2 are constructible, we have

$$\mathrm{Ext}^i_\nabla(E_1, E_2) \simeq \mathrm{Ext}^i_{\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}}(\mathrm{Rsp}_*E_1, \mathrm{Rsp}_*E_2).$$

The case $i = 0$ will give fulfaithfulness and the case $i = 1$ will give essential surjectivity.

The whole thing relies on the next result:

Proof continuing

Theorem

If E is an overconvergent F-isocrystal on $U \subset X$ and $x \notin U$, then

$$\mathrm{RHom}_{\nabla}\left(E, h_{x!}\mathcal{O}_{]x[}\right) = \mathrm{Hom}_{\nabla}\left(\mathcal{R}_x(E), \mathcal{R}_x\right)[-1] \quad \text{and}$$

$$\mathrm{RHom}_{\mathcal{D}^{\dagger}_{\mathcal{X}\mathbb{Q}}}\left(\mathrm{sp}_* E, \tilde{h}_{x+}K(x)\right) = \mathrm{Hom}_{\nabla}(\mathcal{R}_x(E), \mathcal{R}_x)[-1].$$

In this last statement, we wrote $\mathcal{R}_x(E) := \mathcal{R}_x \otimes_{A_K^{\dagger}} M$ with $A_K^{\dagger} := \Gamma(]U[, i_U^{-1}\mathcal{O}_{\mathcal{X}_K})$ and $M := \Gamma(]U[, E)$ (this is the *Robba fiber of E at x*).

On the isocrystal side, the isomorphism is obtained by adjunction because $\mathcal{R}_x$ si nothing but the stalk of $h_{x*}\mathcal{O}_{]x[}$ at the (tube of) the generic point.

End of the proof

On the $\mathcal{D}$-module side, we need to introduce the bounded Robba ring $\mathcal{R}_x^{\mathrm{bd}}$ and the formal completion $\hat{x} \simeq \mathrm{Spf}\mathcal{V}(x)[[t]]$ of x inside $\mathcal{X}$.

Then, if we set $\mathcal{E} := \mathrm{sp}_* E$, we first have by adjunction

$$\mathrm{RHom}_{\mathcal{D}^\dagger_{\mathcal{X}\mathbb{Q}}}(\mathcal{E}, \tilde{h}_{x+}K(x)[1]) = \mathrm{RHom}_{\mathcal{D}^\dagger_{\widehat{x}\mathbb{Q}}}(\mathcal{R}_x^{\mathrm{bd}}(\mathcal{E}), \delta_x)$$

with $\mathcal{R}_x^{\mathrm{bd}}(\mathcal{E}) := \mathcal{R}_x^{\mathrm{bd}} \otimes_{A_K^\dagger} M$ (this is the *bounded Robba fiber at* x) and

$$\delta_x := \mathcal{R}_x^{\mathrm{bd}} / \mathcal{O}_{\hat{x}\mathbb{Q}} \simeq \mathcal{R}_x / \mathcal{O}_{]x[}.$$

This follows from the fact that $\mathcal{R}_x^{\mathrm{bd}}$ is exactly the ring of functions on $\widehat{x}$ that are overconvergent along x. In order to finish the proof, we rely on a theorem of Crew which states that

$$\mathrm{RHom}_{\mathcal{D}^\dagger_{\widehat{x}\mathbb{Q}}}(\mathcal{R}_x^{\mathrm{bd}}(\mathcal{E}), \mathcal{O}_{]x[}) = 0.$$