Université de Rennes 2026-2027

Homological algebra
Homework (due Novermber 3rd)

You can compose in english or in french. You can freely (no need to make a precise reference) use any result proved in the (online) course.

We recall that $\mathbf{\Delta}$ denotes the category of ordered integers $[n] := \{0, 1, \dots n\}$ for $n \in \mathbb{N}$ with order preserving maps as morphisms.

1. Show that, for fixed $m, n \in \mathbb{N}$, a morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ is uniquely determined by the sets

$$I(\alpha) = \{i \in [n], \forall j \in [m], \alpha(j) \neq i\} \quad \text{and} \quad J(\alpha) = \{j \in [m-1], \alpha(j) = \alpha(j+1)\}.$$

Solution: We proceed by induction on n, the case $n = 0$ being trivial. We have $\alpha^{-1}(n) = \{m'+1, \dots, m\}$ with $m' \leq m$ and α is uniquely determined by (m' and) the induced map $\alpha' : [m'] \to [n-1]$. By induction, α' is uniquely determined by $I(\alpha') = I(\alpha) \smallsetminus \{n\}$ and $J(\alpha') = J(\alpha) \smallsetminus \{m', \dots, m-1\}$. It follows that α is uniquely determined by $I(\alpha)$ and $J(\alpha)$.

We shall denote by

$$\delta_i^n : [n-1] \to [n] \quad (\text{resp. } \sigma_i^n : [n+1] \to [n])$$

ith *face* (resp. *degeneracy*) map that forgets (resp. repeats) i.

2. Show that any morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ can be uniquely written as

$$\alpha = \delta_{i_s}^n \delta_{i_{s-1}}^{n-1} \cdots \delta_{i_1}^{n-s+1} \sigma_{j_t}^{m-t} \sigma_{j_{t-1}}^{m-t+1} \cdots \sigma_{j_1}^{m-1}$$

with $m - 1 \geq j_1 > j_2 > \ldots > j_t \geq 0$ and $0 \leq i_1 < i_2 < \ldots < i_s \leq n$ (and $n - s = m - t$).

Solution: It is sufficient to show that, in the above formula, we have

$$I(\alpha) = \{i_1, \dots, i_s\} \quad \text{and} \quad J(\alpha) = \{j_1, \dots, j_t\}.$$

This can be proved by induction on s. Writing $\alpha = \delta_{i_s}^n \alpha'$, we will have

$$\alpha(j) = \delta_{i_s}^n(\alpha'(j)) = \begin{cases} \alpha'(j) \text{ if } \alpha'(j) < i_s \\ \alpha'(j) + 1 \text{ otherwise.} \end{cases}$$

By induction, $I(\alpha') = \{i_1, \dots, i_{s-1}\}$ and $J(\alpha') = \{j_1, \dots, j_t\}$. A quick inspection then shows that the property is hereditary. In order to initialize the induction process, it is also necessary to consider the case $\alpha = \sigma_{j_t}^{m-t} \sigma_{j_{t-1}}^{m-t+1} \cdots \sigma_{j_1}^{m-1}$ which can be done, similarly, by induction on t. Details omitted.

We recall that a *simplicial set* is a sequence of sets S_n endowed with maps $d_n^i : S_n \to S_{n-1}$ for $n > 0$ and $s_n^i : S_n \to S_{n+1}$ for $n \geq 0$ with $0 \leq i \leq n$ satisfying

$$\begin{aligned}
&d_n^i \circ d_{n+1}^j = d_n^{j-1} \circ d_{n+1}^i \quad \text{for } 0 \leq i < j \leq n+1,\\
&s_{n+1}^i \circ s_n^j = s_{n+1}^{j+1} \circ s_n^i \quad \text{for } 0 \leq i \leq j \leq n,\\
&d_{n+1}^i \circ s_n^j = \begin{cases} s_{n-1}^{j-1} \circ d_n^i & \text{for } 0 \leq i < j \leq n \\ \mathrm{Id}_{S_n} & \text{for } 0 \leq j \leq i \leq j+1 \leq n+1 \\ s_{n-1}^{j} \circ d_n^{i-1} & \text{for } 0 \leq j < i-1 \leq n. \end{cases}
\end{aligned}$$

Moreover, if $T_\bullet$ is another simplicial set, then a *map of simplicial sets* $f : S_\bullet \to T_\bullet$ is a family of maps $f_n : S_n \to T_n$ satisfying $f_{n-1} \circ d_n^i = d_n^i \circ f_n$ for $n > 0$ and $f_{n+1} \circ s_n^i = s_n^i \circ f_n$ for $n \geq 0$.

3. Show that simplicial sets form a category **sS**et isomorphic to the category $\widehat{\mathbf{\Delta}} := \mathbf{H}\mathrm{om}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{S}\mathrm{et})$ of presheaves of sets on $\mathbf{\Delta}$.

Solution: If we are given two maps of simplicial sets $f : S_\bullet \to T_\bullet$ and $g : T_\bullet \to R_\bullet$, then

$$g_{n-1} \circ f_{n-1} \circ d_n^i = g_{n-1} \circ d_n^i \circ f_n = d_n^i \circ g_n \circ f_n$$

and

$$g_{n+1} \circ f_{n+1} \circ s_n^i = g_{n+1} \circ s_n^i \circ f_n = s_n^i \circ g_n \circ f_n$$

showing that term by term composition of maps of simplicial sets is well defined. The conditions for **sS**et to be a category (identity and associativity) are then readily checked.

If S is a presheaf of sets on $\mathbf{\Delta}$, we set $S_n := S([n]) = S_n$, $d_n^i := S(\delta_i^n)$ and $s_n^i := S(\sigma_i^n)$. Since

$$\begin{aligned}
&\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad \text{for } 0 \leq i < j \leq n+1,\\
&\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad \text{for } 0 \leq i \leq j \leq n,\\
&\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & \text{for } 0 \leq i < j \leq n \\ \mathrm{Id}_{[n]} & \text{for } 0 \leq j \leq i \leq j+1 \leq n+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & \text{for } 0 \leq j < i-1 \leq n, \end{cases}
\end{aligned}$$

we obtain a simplicial set by applying the contravariant functor S. A morphism of presheaves $f : S \to T$ is a natural transformation and if we set $f_n := f_{[n]}$, we do get a map of simplicial sets because

$$f_{[n-1]} \circ S(\delta_i^n) = S(\delta_i^n) \circ f_n \quad \text{and} \quad f_{[n+1]} \circ S(\sigma_i^n) = S(\sigma_i^n) \circ f_n.$$

This construction preserves also the identity. Moreover, if $g : T \to R$ is another morphism, then $(g \circ f)_{[n]} = g_{[n]} \circ f_{[n]}$ by definition of the composition of natural transformations. This defines a functor $\widehat{\mathbf{\Delta}} \to \mathbf{sS}\mathrm{et}$.

This functor is clearly injective on objects and it follows from question 2 that it is also fully faithful. In order to show that it is an isomorphism, it is sufficient

to prove that it is surjective on objects. Given a simplicial set $S_\bullet$, we must set $S([n]) := S_n$. Moreover, if we have a canonical decomposition

$$\alpha = \delta^n_{i_s}\delta^{n-1}_{i_{s-1}}\cdots\delta^{n-s+1}_{i_1}\sigma^{m-t}_{j_t}\sigma^{m-t+1}_{j_{t-1}}\cdots\sigma^{m-1}_{j_1}$$

as in question 2, then we must set

$$S(\alpha) = s^{j_1}_{m-1}s^{j_2}_{m-2}\cdots s^{j_t}_{m-t}d^{i_1}_{n-s+1}d^{i_2}_{n-s+2}\cdots d^{i_s}_n.$$

We need to show that we will always have $S(\beta\circ\alpha) = S(\alpha)\circ S(\beta)$. By induction, it is sufficient to consider the cases where α and β are either face or degeneracy maps. This is exactly the content of the commutation rules.

We shall make the *identification* $\mathbf{sSet} = \widehat{\mathbf{\Delta}}$ and write $S_n := S([n])$. We shall aslo write $\Delta^n := h_{[n]} : [m] \mapsto \mathrm{Hom}([m],[n])$ so that $\Delta^n \in \mathbf{sSet}$.

For $S \in \mathbf{sSet}$, we consider the following category $\mathbf{\Delta}_{/S}$:

- an object is an $s \in S_n$ for *some* $n \in \mathbb{N}$,
- a morphism from $t \in S_m$ to $s \in S_n$ is a morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ such that $S(\alpha)(s) = t$.

We then consider the diagram in $\mathbf{sSet}$ indexed by $\mathbf{\Delta}_{/S}$ sending $s \in S_n$ to Δ^n and $\alpha : [m] \to [n]$ to $h_\alpha : \Delta^m \to \Delta^n$.

4. Show that if $S \in \mathbf{sSet}$, then $S \simeq \varinjlim_{\mathbf{\Delta}_{/S}} \Delta^n$.

Solution: It follows from Yoneda lemma that

$$S_n = S([n]) \simeq \mathrm{Hom}(h_{[n]}, S) = \mathrm{Hom}(\Delta^n, S)$$

so that any $s \in S_n$ corresponds to a unique morphism $\widetilde{s} : \Delta^n \to S$. Moreover, if we are given $t \in S_m$ and $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, then condition $S(\alpha)(s) = t$ is equivalent to $\widetilde{t} = \widetilde{s}\circ h_\alpha$. It follows that S is a cocone for our diagram. It is clearly universal: an $f_s : \Delta^n \to S'$ corresponds to a unique $f(s) \in S'_n$ by Yoneda lemma again. A compatible family of such f_s will provide a morphism $S \to S'$ and one easily checks that its solves our problem. Details omited.

We let $\mathcal{T}$ be a cocomplete category and $r : \mathbf{\Delta} \to \mathcal{T}$ a functor.

5. Show that
   (a) the functor

$$G : \mathcal{T} \to \mathrm{Hom}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}) = \mathbf{sSet}$$

   obtained by currying from

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathcal{T} \xrightarrow{r^{\mathrm{op}}\times \mathrm{Id}} \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \xrightarrow{\mathrm{Hom}} \mathbf{Set}$$

has an adjoint F,

(b) if $S \in \mathbf{sS}\text{et}$, then $F(S) \simeq \varinjlim_{\mathbf{\Delta}_{/S}} r([n])$ and

(c) if $X \in \mathcal{T}$ and $n \in \mathbb{N}$, then $G(X)_n = \mathrm{Hom}(r([n]), X)$.

Solution: First of all, by definition of currying, if $X \in \mathcal{T}$ and $n \in \mathbb{N}$, then

$$G(X)_n := G(X)([n]) \simeq \mathrm{Hom}(r([n]), X)$$

as claimed in the end. Now, if $S \in \mathbf{sS}\text{et}$, we *set* $F(S) := \varinjlim_{\mathbf{\Delta}_{/S}} r([n])$. Any morphism $f : S \to S'$ will induce a functor $\mathbf{\Delta}_{/S} \to \mathbf{\Delta}_{/S'}$ and provide a morphism

$$F(S) = \varinjlim_{\mathbf{\Delta}_{/S}} r([n]) \to \varinjlim_{\mathbf{\Delta}_{/S'}} r([n]) \simeq F(S').$$

By functoriality of colimits, this defines a functor in the other direction. We can finally prove adjointness: there exists a natural isomorphism

$$\begin{aligned}
\mathrm{Hom}(F(S), Y) &\simeq \mathrm{Hom}\left(\varinjlim_{\mathbf{\Delta}_{/S}} r([n]), Y\right) \\
&\simeq \varprojlim_{\mathbf{\Delta}_{/S}} \mathrm{Hom}(r([n])), Y) \\
&\simeq \varprojlim_{\mathbf{\Delta}_{/S}} G(Y)_n \\
&\simeq \varprojlim_{\mathbf{\Delta}_{/S}} (\mathrm{Hom}(\Delta^n, G(Y)) \\
&\simeq \mathrm{Hom}\left(\varinjlim_{\mathbf{\Delta}_{/S}} \Delta^n, G(Y)\right) \\
&\simeq \mathrm{Hom}(S, G(Y)).
\end{aligned}$$

We finish with some MCQs:

6. (Proof required) In case $r : \mathbf{\Delta} \rightarrowtail \mathbf{sS}\text{et}, [n] \mapsto \Delta^n$ is the Yoneda embedding, then G is

   $\surd$ **faithful** $\surd$ **fully faithful** $\surd$ **an equivalence** ◯ none.

Solution: If $S \in s\mathcal{S}\text{et}$ and $n \in \mathbb{N}$, then there exists a canonical bijection $G(S)_n := \mathrm{Hom}(\Delta_n, S) \simeq S_n$ thanks to Yoneda lemma. This implies that G is isomorphic to the identity and in particular an equivalence. Alternatively, we have $F(S) \simeq \varinjlim_{\mathbf{\Delta}_{/S}} \Delta^n \simeq S$ showing that F is an equivalence, which implies that its adjoint G is an equivalence too.

We recall that a topological space is said to be *compactly generated* if it is a colimit of compact Hausdorff spaces and that there exists topological spaces such as $\mathbb{R}^{\mathbb{R}}$ that are *not* compactly generated.

7. (Proof required) In case $r : \mathbf{\Delta} \rightarrowtail \mathbf{T}\mathrm{op}, [n] \mapsto |\Delta_n|$ sends $[n]$ to the convex envelope of the canonical basis $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$ and α to the linear map $e_i \mapsto e_{\alpha(i)}$, then G is

   $\surd$ **faithful** ◯ fully faithful ◯ an equivalence ◯ none.

   Solution: If X is a topological space, then $G(X)_0 := \mathcal{C}(\{e_0\}, X) \simeq X$ (sending a map to the image of e_0). Moreover, if $f : X \to Y$ is a continuous map, then $G(f)_0$ identifies to the underlying map of f. This shows that f is uniquely determined by $G(f)_0$ and, in particular, by $G(f)$ so that G is faithful.

   In order to show that G is not fully faithful (and therefore not an equivalence either), it is sufficient to show that the counit map $F(G(X)) \to X$ is not always a homeomorphism. But, $F(G(X)) \simeq \varinjlim_{\mathbf{\Delta}_{/G(X)}} |\Delta^n|$ is always compactly generated ($|\Delta^n|$ is closed and bounded in $\mathbb{R}^{n+1}$) although X need not be.

8. (**No** proof required) In case $r : \mathbf{\Delta} \rightarrowtail \mathbf{C}\mathrm{at}, [n] \mapsto [\mathbf{n}]$ is the canonical embedding, then G is

   $\surd$ **faithful** $\surd$ **fully faithful** ◯ an equivalence ◯ none.

   Solution: (sketch) If $\mathcal{C}$ is a small category, then

   $$G(\mathcal{C})_n = \mathrm{Hom}([\mathbf{n}], X) = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\}.$$

   In particular, we may identify $G(\mathcal{C})_0$ (resp. $G(\mathcal{C})_1$) with the set of objects (resp. morphisms) of $\mathcal{C}$. Moreover, $d^i_n : G(\mathcal{C})_n \to G(\mathcal{C})_{n-1}$ is obtained by composing f_i with f_{i+1} for $i \neq 0, n$ and by removing f_1 (resp. f_n) when $i = 0$ (resp. n). Also, $s^i_n : G(\mathcal{C})_n \to G(\mathcal{C})_{n+1}$ consists in inserting the identity at the $i+1$st place.

   Now, if $\mathcal{C}'$ is another category and $\varphi : G(\mathcal{C}) \to G(\mathcal{C}')$ is a map of simplicial sets, one easily checks by induction that

   $$\varphi\left(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\right) = \left(\varphi_0(X_0) \xrightarrow{\varphi_1(f_1)} \varphi_0(X_1) \xrightarrow{\varphi_1(f_2)} \cdots \xrightarrow{\varphi_1(f_n)} \varphi_0(X_n)\right)$$

   so that φ is only determined by φ_0 (on objects) and φ_1 (on morphisms). It is also easy to verify that we always have $\varphi_1(g \circ f) = \varphi_1(g) \circ \varphi_1(f)$ and $\varphi_1(\mathrm{Id}_X) = \mathrm{Id}_{\varphi_0(X)}$. This shows that φ comes from a unique functor $\mathcal{C} \to \mathcal{C}'$ so that G is fully faithful.

   In order to show that G is not an equivalence, it is sufficient to exhibit a simplical set that is not of the above type. One may for example let S_n be the set of all (not necessarily order preserving) maps from $[n]$ to $[1]$. Another possibility is to consider the set of ordered preserving maps $[n] \to [2]$ that factor (in some way) through $[1]$. In both cases, the map associated to some $\alpha : [m] \to [n]$ will be obtained by composition. Details omitted.