Université de Rennes 2025-2026

Homological algebra
Homework (due November 7th)

You can compose in english or in french. You can freely (no need to make a precise reference) use any result obtained prior to the statement of the exercise in the (online) course.

1. Show that, if k is a commutative ring, then $\mathbf{O}\mathrm{p}(k\text{-}\mathbf{M}\mathrm{od})$ is isomorphic to $k[t]\text{-}\mathbf{M}\mathrm{od}$.

Solution: If E is a $k[t]$-module, then it is a k-module by restriction of scalars and the map $u : E \to E, x \mapsto tx$ is k-linear. Conversely, if E is a k-module and $u \in L(E)$, then there exists a unique morphism of k-algebras $k[t] \to L(E), t \mapsto u$. This defines the structure of a $k[t]$-module on E. Concretely, $fx := f(u)(x)$ for $f \in k[t]$ and $x \in E$. Thus, we have an explicit bijection on objects and we turn now to morphisms. First of all, functoriality is automatic. Now, if we are given a morphism of $k[t]$-modules $\varphi : E \to F$ and we denote by $u \in L(E)$ and $v \in L(F)$ the corresponding endomorphisms, then we have $(\varphi \circ u)(x) = \varphi(tx) = t\varphi(x) = (v \circ \varphi)(x)$ for $x \in E$ so that $\varphi \circ u = v \circ \varphi$. It remains to show that, any k-linear map $\varphi : E \to F$ satisfying $\varphi \circ u = v \circ \varphi$ is a morphism of $k[t]$-modules. We give ourselves $f \in k[t]$ and $x \in E$ and we need to prove that $\varphi(fx) = f\varphi(x)$, or equivalently $\varphi(f(u)(x)) = f(v)(\varphi(x))$. In other words, we have to show that $\varphi \circ f(u) = f(v) \circ \varphi$. By linearity, we may assume that $f = t^n$ for some $n \in \mathbb{N}$ and the assertion reduces to $\varphi \circ u^n = v^n \circ \varphi$ which is easily obtained by induction.

2. Show that a split monomorphism is a regular monomorphism and that a regular monomorphism is a monomorphism (and dual).

Solution: Assume $i : X \to Y$ is a split monomorphism. By definition, it admits a retraction $r : Y \to X$ (so that $r \circ i = \mathrm{Id}_X$) and we will prove that the diagram

$$X \xrightarrow{\ i\ } Y \overset{i\circ r}{\underset{\mathrm{Id}_Y}{\rightrightarrows}} Y$$

is left exact. If we are given a morphism $f : Z \to Y$ such that $i \circ r \circ f = f$, and we set $g = r \circ f : Z \to X$, then $i \circ g = f$. Assume conversely, that g satisfies $i \circ g = f$. Then, $g = r \circ i \circ g = r \circ f$ which shows uniqueness.

Assume now that $i : X \to Y$ is a regular monomorphism so that there exists a left exact diagram

$$X \xrightarrow{\ i\ } Y \overset{u}{\underset{v}{\rightrightarrows}} Y.$$

We give ourselves $g_1, g_2 : Z \to X$ such that $i \circ g_1 = i \circ g_2$ and we denote this common map by f so that $f = i \circ g_k$ for $k = 1, 2$. We have $u \circ f = u \circ i \circ g_k =$

$v \circ i \circ g_k = v \circ f$. Therefore, there exists a *unique* $g : Z \to X$ such that $f = g \circ i$. It follows that $g = g_k$ and then $g_1 = g_2$.

By duality, a split epimorphism is a regular epimorphism and a regular epimorphism is an epimorphism.

3. Show that a morphism $X \to Y$ in a category $\mathcal{C}$ is a monomorphism (resp. an epimorphism) if and only if the induced functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ (resp. ${}_{Y\backslash}\mathcal{C} \to {}_{X\backslash}\mathcal{C}$) is fully faithful.

Solution: If we denote the morphism by $i : X \to Y$, then an object in $\mathcal{C}_{/X}$ is a morphism $f : Z \to X$ and the induced functor sends f to the composite $i \circ f : Z \to Y$. A morphism between $f : Z \to X$ and $f' : Z' \to X$ is a morphism $g : Z \to Z'$ such that $f' \circ g = f$. It is sent to the same morphism g but the condition now reads $i \circ f' \circ g = i \circ f$. To make it clear, we have

$$\mathrm{Hom}_{\mathcal{C}_{/X}}(f, f') \subset \mathrm{Hom}_{\mathcal{C}_{/Y}}(i \circ f, i \circ f') \subset \mathrm{Hom}_{\mathcal{C}}(Z, Z').$$

The induced functor is fully faithful when the first inclusion is an equality. It means that condition "$i \circ f' \circ g = i \circ f \Rightarrow f' \circ g = f$" is always satisfied. By definition, this is automatic when i is a monomorphism. Conversely, the condition in the case $Z = Z'$ and $g = \mathrm{Id}_Z$ will tell us that i is a monomorphism.

By duality, $\pi : X \to Y$ is an epimorphism in $\mathcal{C}$ if and only it is a monomorphism (from Y to X) in $\mathcal{C}^{\mathrm{op}}$ if and only if $\mathcal{C}^{\mathrm{op}}_{/Y} \to \mathcal{C}^{\mathrm{op}}_{/X}$ is fully faithful, or equivalently since it boils down to the same condition, if and only if ${}_{Y\backslash}\mathcal{C} = (\mathcal{C}^{\mathrm{op}}_{/Y})^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}}_{/X})^{\mathrm{op}} = {}_{X\backslash}\mathcal{C}$ is fully faithful.

4. Show that colimits of *sets* are stable under pullback.

Solution: It is sufficient to consider the cases of a coproduct or a cokernel.

If $X := \coprod_{i \in I} X_i$ is a disjoint union of sets and we are given two maps $f : X \to Y$ and $g : Z \to Y$, then clearly

$$\coprod_{i \in I} (X_i \times_Y Z) = X \times_Y Z.$$

Actually, on both sides, an élément is a couple (x, z) with $x \in X$, $z \in Z$ and $f(x) = g(z)$.

We consider now a pair of maps $u, v : X' \to X$, a cocone which is another pair of maps $f' : X' \to Y, f : X \to Y$ satisfying $f \circ u = f \circ v = f'$ and finally a map $g : Z \to Y$. We have coker $(u, v) = X/\widetilde{R}$ where $\widetilde{R}$ is the equivalence relation generated by $u(x')Rv(x')$ whenever $x' \in X'$. Since $f \circ u = f \circ v$, the map f factors through the surjection $X \twoheadrightarrow X/\widetilde{R}, x \mapsto \overline{x}$ and provides $\overline{f} : X/\widetilde{R} \to Y$. We can then consider the canonical map

$$\Pi : X \times_Y Z \to X/\widetilde{R} \times_Y Z, \quad (x, z) \mapsto (\overline{x}, z).$$

It is well defined and surjective: this follows from the fact that, since $\overline{f}(\overline{x}) = f(x)$, we have

$$\forall x \in X, z \in Z, \quad (x, z) \in X \times_Y Z \Leftrightarrow (\overline{x}, z) \in X/\widetilde{R} \times_Y Z.$$

The equivalence relation $\widetilde{S}$ on $X \times_Y Z$ generated by $(u(x'), z)S(v(x'), z)$ whenever $(x', z) \in X' \times_Y Z$ is given by

$$(x_1, z_1)\widetilde{S}(x_2, z_2) \Leftrightarrow x_1\widetilde{R}x_2 \text{ and } z_1 = z_2.$$

Thus, Π provides a bijection

$$\text{coker } (U, V) = (X \times_Y Z)/\widetilde{S} \simeq X/\widetilde{R} \times_Y Z = \text{coker } (u, v) \times_Y Z.$$

5. Show that, if k is a commutative ring, then $k[t]$ (resp. $k[t]_t$) endowed with

$$\mu : t \mapsto t \otimes 1 + 1 \otimes t \quad (\text{resp. } \nu : t \mapsto t \otimes t)$$

is an abelian group of the category opposite to the category of k-algebras.

Solution: Commutativity is clear in both cases and makes it faster to check the other properties. In order to prove associativity, we compute

$$\begin{aligned}
((\text{Id} \otimes \mu) \circ \mu)(t) &= (\text{Id} \otimes \mu)(t \otimes 1 + 1 \otimes t) \\
&= t \otimes \mu(1) + 1 \otimes \mu(t) \\
&= t \otimes 1 \otimes 1 + 1 \otimes (t \otimes 1 + 1 \otimes t) \\
&= t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t.
\end{aligned}$$

By symmetry, we see that

$$((\text{Id} \otimes \mu) \circ \mu)(t) = (\mu \otimes \text{Id}) \circ \mu)(t).$$

In the same way, we have

$$((\text{Id} \otimes \nu) \circ \nu)(t) = t \otimes t \otimes t = (\nu \otimes \text{Id}) \circ \nu)(t).$$

The unit is $\epsilon : k[t] \to k, t \mapsto 0$ (resp. $\eta : k[t]_t \to k, t \mapsto 1$) as the following shows:

$$((\epsilon \otimes \text{Id}) \circ \mu)(t) = (\epsilon \otimes \text{Id})(t \otimes 1 + 1 \otimes t) = \epsilon(t) \times 1 + \epsilon(1) \times t = 0 \times 1 + 1 \times t = t.$$

$$(\text{resp. } (\eta \otimes \text{Id}) \circ \nu)(t) = (\eta \otimes \text{Id})(t \otimes t) = \eta(t) \times t = 1 \times t = t).$$

Finally, the inverse is given by $i : t \mapsto -t$ (resp. $j : t \mapsto t^{-1}$): if we to compose μ (resp. ν) with the map

$$\begin{aligned} k[t] \otimes_k k[t] &\longrightarrow k[t] \\ f \otimes g &\longmapsto f(-t)g(t) \end{aligned} \qquad \left(\text{resp.} \quad \begin{aligned} k[t]_t \otimes_k k[t]_t &\longrightarrow k[t]_t \\ f \otimes g &\longmapsto f(t^{-1})g(t) \end{aligned} \right),$$

then we get the constant map 0 (resp. 1).